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THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teachers' agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

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THE NOVEMBER MEETING OF THE SOUTHERN CALIFORNIA SECTION

The eighth regular meeting of the Southern California Section was held at Fullerton Junior College, Fullerton, California, on Saturday, November 3, 1928. Professor E. T. Bell presided.

The attendance was forty-eight, including the following thirty-one members of the Association: O. W. Albert, E. E. Allen, L. D. Ames, M. A. Basoco, Harry Bateman, Clifford Bell, E. T. Bell, Grace E. Berry, W. N. Birchby, Jessie R. Campbell, P. H. Daus, Iva B. Ernsberger, Raymond Garver, Harriet E. Glazier, W. L. Hart, E. R. Hedrick, G. H. Hunt, W. E. Mason, A. W. Prater, Lena E. Reynolds, W. P. Russell, G. E. F. Sherwood, H. M. Showman, D. V. Steed, Morgan Ward, L. E. Wear, Mabel G. Whiting, W. M. Whyburn, Clyde Wolfe, Euphemia R. Worthington.

The following program was presented:

1. "A note on the evaluation of two definite integrals," by Professor H. Bateman, California Institute of Technology.
2. "An application of a Tschirnhaus transformation to elementary theory of equations," by Professor Raymond Garver, University of California at Los Angeles.
3. "A complex space," by Professor L. E. Wear, California Institute of Technology.
4. "Boundary value problems for second order differential systems," by Professor W. M. Whyburn, University of California at Los Angeles.
5. "Relations satisfied by the binomial series," by Dr. Morgan Ward, California Institute of Technology.
6. "Conservative new-type examinations for college mathematics and their coefficient of reliability," by Professor W. L. Hart, University of Minnesota.

Abstracts of these papers follow:

1. The well known inversion formula of Fourier may be generalized to meet the case in which the product $\cos(xt)\cdot\cosh(xt)$ occurs in the first integral instead of $\cos(xt)$. The inversion formula is of a slightly different type and is applicable only to a certain class of functions. A definite integral mentioned in a previous verification of the inversion formula in a special case is now evaluated rigorously. A similar integral is also evaluated to check the corresponding generalization of Fourier's sin-formula.

2. By applying a simple Tschirnhaus transformation to the roots of the reduced cubic, the cubic is transformed to the form $z^3+Az^2-\Delta=0$, where Δ is the discriminant of the cubic. In this form, the relation between the nature of the roots and the discriminant is readily illustrated graphically.

3. A point in three-dimensional space may be determined by a complex variable z and a real variable t. An equation connecting z and t will be a space curve. Certain equations are discussed and the map equations of some surfaces given.

4. In an article that is to appear in the Transactions of the American Mathematical Society, the author introduces a transformation that when applied to a class of non-linear differential equations of the second order yields existence and oscillation theorems for these equations and boundary conditions of the Sturmian type. The class of non-linear systems treated includes the linear system, and for this case the methods of the paper yield a very simple derivation of many of the properties that were established by Sturm, Bôcher, and others. The present paper is confined primarily to the linear case of the above article.

5. Some functional equations are considered and some of their properties developed. The connection between these functions and the binomial series is explained.

6. Professor Hart argues against the use of the typical varieties of questions found in so-called objective examinations as the sole foundation for a reformed system of examinations in mathematical examinations. He advises the use of an alternative variety of short-answer questions of a direct nature, as contrasted with the indirectness of the typical objective examination. For a particular new-type examination of this conservative type which Professor Hart gave to six hundred students, the coefficient of reliability is approximately .9.

P. H. DAUS, *Secretary.*

GRAPHICAL INTEGRATION AND DIFFERENTIATION OF FUNCTIONS IN A POLAR CÖORDINATE SYSTEM

By E. A. KHOLODOVSKY, New York, N. Y.

1. In his book on *Graphical Methods*[1] (Columbia University lectures) Carl Runge gives methods of graphical integration of functions in a Cartesian coördinate system, that is, the methods of finding graphically the curve whose equation in Cartesian coördinates could be written

$$y = \int_{x_0}^{x} f(x)dx,$$

when the curve $y=f(x)$ is given graphically, and also methods of graphical differentiation.

In some cases the solution of a problem requires graphical integration and differentiation when the equation of the curve, which is given graphically, would be written in polar coördinates. The methods of this graphical integration and differentiation in a polar coördinate system will be expounded in this article.

[1] For the literature of the subject see: *Encyklopädie der Mathematischen Wissenschaften* B. II$_3$. N2 (or the French edition); C. Runge, *Graphische Methoden* (1914); B. Mehmke, *Leitfaden zum graphischen Rechnen* (1924); J. Massau, *Mémoire sur l'intégration graphique* (1885); Fr. Willers, *Graphische Integration* (1920); D'Ocagne, *Calcul graphique et nomographie* (1908).

GRAPHICAL INTEGRATION

2. *The problem of graphical integration.* The problem of integration can be formulated as follows. Let $\rho=\phi(\theta)$ be the polar equation of the given curve AB (or a broken line) with the point 0 as the pole. It is required to find the curve of variation of the area of a sector bounded by the given curve and two vectors from the point O when the polar angle θ increases (Fig. 1); in other terms to find the function

$$\Phi(\theta) = \frac{1}{2}\int_{\theta_0}^{\theta} \phi^2(\theta)d\theta.$$

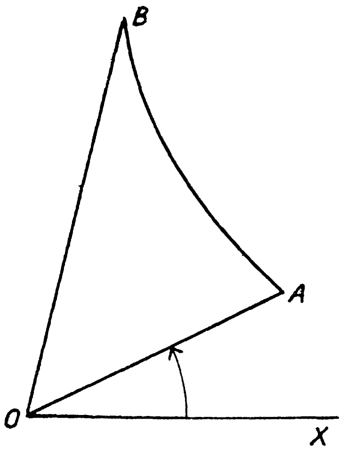

FIG. 1.

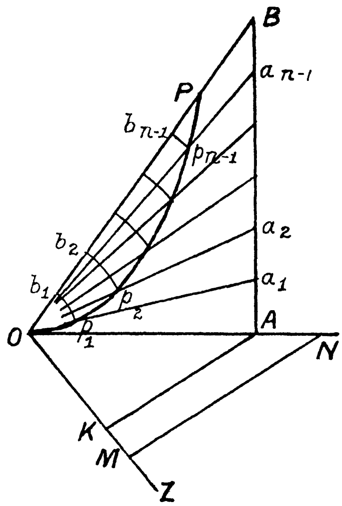

FIG. 2.

The curve may be closed with the point O inside the curve (Fig. 15) or outside (Fig. 9).

If the function is given by an equation $\rho=\phi(\theta)$ we can graph the curve representing the function and then apply the graphical method of integration.

3. *Area of a triangle with a constant base.* Let us begin with a simple particular case of finding the curve of variation of the area of a right triangle OAB (Fig. 2) with a constant base $OA=a$ and increasing altitude AB. On an arbitrary axis OZ put the segment OK equal to the unit of length and the segment $OM=\frac{1}{2}AB$. Join the points A and K and draw MN parallel to KA, N being the point of intersection of this line with the line OA. On the side OB we lay the length of the segment ON; that is we make $OP=ON$. The number measuring the length of the segment OP gives the measure of the area of the triangle OAB in quadratic measure, for $ON/OM=OA/OK$, $ON=\frac{1}{2}AB\cdot OA$. If we divide AB in n equal parts and join the points of division $a_1, a_2, \cdots, a_{n-1}$ with the point O, we obtain the triangles OAa_1, $Oa_1a_2, \cdots, Oa_{n-1}b$ with the area of each equal to OP/n. Dividing OP in n equal parts Ob_1, $b_1b_2, \cdots,$

$b_{n-1}P$ and drawing circles with the center at O through the points $b_1, b_2, \cdots,$ b_{n-1}, we get the points $p_1, p_2, \cdots, p_{n-1}$, of intersection of these circles with corresponding rays $Oa_1, Oa_2, \cdots, Oa_{n-1}$. The vectors $Op_1, Op_2, \cdots, Op_{n-1}$ measure correspondingly the areas of the triangles $OAa_1, OAa_2, \cdots, OAa_{n-1}$, and thus we get the points $O, p_1, p_2, \cdots, p_{n-1}$, of the integral curve.

If we take the point O as the pole and OA as the polar axis, the polar equation of AB is $\rho = a/\cos\theta$ $(a = OA)$, and the equation of the obtained integral curve is

$$\Phi(\theta) = \frac{1}{2}\int_0^\theta \frac{a^2 d\theta}{\cos^2\theta} = \frac{a^2}{2}\tan\theta.$$

4. *When the given curve is represented by a straight line.* We shall find the integral curve representing the variation of the area bounded by an arbitrary straight line AB and vectors OA and OB, when the vector OB rotates from coincidence with the vector OA to the position OB (Fig. 3).

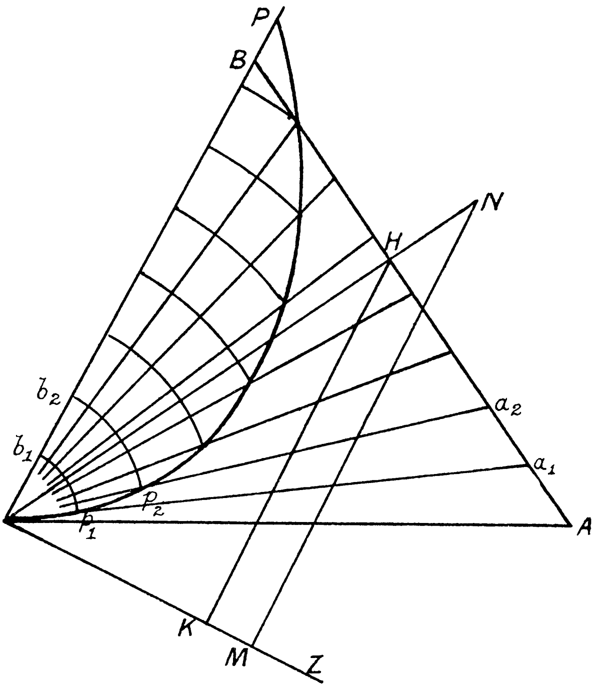

FIG. 3.

Construction: (1) Draw the perpendicular OH to AB, point H being the foot of this perpendicular. (2) On an arbitrary axis OZ construct $OK = 1$ and $OM = \frac{1}{2}AB$. (3) Join the points K and H and draw through the point M a parallel to KH. The point N is the intersection of this line with the perpendicular OH. (4) On OB we lay the length $OP = ON$. The length of the vector OP is the measure of the area of the triangle OAB (as in §3).

Dividing AB and OP in n equal parts we obtain the integral curve in the same way as in §3.

5. *The direction of the axis OZ.* The direction of the axis OZ on which we construct the unit-segment OK is arbitrary, but we shall obtain the most exact sketch if the angle ONM and consequently the angle OHK is greatest within the limits 0 and $\pi/2$. This we obtain when $HK \perp OH$ in case $OH<1$ and when $HK \perp OZ$ in case $OH<1$ (Fig. 3). The maximum of

$$\tan OHK = (\sin HOK)/(h - \cos HOK)$$

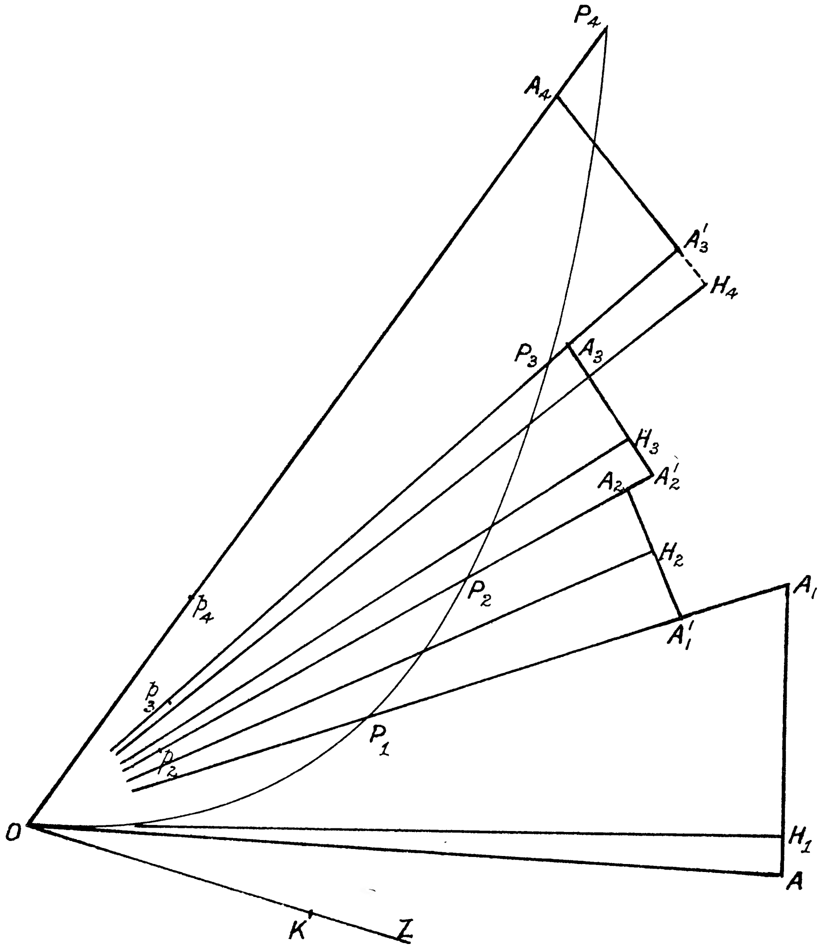

FIG. 4.

occurs when $\cos HOK = h$ in first case and when $\cos HOK = 1/h$ in second case. $h = OH$. Therefore the best position for the axis OZ is one making approximately a right angle with the direction HK in case $OH>1$.

6. *When the given curve is represented by a broken line.* We shall demonstrate the methods of finding the integral curve in case the area is bounded by a broken line and vectors issued from the point O.

First method (Fig. 4): By joining the vertices of the broken line $AA_1A_1'A_2A_2'A_3A_3'A_4$ with the point O we divide the given area into triangles and we can find the area of each triangle separately as in §4. In this way we obtain on the rays OA_1, OA_2, OA_3, OA_4 the vectors OP_1, Op_2, Op_3, Op_4 measuring correspondingly the areas of the triangles OAA_1, $OA_1'A_2$, $OA_2'A_3$, $OA_3'A_4$. Adding to the vector Op_2 on the ray OA_2 the length OP_1, $Op_2+OP_1=OP_2$; adding to the

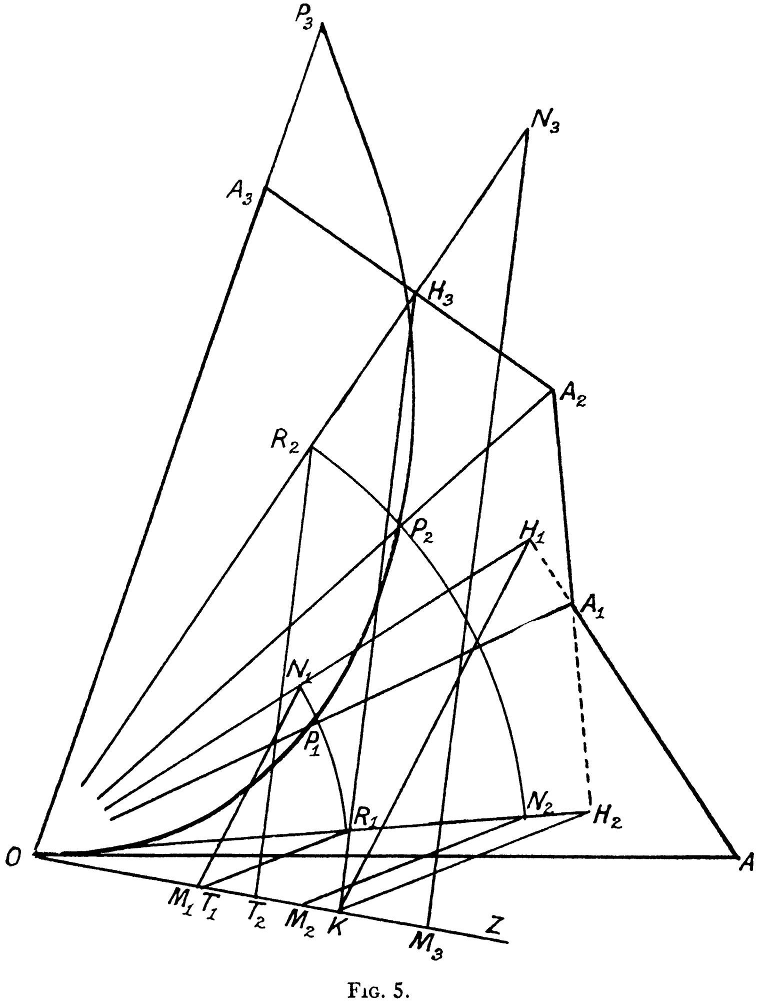

FIG. 5.

vector Op_3 on the ray OA_3 the length OP_2, $Op_3+OP_2=OP_3$; and so on; we obtain the points P_1, P_2, P_3, $\cdots$, of the integral curve.

Following the method of §3 and §4 we can obtain, if necessary, the intermediate points of the curve. We take for different triangles different axes OZ to obtain most accurate results.

Second method (Fig. 5). Instead of constructing vectors measuring the area of each triangle separately it is possible to obtain directly the vectors OP_1, OP_2, $\cdots$, which measure the areas OAA_1, $OAA_1A_1'A_2$, $\cdots$.

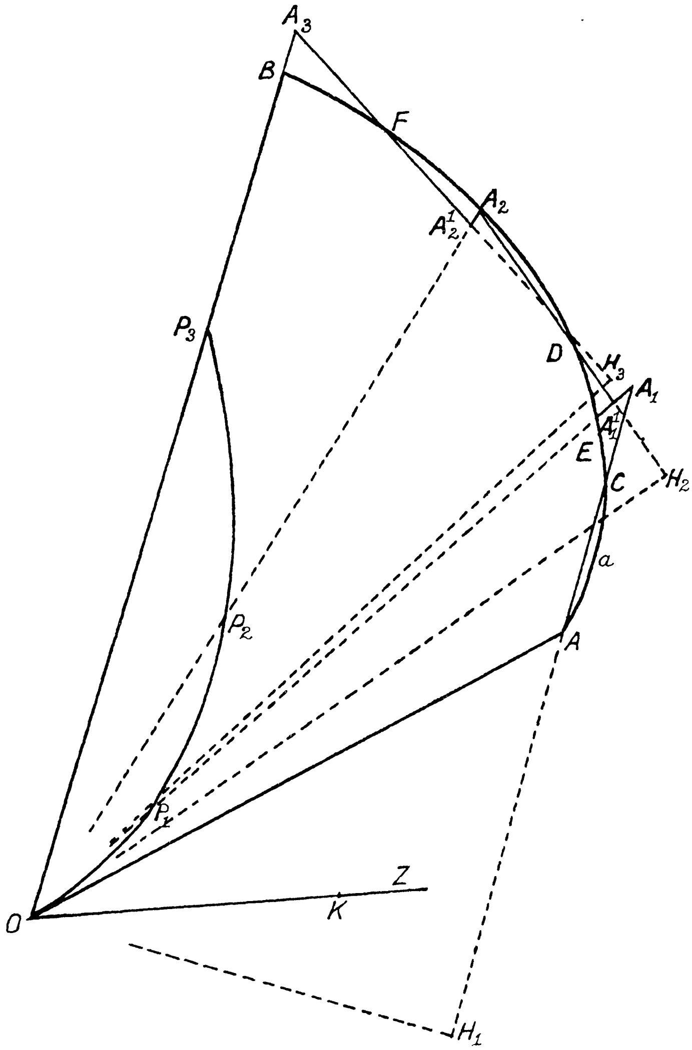

FIG. 6.

Construction: (1) Find the vector OP_1 measuring the area of the triangle OAA_1 as before. (2) On the line OH_2 perpendicular to A_1A_2 lay the length $OR_1 = OP_1$. (3) Join the points K and H_2 and draw the line $R_1T_1 \| KH_2$. (4) From the point T_1 lay off the length $T_1M_2 = \frac{1}{2}A_1A_2$. (5) Draw M_2N_2 parallel to KH_2, N_2 being the intersection of this line with the perpendicular OH_2. (6) On the line OA_2 construct the segment $OP_2 = ON_2$.

The measure of the vector OP_2 is the measure of the area OAA_1A_2, for

$$ON_2/OH_2 = OM_2/OK\ ; \quad ON_2 = OH_2(T_1M_2 + OT_1) = OH_2(A_1A_2/2) + OH_2 \cdot OT_1.$$

But $OH_2/OR_1 = OK/OT_1$, $OH_2 \cdot OT_1 = OR_1 = OP_1$, and $OP_2 = ON_2 = OH_2\ (A_1A_2/2) + OP_1$, the area of the triangles OA_1A_2 and OAA_1.

Proceeding in the same way, on OH_3 we construct $OR_2 = OP_2$; then we draw $R_2T_2 \| KH_3$ and lay off $T_2M_3 = (A_2A_3/2)$. Next we draw $M_3N_3 \| KH_3$ and construct $OP_3 = ON_3$, and so on.

7. *General case.* If the area is bounded by a curve AB and vectors OA and OB (Fig. 6) we replace the given curve by a broken line $AA_1A_1'A_2A_2'A_3$ as in integration in a Cartesian coördinate system, so that the area which is added is compensated by an area which is subtracted. Thus we subtract the area AaC and add the area CA_1E, etc.

The integral curve of this broken line gives us an approximate integral curve for the given curve which may be corrected taking in consideration added and subtracted areas.

With a careful drawing we can obtain a very close approximation. When drawing, it is more convenient to construct first the broken line and, joining all vertices $A_1, A_2, \cdots$ with O, then to draw all perpendiculars $OH_1, OH_2, \cdots$ and bisect all lines $AA_1, A_1'A_2, \cdots$, and afterwards to proceed to locate the points of the integral curve with the OZ axis suitably chosen.

The radii vectors OP_1, OP_2, OP_3, (Fig. 6) of the integral curve corresponding to the points E, A_2, B, of the given curve, where the added and subtracted areas are compensated, represent the exact value of the required sectorial area as nearly as the added areas equal the corresponding subtracted ones.

If we are not satisfied with estimating the equivalence of these areas by eye, we may draw mechanically a broken stepping line to take place of the given curve applying the same construction as in the case of a Cartesian system. (See any of the books referred to at the beginning of this article.)

In Fig. 7 the stepping line $A_1A_2 \cdots A_8$ replaces the given curve $AabcdeA_8$. The area A_1A_2a equals the area aA_3b, the area bA_4c equals the area cA_5d, the area dA_6e, the area eA_7A_8. At the points A_1, b, d, A_8 of the given curve, the corresponding radii vectors of the integral curve represent the exact value (in the limits of exactitude of the drawing) of the required sectorial area.

The stepping line has the advantage that the altitudes of the triangles all coincide with the straight lines OA_1, or OB perpendicular to OA_1, though it increases the number of segments required for construction. We may diminish the number of segments without decreasing the accuracy of substituted areas as shown in Fig. 8: $A_2C_1 = C_1A_3$, $A_4C_2 = C_2A_5$, $A_7C_3 = C_3A_8$. Instead of 8 triangles we have only 5, substituting the broken line $A_1B_1B_2A_6B_3A_9$ for the stepping line.

8. *A closed curve.* If it is necessary to find the area of a closed curve $ABCD$ (Fig. 9), we draw the tangents OA and OC to the curve and we construct the

integral curves for the curves ABC and CDA. The difference of the vectors for the same angle will give the measure of the corresponding part of the area bounded by the closed curve and corresponding radius vector. So Op is the measure of the area ABD.

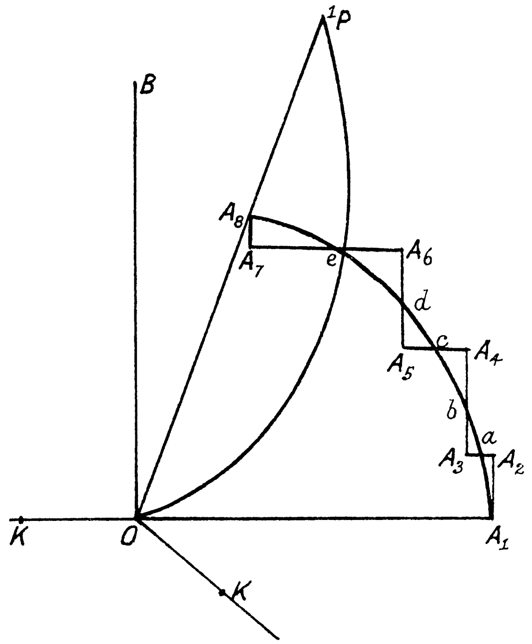

FIG. 7.

9. *A particular case.* We proceed in the same manner, if the vector intersects the given curve in several points (Fig. 10). Drawing the tangents OC and OB we find by subtraction the area CBD. When this area is subtracted from the area OAB we get the required area.

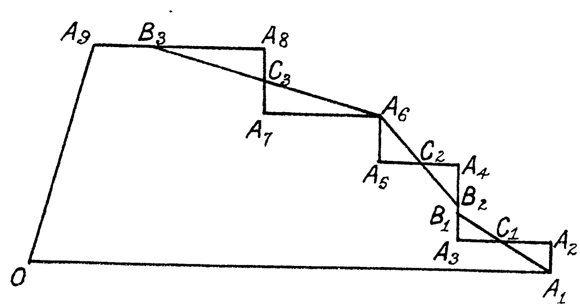

FIG. 8.

10. *Adding of a constant area.* Usually the integral curve in beginning is very flat which makes it difficult to define the points of the given curve corresponding to different radii vectors of the integral curve. We can avoid this

difficulty by adding an arbitrary constant area. This is accomplished by adding to each radius vector (constructed as shown previously) the same length ρ_0. The points obtained belong to the curve whose equation is $\rho = \Phi(\theta) + \rho_0$.

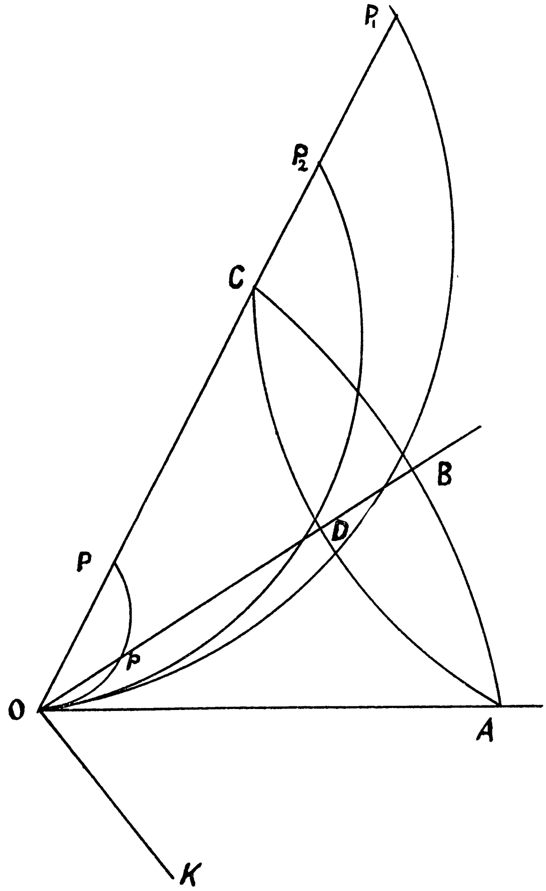

FIG. 9.

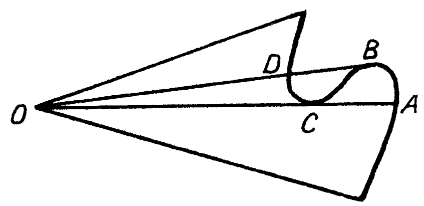

FIG. 10.

We may proceed in another manner: we put on the initial ray OA (Fig. 11) the length $OP_0 = \rho_0$ measuring this added constant area and then proceed as shown in §6. On OH_1 lay the length $OR_0 = OP_0$; join K and H_1; draw R_0T_0 parallel to KH_1; lay $T_0M_1 = \frac{1}{2}AA_1$; draw M_1N_1 parallel to KH_1; and put $OP_1 =$

ON_1. It is easy to see that the vector OP_1 measures the area of the triangle OAA_1 plus the area measured by the vector OR_0. $OP_1 = p_1A_1 + Op_1$. Then we continue as in §6. Every radius vector of the integral curve is increased by the same constant.

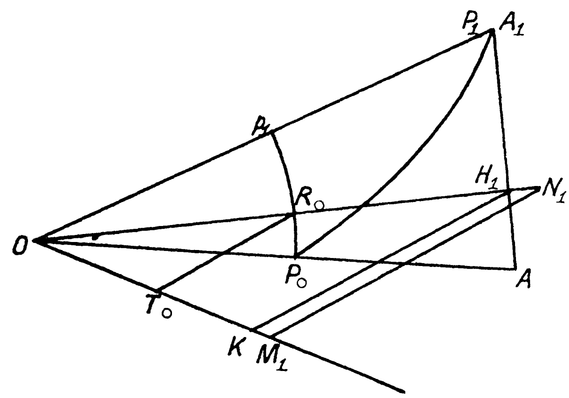

FIG. 11.

11. *Division of the area of a sector in equal sectors.* By use of graphical integration we can divide the area of a sector bounded by a curve and two radii vectors in n equal sectors. After constructing the integral curve $OP_1P_2 \cdots P_n$, we divide the vector OP_n representing the value of the whole area into n equal parts $Ob_1 = b_1b_2 = \cdots = b_{n-1}P_n$ and draw circles through the points of division $b_1, b_2, \cdots, b_{n-1}$, taking O as the center. The circles intersect the integral curve in points $p_1, p_2, \cdots, p_{n-1}$. Drawing rays from the point O through these points $p_1, p_2, \cdots, p_{n-1}$ meeting the given curve in points $a_1, a_2, \cdots, a_{n-1}$, we get the sectors $OAa_1, Oa_1a_2, \cdots, Oa_{n-1}B$ of equivalent areas. If for the construction of the integral curve we added a constant area, as it was shown in §10, we divide in n equal parts the segment $P_0'P$ (Fig. 12) representing the area of the given sector, $OP_0' = OP_0$ being the measure of the added constant area.

In a similar way we may divide the given area in n sectors proportional to arbitrary numbers dividing the vector which measures the whole area in n parts proportional to these given numbers.

12. *The influence of a change of scale.* Let OH be the length of the perpendicular drawn from the point O to the base of a triangle, OM one half the length of its base, OK and OK_1 two different units of length (Fig. 13). Constructing as in §4, we draw MN parallel to KH and MN_1 parallel to K_1H. We obtain

$$ON/OH = OM/OK, ON_1/OH = OM/OK_1;$$

hence $ON/ON_1 = OK_1/OK$. The lengths of the vectors which represent the value of the areas are inversely proportional to the length of the segments taken as units. As all vectors are changed in the same ratio, the integral curves give similar figures (Fig. 14).

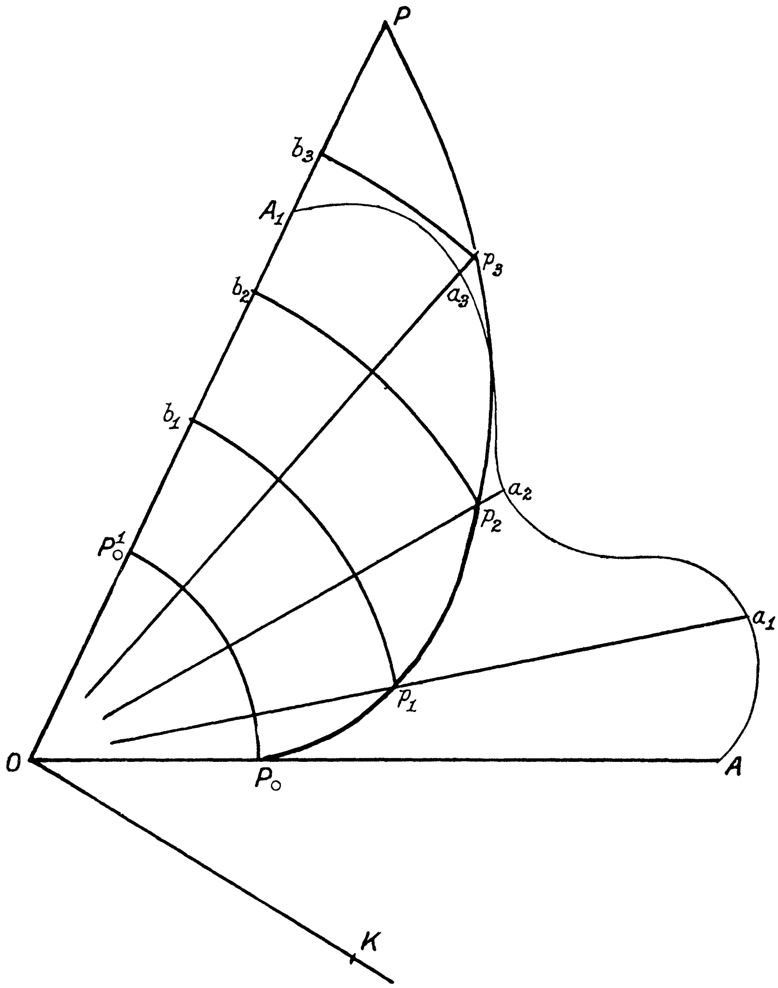

FIG. 12.

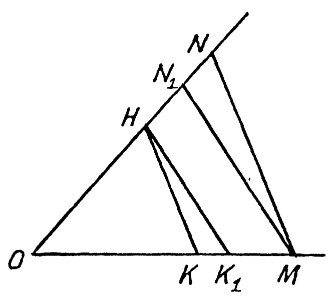

FIG. 13.

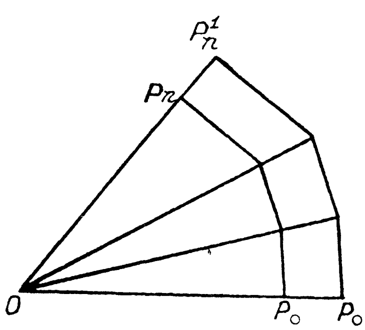

FIG. 14.

13. *Example.* As an example we shall find the positions in its orbit of a body moving according to the law of Kepler (with a constant sectorial velocity) at any given epoch (Fig. 15). Let, for instance, the orbit be an elliptic one and the

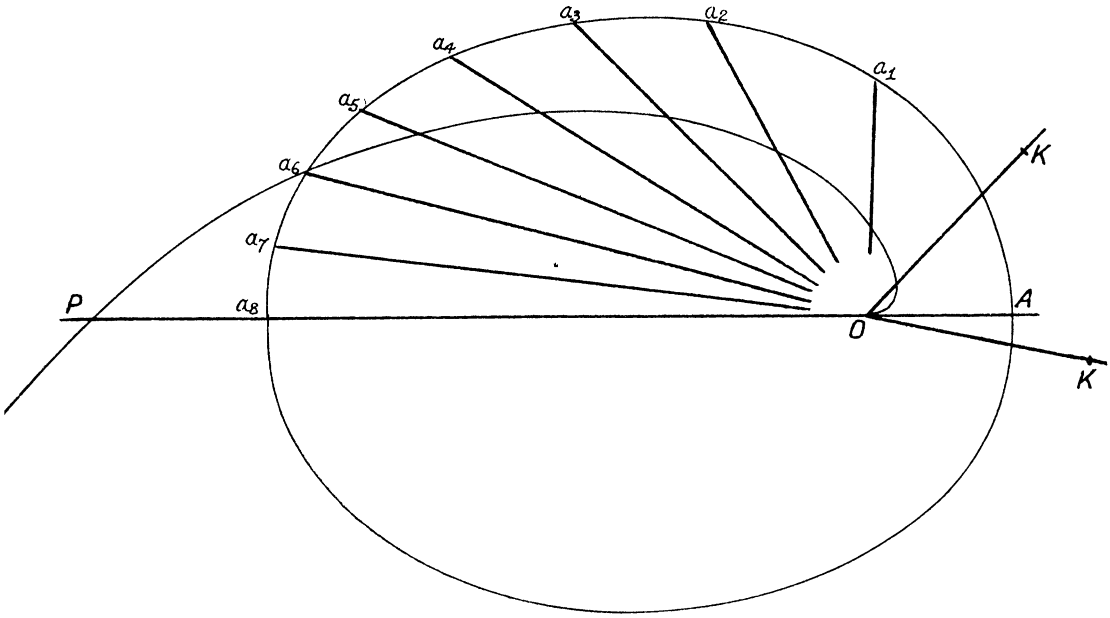

FIG. 15.

point O the focus (center of force). Let the times t and t' of any two positions of the body in its orbit be given, say at the points A and a_5. We construct the integral curve for this ellipse and we divide the radius vector of this curve measuring the sectorial area of the ellipse from the point A to the point a_5 in n equal parts, as was shown in §11. Constructing sectors with equal areas OAa_1, Oa_1a_2, Oa_2a_3, $\cdots$, Oa_4a_5, we get on the given orbit the arcs Aa_1, a_1a_2, a_2a_3, $\cdots$, a_4a_5 which the moving body passes in equal intervals of time, namely $(t'-t)/n$. We may continue constructing equivalent sectors and so extrapolate the points a_i. We may subdivide any of these intervals in as many equal subintervals as we please.

It is sufficient to construct the integral curve only for the upper half of the ellipse. To continue the integral curve we draw the rays dividing the area of the ellipse in equal parts (or any rays) and construct the vectors of corresponding length.

This method may be applied to problems of mechanics where the moving point is subject to a central force; under these circumstances it is moving according to the law of areas.

14. In some cases (in graphical integration of differential equations, for instance) it is necessary to find graphically the curve, the polar equation of

which could be written $\rho = \Phi(\theta) + C$, where $\Phi(\theta) = \int\phi(\theta)d\theta$, and $\rho = \phi(\theta)$ is the polar equation of a curve which is given graphically; C is the initial value of the integral $\int\phi(\theta)d\theta$. To make this construction we put $\phi(\theta) = \frac{1}{2}\psi^2(\theta)$ and construct the curve $\rho = \psi(\theta)$. The integral curve of this curve, $\rho = \frac{1}{2}\int\psi^2(\theta)d\theta$, is the required curve, as $\frac{1}{2}\int\psi^2(\theta)d\theta = \int\phi(\theta)d\theta$.

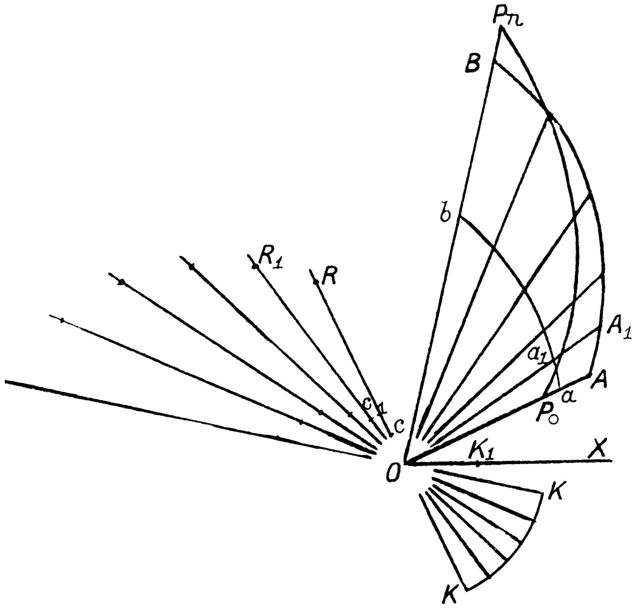

FIG. 16.

Construction of the curve $\rho = \psi(\theta)$ (Fig. 16): Let AB be the given curve, the polar equation of which is $\rho = \phi(\theta)$. Draw OR perpendicular to OA, put $OK = 2$ and $OR = OA$; taking KR as diameter, draw a circle. The intersection of this circle with the ray OA, the point a, is the point of the curve $\rho = \psi(\theta)$ corresponding to the point A, as $(Oa)^2 = OK \cdot OR = 2\phi(\theta)$. In the same way we construct the points $a_1, a_2, \cdots, b$, corresponding to the points $A_1, A_2, \cdots, B$ and so obtain the curve $\rho = \psi(\theta)$.

Then we construct the integral curve P_0P_n of the curve ab (see §10). $OP_0 = C$ is the given initial value of the integral $\int\phi(\theta)d\theta$. This curve P_0P_n represents graphically the function $\rho = \int\phi(\theta)d\theta + C$.

It is obvious that the integral $y = \int_{x_0}^{x_1} f(x)dx$ geometrically can be interpreted either as the area bounded by the curve $y = f(x)$ (in Cartesian coördinates), the X axis and the ordinates $x = x_0$, $x = x_1$, or as the sectorial area bounded by the curve $y = \sqrt{2f(x)}$ and the radii vectors $x = x_0$, $x = x_1$. (In polar coördinates, x is the polar angle and y is the radius vector). We choose the construction which suits us best. For instance, the function whose analytical expression is

$$y = \frac{1}{2}\int_0^{x_1} x^2 dx = \frac{1}{6}x^3$$

may be considered as the area bounded by the parabola $y=\frac{1}{2}x^2$ (in Cartesian coördinates), the X axis and the ordinate $x=x_1$, or as the area bounded by the spiral of Archimedes $\rho=\theta$ (in polar coördinates) and the vector $\theta=\theta_1$ (the letters θ and ρ being substituted for the letters x and y). Both areas are equal.

GRAPHICAL DIFFERENTIATION

15. *Drawing of tangents.* The solution of the problem of graphical differentiation of a function represented by a curve in a polar coördinate system is based, as in the Cartesian system, on the construction of tangents to the given curve. The drawing of tangents, from the standpoint of precision, is more difficult than the substitution of the broken line for a curve required in graphical integration.

C. Runge and R. Mehmke propose several methods of drawing tangents with sufficient accuracy. Some of these methods refer to drawing tangents at a given point of the curve, others to defining the point of contact of a tangent parallel to a given direction.

C. Runge in his *Graphical methods* proposes to give the direction of a tangent and then determine the point of contact by drawing a number of chords parallel to the given direction; the intersection of the curve through the midpoints of these chords with the given curve defines the point of contact. See also methods proposed by R. Mehmke (l.c., pp. 108–111).[2]

16. *The general methods of graphical differentiation.* The problem is to find graphically the curve $\rho=\phi(\theta)$ when we are given the graph of the curve $\rho=\Phi(\theta)$, where $\Phi(\theta)=\frac{1}{2}\int_{\theta_0}^{\theta}\phi^2(\theta)d\theta$. Let OP_iP_n be the given curve (Fig. 17) and $\rho=\Phi(\theta)$

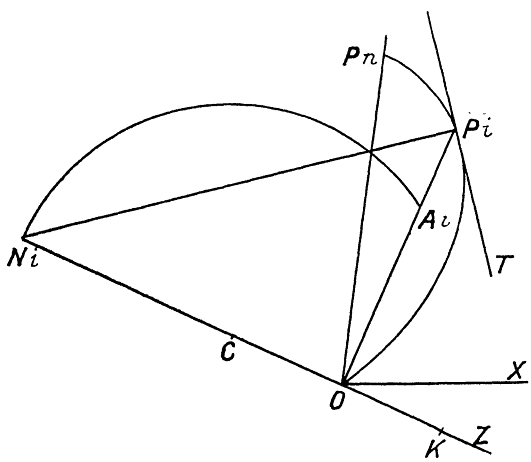

FIG. 17.

[2] There exist several instruments for drawing a tangent (or normal) at a given point of a curve, as: the "Tangentenzeichner" of A. Pflüger (see Zeitschrift des Ver. der Deutschen Ing., Bd. 58, 1914); the "Spiegellineal" of E. Reusch (see Carls Repertorium für Experimentale Physik," B. 16, 1880); the "Spiegelderivator of A. Wagener (see Physical Zeitschrift, Bd. 10, 1909). The last one is very expensive but with accuracy not less than that of other geometrical constructions.

be its equation in polar coördinates with the point O as the pole and OX the polar axis. We wish to find the point A_i of the curve $\rho=\phi(\theta)$ corresponding to the point P_i of the curve $\rho=\Phi(\theta)$, when $\Phi(\theta)=\frac{1}{2}\int_{\theta_0}^{\theta}\phi^2(\theta)d\theta$.

From the last equation we get

$$\Phi'(\theta) = \tfrac{1}{2}\phi^2(\theta), \quad \phi^2(\theta) = 2\Phi'(\theta).$$

First method: (Fig. 17). (1) Draw a tangent to the given curve at the point P_i. (2) Draw the ray OP_i and construct OZ perpendicular to OP_i. (3) On OZ lay the length OK equal to two units of length. $OK=2$. (4) Construct the normal P_iN_i to the curve at the point P_i, the point N_i being the intersection of the normal with OZ. (5) Bisect N_iK in C and taking the point C as the center describe a circle with the radius CK. This circle intersects OP_i or its prolongation at the point A_i. The point A_i is the required point of the curve $\rho=\phi(\theta)$ when $\angle P_iOX=\theta$, for $ON_i=\Phi'(\theta)$ (the subnormal) and $(OA_i)^2=OK\cdot ON_i$ $=2\Phi'(\theta)$; $OA_i=\phi(\theta)$.

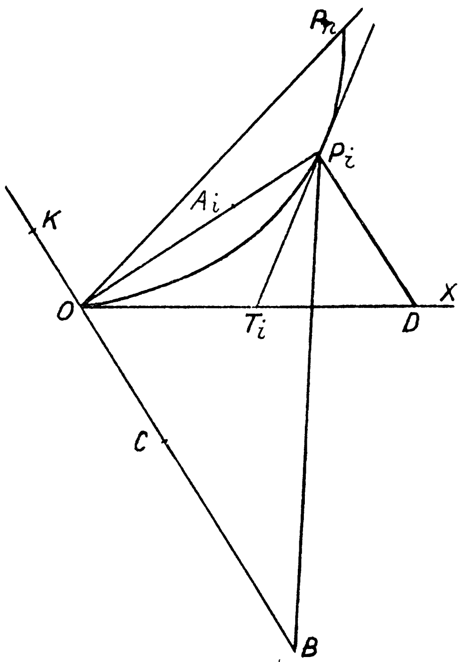

FIG. 18.

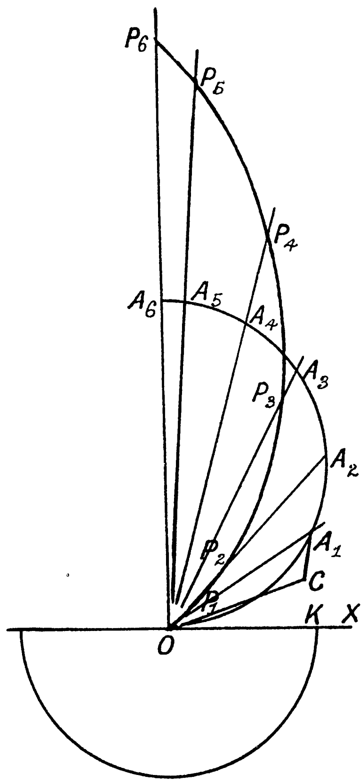

FIG. 19.

Second method: In some cases the subnormal is very long and the dimensions of the sketch do not allow the construction of the point N_i in the direction from O to N_i. We can apply then another method.

Construction (Fig. 18): P_iT_i is the tangent; $P_iD \perp OP_i$; $OB \perp OP_i$; $OK=2$; $\angle OP_iB = \angle T_iP_iD = \frac{1}{2}\pi - \mu$; $KC=CB$.

μ is the angle between the positive directions of the tangent T_iP_i and the radius vector OP_i. The point A_i is the point of intersection of the radius vector OP_i with the circle of radius CK and with the point C as center.

The point A_i is the required point of the curve $\rho=\phi(\theta)$, because $OB=OP_i \tan OP_iB=\Phi(\theta)\cot\mu=\Phi'(\theta)$ (for $\tan\mu=\rho/\rho'$).

$$(OA_i)^2=OK\cdot OB=2\Phi'(\theta);\; OA_i=\phi(\theta).$$

17. *Construction of a differential curve.* Given the curve OP_3P_6 (Fig. 19). We draw the vectors OP_1, OP_2, $\cdots$, OP_5 and construct the points A_1, A_2, $\cdots$, A_5 of the differential curve corresponding to the points P_1, P_2, $\cdots$, P_5. Through these points we draw a smooth curve which is the required differential curve.

It is impossible to draw a tangent between the points O and P_1 of the given curve. But as the vector OP_1 is approximately $\frac{1}{3}$ of the vector OP_2 we conclude that the area measured by the vector OP_1 is approximately half the area of the sector OA_1A_2. We construct a triangle OCA_1 with this area and replace the broken line OCA by a smooth curve so that the added and subtracted areas are equal.

In §18 we shall indicate more precise methods of drawing in such cases.

18. *Some secondary methods.* The inexactness of drawing tangents in some cases may be corrected by some auxiliary methods. When only the angle POX of the sector and its area represented by a segment OP are given (Fig. 20), the problem is evidently indefinite and there is an infinity of sectors satisfying these conditions. Adding some supplementary conditions, however, the problem can be made definite.

Let us consider a few such cases. We shall give the construction only, without the proof which is quite elementary. In all cases we are given the angle POX of the sector and the segment OP measuring the area of this sector.

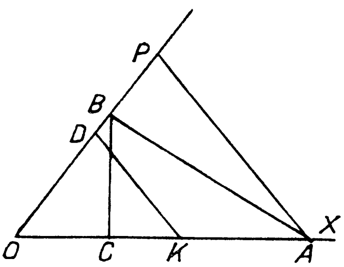

FIG. 20.

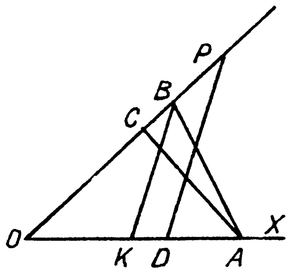

FIG. 21.

(1) It is required that the sector be represented by a triangle with the fixed end-point B (Fig. 20).

Construction: $OK=2$; $BC\perp OX$; $OD=BC$; $PA\|DK$. A is the vertex of the required triangle OAB.

If instead of the point B the point A is given (Fig. 21), to find the point B we construct: $OK=2$; $AC\perp OP$; $OD=AC$; $KB\|DP$. The point B is the required point.

(2) It is required that the sector be represented by a triangle OAB with AB parallel to a given direction $A'B'$ (Fig. 22). *Construction*: $A''P \| A'B'$; $OH' \perp A''P$; $OF = A''P$; $H'E \| FP$; $OK \perp OH'$; $OK = 2$; $OD = OE$; $KC = CD$; a circle

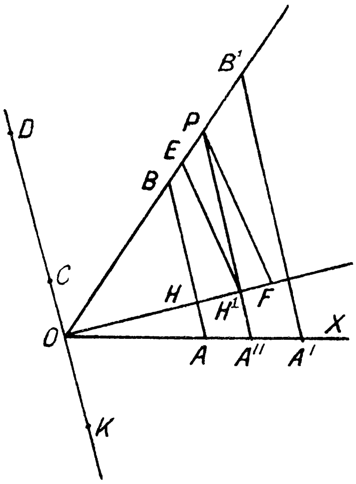

FIG. 22.

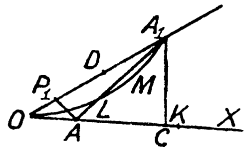

FIG. 23.

with C as the center and radius CD intersects OH' in H; a parallel to $A'B'$ through the point H gives AB. Instead of direction $A'B'$ the direction OH of the perpendicular to AB may be given.

(3) In a similar way a construction can be made when the length AB or OH is given.

These constructions can help us in drawing the differential curve or in correcting it. So in the example of §17 we can construct a triangle OA_1A (Fig. 23) the area of which is measured by the vector OP_1 and one of the vertices of which is the point A_1, found previously by the method of tangents. Then we draw the curve $OLMA_1$, so that the subtracted area OAL is equal to the added area of LMA_1.

19. *Combination of the methods.* If the given curve is of such form that only a few points, or even a single point, of contact of the curve with the tangent can be determined with sufficient accuracy, we can find the points (or the point) of the required curve $\rho = \phi(\theta)$ corresponding to these points of contact by the method of §16 and continue the construction of the curve by the methods of §18. If, for instance, the point A_i of the curve $\rho = \phi(\theta)$ (Fig. 24) corresponding to the point P_i of the given integral curve was found by means of tracing a tangent we can construct a broken line beginning with the point A_i in both directions using only the values of the radius vector of the curve OP_iP by the method of §18, case 1, without taking into consideration the tangents to the curve. To find the point A_{i-1} corresponding to the point P_{i-1} we take on the ray OP_i the value of the area of the triangle $OA_{i-1}A_i$ which is equal to the difference

of vectors $OP_i - OP_{i-1} = Op_i$ and then construct as in §18. To find the point A_{i+1} corresponding to the point P_{i+1} we take on the ray OP_{i+1} the segment $OP_{i+1} - OP_i = Op_{i+1}$ measuring the area of the triangle OA_iA_{i+1}, and so on. The broken line through these points A_i is the approximation of the differential curve. Then we draw through the point A_i a smooth curve so that the areas bounded by the broken line and this curve are equal.

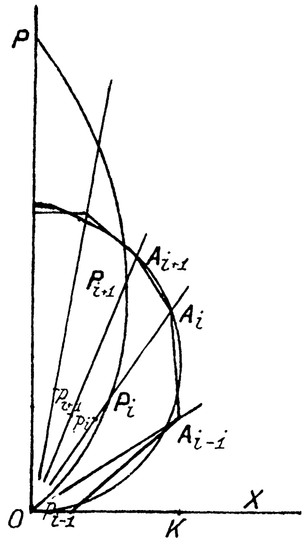

FIG. 24.

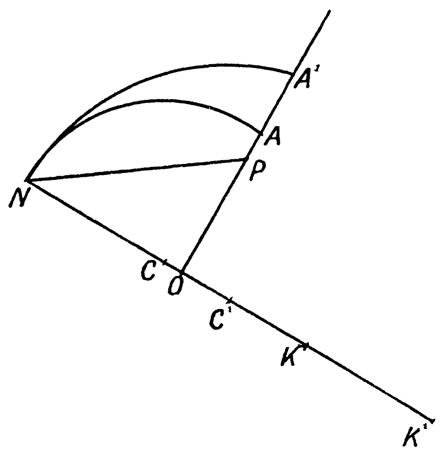

FIG. 25.

20. *The influence of a change of scale.* Let OP be the vector measuring the area of a sector (Fig. 25). By the method shown in §16 we construct the corresponding point A of the differential curve when the unit of length is represented by half the segment OK, and the point A' when the unit of length is represented by half the segment OK'. Then

$$(OA)^2 = NO\cdot OK\ ; \quad (OA')^2 = NO\cdot OK'\ ; \quad OA/OA' = \sqrt{(OK/OK')}\,.$$

The radii vectors of the differential curve are proportional to the square roots of the values of the corresponding units. We could conclude this from considering the influence of altering the scale in the case of integration (§12). Hence, if we increase the scale k times, a figure similar to the first is obtained, the linear dimensions of which are $\sqrt{k}$ times greater.

21. *Example.* Let us find the orbit of a body which is moving according to the law of constant areal velocity when one point of this orbit (Z) (Fig. 26) and the directions (OS_1, OS_2, $\cdots$, OS_7) in which this moving body is seen in different intervals of time from the center of force (O) are known. Let the intervals of time from the first observation to the second be a_1, from the second to the third be a_2, and so on.

We take an arbitrary length OP_7 as the measure of the sectorial area bounded by the required orbit and the rays OS_1 and OS_7 and divide it in parts proportional to the numbers $a_1, a_2, \cdots$, and we draw through these points of division circles which intersect the corresponding rays $OS_1, OS_2, \cdots, OS_6$ at the points $P_1, P_2, \cdots, P_6$. In this way we obtain the integral curve $P_1P_2 \cdots P_7$ of the unknown orbit.

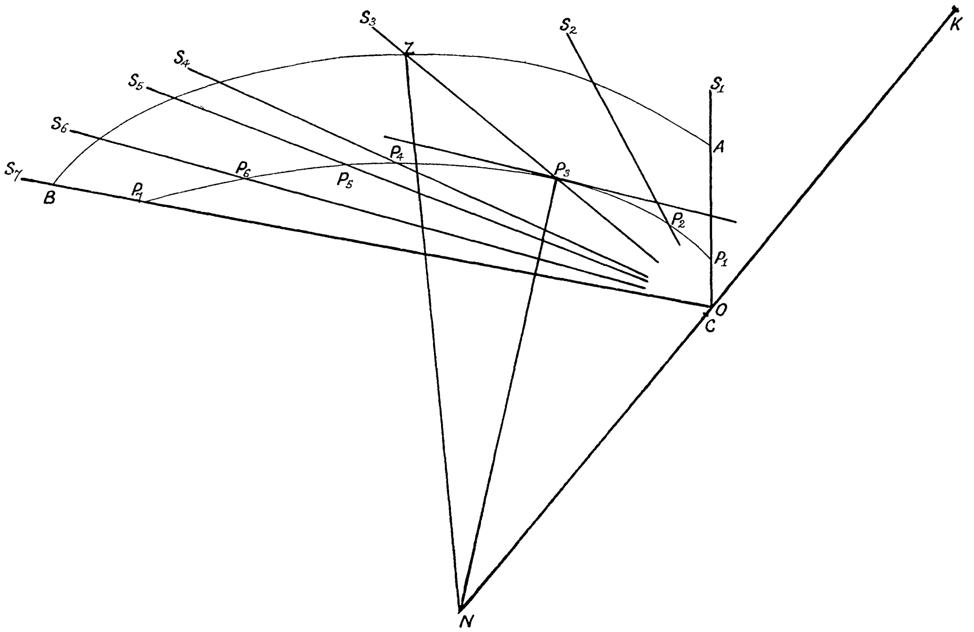

FIG .26.

The unit of length of the integral curve is not yet defined. To find this unit we draw a tangent to the integral curve at the point P_3 corresponding to the given point of the orbit, Z, the perpendicular to the radius vector OP_3, ON, the normal P_3N. Then $ON = \Phi'(\theta)$ where $\rho = \Phi(\theta)$ is the equation of the integral curve P_1P_7. We know that the point Z of the differential curve is lying on the circle through the point N and with the center on the line ON (see §16). Therefore the intersection of the mid perpendicular to the segment NZ and the line ON, the point C, is the center of this circle with the radius CN. If $CK = CN$, the point K determines the length $OK = 2$ units of length.

Then we construct the differential curve AB which is the required orbit.

If no point of the orbit is known, we shall be able to find only a figure similar to the figure bounded by the true trajectory.

HERONIAN TRIANGLES

By WM. FITCH CHENEY, JR., Tufts College

1. *Introduction.* A triangle whose sides and area are rational is called a rational triangle. For any rational triangle there exists a single infinity of other rational triangles similar to it. In any such singly infinite set of similar rational triangles there is always just one whose three sides are integers which have no factor common to all three. This we shall call the Heronian triangle of the set. It is so named after Hero, who is credited with the formula for the area, K, of a triangle of sides a_1, a_2, a_3:

$$K = (ss_1s_2s_3)^{1/2}, \quad \text{where} \quad s = \tfrac{1}{2}\sum_{i=1}^{3} a_i \quad \text{and} \quad s_i = s - a_i.$$

While some authors admit in the class of plane triangles those which have zero area and a zero altitude, such flat triangles are not included in the theorems of this article. The class of Heronian triangles is then to include all triangles of non-zero, rational area and integral sides which have no factor common to all three. This paper presents some theorems regarding factors of the sides and area of Heronian triangles.

2. *Rational triangles.* In the July, 1921, issue of this Monthly, Professor L. E. Dickson of the University of Chicago showed that all shapes of rational triangles could be formed by the juxtaposition of pairs of rational right triangles.[1] This process leads, however, to duplication, producing the same shape of rational triangle in three different ways, depending on which altitude of the oblique triangle is the line of juxtaposition between the right triangles.

Since for any triangle of sides a_i and angles A_i, $\tan \frac{1}{2}A_i = K/ss_i$, the tangents of the half-angles of any rational triangle must all be rational. If we set $t_i = \tan \frac{1}{2}A_i$, we know from trigonometry that $t_1t_2 + t_2t_3 + t_3t_1 = 1$. Hence if we have a triangle for which we know two t's rational, the third must also be rational, and the triangle will be similar to a Heronian triangle.

A triangle T can be constructed similar to any given triangle and having the vertex of its largest angle in the first quadrant of a set of rectangular coordinates, and its other two vertices at $(-1, 0)$ and $(1, 0)$. For any given triangle this triangle, T, is unique. The vertex of its largest angle must be within or on the circle centered at $(-1, 0)$ with radius 2, and consequently its incenter must be within or on the loop of the cubic curve

[1] So far as the author has been able to determine, the only information hitherto published about Heronian triangles is typified by the article of Professor Dickson, referred to above, and by chapter fifteen of the first volume of Dr. Hermann Schubert's *Mussestunden.* The standard method has been to construct a Heronian triangle by the juxtaposition of two right triangles, each having three integral sides. But there are Heronian triangles which can not be so constructed, as for example, one of sides 25, 34 and 39 units. The author has found no reference to the nature of the factors of the sides of Heronian triangles, which consideration constitutes the major portion of the present paper.

$$y^2 = (1 + x)(1 - x)^2(3 - x)^{-1},$$

and also in the first quadrant. This cubic curve is the locus of the intersections of the angle-bisectors of all triangles having two vertices at $(-1, 0)$ and $(1, 0)$ and the third on the circle $(x+1)^2+y^2=4$.

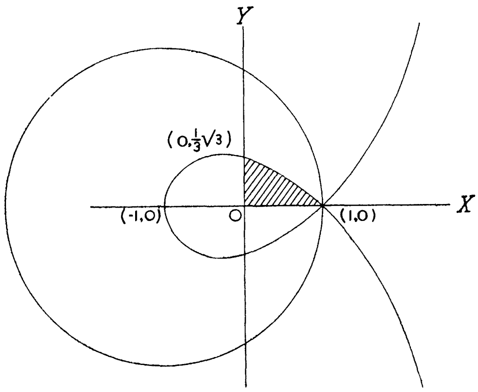

FIG. 1.

Consequently we may take any number x such that $0 \leqq x < 1$, and with it any positive number $y \leqq (1-x)\ (1+x)^{1/2}\ (3-x)^{-1/2}$, as the coordinates of the incenter of a triangle.

Theorem 1. *Each different point P of the shaded area in the accompanying diagram determines uniquely a differently shaped triangle T, of which it is the incenter.*

Since the bisectors of the interior angles of T all pass through P, the tangents t_i of the half-angles are expressible rationally in terms of the coordinates of P, and conversely.

Theorem 2. *The necessary and sufficient condition that T be a rational triangle is that the coordinates of P shall be rational.*

Thus all different shapes of rational triangles are obtained once each by letting P stop once on each point of the shaded area in the diagram which has rational coordinates. Since the set of all points with rational coordinates is denumerable, we have the two theorems:

Theorem 3. *The set of all Heronian triangles is denumerable.*

Theorem 4. *The set of all rational triangles is denumerable.*

3. *Properties of the t_i.* We have shown that any point P of a denumerable set of points furnishes the three t_i of a uniquely determined set of similar rational triangles. This set contains just one Heronian triangle. To this latter triangle we shall now confine our attention. If its sides be a_i $(i=1, 2, 3)$, its area K, its semi-perimeter s, and $s_i = s - a_i$, Hero's formula then reads:

$$(A) \qquad K = (ss_1s_2s_3)^{1/2}.$$

We also know from trigonometry that

$$(B) \qquad K = ss_it_i$$

and that

$$(C) \qquad t_1t_2 + t_2t_3 + t_3t_1 = 1.$$

Obviously

$$(D) \qquad s = \sum_{i=1}^{3} s_i.$$

Dividing the cube of (B) by the square of (A) gives

$$(E) \qquad K = s^2t_1t_2t_3,$$

which we may combine with (B) to eliminate K and get

$$(F) \qquad s_i = st_jt_k \quad (i,j,k = 1,2,3\ ;\ i \neq j \neq k).$$

Replacing s_i in (F) by its definition value $s-a_i$, we may obtain

$$(G) \qquad a_i = s(1 - t_jt_k).$$

Since the sides of our triangle are each positive, this gives:

Theorem 5. *The product of any two t's must be less than* 1.

4. *The numerators and denominators of rational t_i.* Now since each t_i in any Heronian triangle is rational, we shall write it as a fraction *in lowest terms,*

$$(H) \qquad t_i \equiv n_i/d_i \qquad (i = 1,2,3),$$

where d_i and n_i have no common factor. Setting (H) into (C) gives

$$(I) \qquad d_1n_2n_3 + d_2n_3n_1 + d_3n_1n_2 = d_1d_2d_3.$$

By transposition and division (I) may be converted into

$$(d_jd_k - n_jn_k)/n_i = (d_jn_k + d_kn_j)/d_i,$$

and since the denominators of these two equal fractions have no common factor, each fraction must reduce to an integer. This integer we shall call v_i. Accordingly we write

$$(J)\ (1) \qquad d_iv_i = d_jn_k + d_kn_j \qquad \text{and}\ (2) \qquad n_iv_i = d_jd_k - n_jn_k.$$

If these two equations be squared and added, there results

$$(K) \qquad v_i^2(d_i^2 + n_i^2) = (d_j^2 + n_j^2)(d_k^2 + n_k^2).$$

Solving for v_i and deriving v_j by symmetric permutation of subscripts produces the remarkably fruitful equation

$$(L) \qquad v_iv_j = d_k^2 + n_k^2.$$

Now it is well known in number theory that the sum of two relatively prime squares can only have factors each of which is the sum of two relatively prime squares. For our purposes this means:

Theorem 6. *Every factor of each v is expressible as the sum of two relatively prime squares.* In other words, no v has any factor of the form $4n-1$. Further, from (L) follows:

Theorem 7. *Each v is prime to each of the d's and n's whose subscripts differ from its own.* A logical consequence of this is:

Theorem 8. *Any factor common to two v's is prime to all the d's and n's.*

Now since every d and n is 1 or greater, the minimum value of d^2+n^2 is 2. (In this case only, our Heronian triangle is a right triangle.) In this light, (L) demonstrates:

Theorem 9. *In any Heronian triangle at least two v's exceed* 1.

If the three v's have a common factor f, it must by $(J, 1)$ divide $d_jn_k+d_kn_j$, and hence must also divide $\sum n_i(d_jn_k+d_kn_j)$. But by (I) this equals $2d_1d_2d_3$. By theorem 8, f is prime to all the d's, and must therefore be either 1 or 2. If f were 2, then by theorem 8, all the d's and n's would have to be odd. This would make the left member of (L) a multiple of 4 while the right side would be twice an odd number, which is impossible. Hence the supposition that f might be 2 can not hold, and there results:

Theorem 10. *The three v's can not have any common factor.*

If the t's in (G) be replaced by their values from (H) and the result reduced by $(J, 2)$, it appears that

$$(M) \qquad a_i = sn_iv_i/d_jd_k,$$

which must be an integer. A similar substitution of (H) in (E) gives

$$(N) \qquad K = s^2n_1n_2n_3/d_1d_2d_3,$$

which we shall use later.

5. *The factors of d_i and n_i.* Since any factor common to three terms must divide their sum, and since each d is prime to its corresponding n, there follow from (I) three important theorems:

Theorem 11. *Any factor common to two n's divides the third d.* We shall represent the greatest common factor of d_i, n_j, and n_k by e_i.

Theorem 12. *Any factor common to a d and an n divides the third n.*

Theorem 13. *Any factor common to two d's must divide the third d.* We shall represent the greatest common factor of the three d's by E. In the most general case we may then write:

$$(O) \quad (1) \qquad d_i \equiv c_ie_iE \qquad \text{and} \quad (2) \qquad n_i \equiv b_ie_je_k,$$

where i, j, k represent the three different numbers 1, 2, 3 in some order. More specifically,

$$(O') \qquad t_1 \equiv b_1e_2e_3/c_1e_1E, \qquad t_2 \equiv b_2e_3e_1/c_2e_2E, \qquad t_3 \equiv b_3e_1e_2/c_3e_3E,$$

where each fraction is in lowest terms, and no two of the b's and c's have any common factor.

If we substitute from (O) into (M) and cancel, we get

$$(P) \qquad a_i = sb_iv_i/c_jc_kE^2.$$

By the definition of a Heronian triangle, the three a's are integers with no factor common to all three. And since b_iv_i is prime to $c_jc_kE^2$, it follows that $c_1c_2c_3E^2$ must equal the numerator of s. Now the denominator of s must be either 1 or 2, and the latter case may be shown impossible; for if the denominator of s were 2, then b_1v_1, b_2v_2, and b_3v_3 would all have to be even to satisfy (P). But by definition, only one b can be even, and since by theorem 7 the other two v's must then be odd, we are led to an absurdity. The only alternative is

$$s = c_1c_2c_3E^2, \tag{Q}$$

which proves:

Theorem 14. *The semiperimeter of any Heronian triangle is an integer.* Since the sum of the three sides is thus even and the three have no common factor, there also appears:

Theorem 15. *Any Heronian triangle has just one even side.*

Substituting (O) and (Q) in (M) and (N) gives, by cancelling,

$$a_i = b_ic_iv_i, \tag{R}$$

$$K = b_1b_2b_3c_1c_2c_3e_1e_2e_3E, \tag{S}$$

which last leads at once to the theorems:

Theorem 16. *The area of any Heronian triangle is an integer.*

Theorem 17. *The area of any Heronian triangle is the least common multiple of the d's and the n's.*

We shall now show that the factor 2 must appear in the area. If the area were odd, all the d's and n's would have to be odd. Now $(d_1+n_1)(d_2+n_2)$ is the product of two even numbers, and hence a multiple of 4, while n_1n_2 is odd, so that $(d_1+n_1)(d_2+n_2)-2n_1n_2$ must be twice an odd number. But this equals the sum of the two even numbers $d_1d_2-n_1n_2$ and $d_1n_2+d_2n_1$, so that one of them is twice an odd number while the other is a multiple of 4. Then by (J), v_3 is twice an odd number and either d_3 or n_3 must be even, contrary to the assumption that all the d's and n's were odd. We may consequently state:

Theorem 18. *The area of any Heronian triangle is even.*

We next show that the factor 3 must appear in the area. We first assume the contrary and then show that such an assumption leads to a contradiction. If 3 were not a factor of the area, no d or n could be a multiple of 3 (by theorem 17). Now the squares of all numbers prime to 3 are congruent modulo 3, so that $d_1^2-n_1^2$ and $d_2^2-n_2^2$ would be multiples of 3. Then $(d_1^2-n_1^2)d_2n_2+(d_2^2-n_2^2)d_1n_1$ would be a multiple of 3. But this equals $(d_1d_2-n_1n_2)(d_1n_2+d_2n_1)$, which by (J) equals $d_3n_3v_3^2$. Since by theorem 6, v_3 can not be a multiple of 3, either d_3 or n_3 would then have to be a multiple of 3, contrary to the assumption that no d or n could be a multiple of 3. As a result of this consideration we state:

Theorem 19. *The area of any Heronian triangle is a multiple of* 3.

We shall next conduct a similar investigation of the factor 5. If the area is not a multiple of 5, then by theorem 17, all the d's and n's are prime to 5. Now the fourth powers of all numbers prime to 5 are congruent modulo 5. Consequently 5 divides $d_1^4 - n_1^4$, which by (L) equals $(d_1^2 - n_1^2)v_2v_3$. Then $(d_1^2 - n_1^2)d_2n_2v_1v_2v_3$ is a multiple of 5, and so similarly is $(d_2^2 - n_2^2)d_1n_1v_1v_2v_3$. Then their sum,

$$[(d_1^2 - n_1^2)d_2n_2 + (d_2^2 - n_2^2)d_1n_1]v_1v_2v_3,$$

which equals

$$(d_1d_2 - n_1n_2)(d_1n_2 + d_2n_1)v_1v_2v_3$$

and which reduces by (J) to $d_3n_3v_1v_2v_3^3$ must also be a multiple of 5. Hence if the factor 5 does not appear in any of the d's or n's, it must appear in one of the v's. This gives:

Theorem 20. *In any Heronian triangle, if the area is not a multiple of* 5, *one of the sides is.*

From (R) we see that any factor common to a_i and a_j must divide v_i and v_j, so that by theorem 6 we may state:

Theorem 21. *Every factor common to two sides of a Heronian triangle must be expressible as the sum of two relatively prime squares.* Furthermore, by theorem 8, we state:

Theorem 22. *No factor common to two sides of a Heronian triangle can be a factor of the area.* Also, by (R) and (S), we have:

Theorem 23. *Any number of the form* $4n-1$ *which is a factor of a side of a Heronian triangle, must also be a factor of the area.*

6. *Construction of Heronian triangles.* Probably the easiest way to make a list of Heronian triangles is to develop the method suggested in paragraph 2. We may choose any two fractions in lowest terms for t_1 and t_2, subject to the conditions that $0 < t_1 \leqq t_2 < 1$ and $t_1 \leqq (1-t_2^2)/2t_2$. This keeps the point P of paragraph 2 in the shaded area of the loop in the diagram. Since Heronian triangles form a denumerable set, we may put them in the following order:

(a) We define the index, m, of a Heronian triangle as $d_1+n_1+d_2+n_2$.

(b) Of two triangles of different indices, that with the smaller index shall precede.

(c) Of two triangles of the same index but different t_1's, that with the smaller t_1 shall precede.

(d) Of two triangles of the same index and the same t_1, that with the smaller t_2 shall precede.

Having given t_1 and t_2, we first compute t_3 from (C). We next use (J) to obtain the v's, determine the e's and E from their definitions under theorems 11 and 13, and thence derive the b's and c's. We are then in a position to write the a's and K from (R) and (S).

The following is a list of all Heronian triangles for which the index m does not exceed 12:

m	t_1	t_2	t_3	v_1	v_2	v_3	E	e_1	e_2	e_3	b_1	b_2	b_3	c_1	c_2	c_3	a_1	a_2	a_3	K
6	1/2	1/2	3/4	5	5	1	2	1	1	1	1	1	3	1	1	2	5	5	6	12
7	1/3	1/2	1/1	1	2	5	1	1	1	1	1	1	1	3	2	1	3	4	5	6
8	1/4	1/2	7/6	5	17	1	2	1	1	1	1	1	7	2	1	3	10	17	21	84
	1/3	1/3	4/3	5	5	2	3	1	1	1	1	1	4	1	1	1	5	5	8	12
9	1/5	1/2	9/7	5	26	1	1	1	1	1	1	1	9	5	2	7	25	52	63	630
	1/4	1/3	11/7	10	17	1	1	1	1	1	1	1	11	4	3	7	40	51	77	928
	1/3	2/3	7/9	13	10	1	3	1	1	1	1	2	7	1	1	3	13	20	21	126
10	1/6	1/2	11/8	5	37	1	2	1	1	1	1	1	11	3	1	4	15	37	44	264
	1/5	1/3	7/4	5	13	2	1	1	1	1	1	1	7	5	3	4	25	39	56	420
	1/4	1/4	15/8	17	17	1	4	1	1	1	1	1	15	1	1	2	17	17	30	120
	1/4	2/3	10/11	13	17	1	1	2	1	1	1	1	5	2	3	11	26	51	55	660
	2/5	1/2	8/9	5	29	1	1	1	2	1	1	1	4	5	1	9	25	29	36	360
11	1/7	1/2	13/9	5	50	1	1	1	1	1	1	1	13	7	2	9	35	100	117	1,638
	1/6	1/3	17/9	10	37	1	3	1	1	1	1	1	17	2	1	3	20	37	51	306
	1/5	1/4	19/9	17	26	1	1	1	1	1	1	1	19	5	4	9	85	104	171	3,420
	1/5	2/3	1/1	1	2	13	1	1	1	1	1	2	1	5	3	1	5	12	13	30
	1/3	2/5	13/11	29	10	1	1	1	1	1	1	2	13	3	5	11	87	100	143	4,290
	1/2	3/5	7/11	34	5	1	1	1	1	1	1	3	7	2	5	11	68	75	77	2,310
12	1/8	1/2	3/2	1	13	5	2	1	1	1	1	1	3	4	1	1	4	13	15	24
	1/7	1/3	2/1	1	5	10	1	1	1	1	1	1	2	7	3	1	7	15	20	42
	1/6	1/4	23/10	17	37	1	2	1	1	1	1	1	23	3	2	5	51	74	115	1,380
	1/6	2/3	16/15	13	37	1	3	2	1	1	1	1	8	1	1	5	13	37	40	240
	1/5	1/5	12/5	13	13	2	5	1	1	1	1	1	12	1	1	1	13	13	24	60
	1/4	2/5	18/13	29	17	1	1	2	1	1	1	1	9	2	5	13	58	85	117	2,340
	1/4	3/4	13/16	25	17	1	4	1	1	1	1	3	13	1	1	4	25	51	52	784
	2/7	1/2	12/11	5	53	1	1	1	2	1	1	1	6	7	1	11	35	53	66	924
	1/3	3/5	6/7	17	5	2	1	3	1	1	1	1	2	1	5	7	17	25	28	210
	2/5	2/3	11/16	13	29	1	1	1	1	2	1	1	11	5	3	8	65	87	88	2,640

It is interesting to note, in the foregoing list of the first twenty-eight Heronian triangles, that only two are right triangles. (A right Heronian Triangle is characterized by $t_3 = 1$.)

Although each of the triangles listed above has at least one altitude which is an integer, this is not always true, as may be seen in the case of the Heronian triangle whose sides are 39, 58, and 95 and whose area is 456.

HOW CAN INTEREST IN CALCULUS BE INCREASED?[1]

By ROSCOE WOODS, University of Iowa

As teachers I presume that all of us have certain aims in our work. There is no doubt but that these aims vary with the individual. However, one of the important facts to keep before ourselves is that mathematics, due to its inherent nature, needs a teacher more than any other subject. If we do not have definite aims of a high type as to our functions as teachers, we are not going to create much enthusiasm for our subject. Unless we are convinced that the

[1] Read before the Iowa Section of the Mathematical Association of America, May 4, 1928.

study of mathematics can do something for the student beyond the acquisition of certain mechanical skills and facts required for the next course and unless we believe that the student acquires certain fundamental thought processes from mathematics, we will not engender a great deal of interest in our students.

The study of mathematics ought to help the student to express himself in clear, forcible, and concise language. It should unfold his rational faculties and teach him how to use them. In the various subjects that we set before him and in the problems that he solves, he ought to be able to penetrate from the surface to the central idea. We as teachers should be convinced and should teach that one of the primary aims of all learning, whether of a mathematical nature or not, is to see relations that exist between things and to correlate facts with principles that seem to be unrelated. This is another way of saying that mathematics should be so taught that the student is brought to realize that he has faculties that can discern the good and the bad intellectually and that this discernment is the foundation of the development of character for which all education should exist.

In the teaching of the calculus these general aims should be more nearly attained than in any other single subject. Perhaps, to be more specific, we ought to ask ourselves why we teach calculus, to liberal arts students, in preference to some other subject which would contribute toward their culture and mental development. The answer to this question presents a multiplicity of notions. In a broad way we are accustomed to think of the practical side of calculus, when engineers are mentioned; of its applications, when other science students are named; and of the cultural side, when liberal arts students are considered. Let us enumerate some of the aims for teaching calculus. Do we teach it with the idea that the student will recognize forms and acquire facts and technique? Do we teach it with the idea that it unifies all that the student has learned before and that it is the key stronghold from which all other fields of mathematics are to be explored? Do we teach it solely as a preparation of the engineer and of the teacher of elementary mathematics? Do we teach it as a means of interpreting physical phenomena? Do we teach it simply as a tool whereby other achievements are accomplished? Or do we teach it as a formal exercise in logic with the idea that it may stimulate certain thought processes?

Some years ago Professor Osgood of Harvard,[2] in an address before the American Mathematical Society, maintained that the calculus should be taught as a means of interpreting physical phenomena. He claimed that there was only one calculus for all students, no matter what the ultimate aim or profession of the student. He presents the course formally and uses the differential as soon as it is possible to introduce it. Professor Rietz has likewise written an article, which appeared in this Monthly,[3] in which he points out that the aim in teaching calculus should be to implant certain notions that the student should

[2] W. F. Osgood, Bulletin of the American Mathematical Society, vol. 13, p. 449.

[3] H. L. Rietz, this Monthly, vol. 26 (1919), p. 341.

remember ten years afterward. Professor Bliss of Chicago[4] has written a monograph on the subject, *The function concept and the calculus.* All of these papers are well worth reading and will help you to decide why the calculus should be taught.

At first sight it seems that it ought to be a fairly simple matter for us to inculcate in the student the three central ideas of the calculus; namely, the derivative, the anti-derivate or the indefinite integral, and the definite integral interpreted as the limit of a sum. But the appreciation of these ideas presupposes a considerable amount of preparation on the part of the student. For the study of the calculus, work, intelligence and a certain spirit of adventure are needed. To pursue this study properly and with the greatest pleasure, the student must also possess the spirit of exploration. A few students possess this spirit but the general attitude is a passive one.

Some of the factors that give rise to the lethargic attitude are: (1) the trend of high school instruction, (2) preparation of the teachers of elementary mathematics, (3) training of college freshman, (4) the fact that no organized attempt is made to acquaint students with the content and purposes of the calculus, (5) the type of text book at our disposal, and (6) the general attitude and habits of the American student. Let us examine these six reasons briefly.

(1) The best index to the trend in high school instruction is to note the quality of the product produced. The way in which the high school graduate handles his complex simplification problems, the manner in which he analyzes problems in story form, and his feeble attempt to give definitions and logical explanations accurately usually indicate a lack of training rather than a lack of intelligence. Also there are certain habits that are indicative of training. For instance, the failure to read carefully the text is very common and usually leads to poor grades in college. This probably arises from the fact that the problems assigned in high school were of a single type and needed only the application of the illustrative exercise. Let us give another illustration of habit formation. Very few high school students have any idea how to study for recitation or how to prepare for a written lesson or how to use their time during the examination period. In other words, the fad of omitting the hard topics or of giving them little emphasis seriously handicaps the student's future.

(2) In general our elementary teachers are not highly trained. For instance only 47 percent in this state have had training in calculus. We should blush with shame when we compare the training of our elementary teachers with that of teachers in Europe.[5] There is no surer way of dampening a student's ardor for mathematics than to give him a teacher who is unprepared and who has no inherent interest in mathematics. We know that it is not uncommon for a

[4] G. A. Bliss, in *Monographs on Modern Mathematics* (1915), Edited by J. W. A. Young.

[5] Report of the National Committee on *The Reorganization of Mathematics in Secondary Education*, Chapter 14.

high school principal to ask a teacher who majored in English, for instance, to take a mathematics class in the high school. There seems to be an impression abroad that anyone can teach mathematics and just as long as this impression prevails, just so long will this vicious practice of assigning majors in other fields to teach mathematics classes continue.

(3) There are very few freshmen who are adequately prepared for and who can successfully master college algebra, trigonometry, and analytical geometry in one year, four periods per week. It requires time to acquire mathematical ideas and much training for mastery. To be frank we shall have to admit that college algebra is a conglomeration of topics. We admit that they are needed and badly needed; but they are not built around a central idea. I refer to the rational functions. In trigonometry we give the student a smattering of the circular functions, but we do not let him hear of their inverses. Analytical geometry offers the first opportunity to the student to use all his mathematical knowledge. It is the proving ground. The average student has just about enough time to learn the terminology connected with the course, but not enough of the technique required for enjoyment. He learns that there are such things as the straight line and the standard forms of the conics. He is allowed a glimpse of polar coordinates, higher plane curves, and transformation of coordinates. His solid analytical geometry is sealed with a great seal. He finishes his course with no idea of what the fundamental problem of analytical geometry is. He has graphed a few curves but he has not been given time to study the graphs after they have been made.

(4) We never point out to the student that all his work thus far has to do with functions of different kinds. We never classify them for him. He has been learning some of their properties through algebraical and graphical methods, but he has not been brought face to face with the tangent problem in his analytics, nor have his algebra and trigonometry convinced him that he needs to have a method for studying the variation of these functions. At no stage in the freshman's preparation do we acquaint him with the aims and purposes and the grandeur of the calculus. If he elects mathematics in the second year, he faces the same kind of useless (?) computation that he has done all his life with no apparent reason or incentive. Mathematical enthusiasm does not thrive in such soil.

(5) The absence of good exposition in our calculus texts is apparent. In order to meet the demand of the technical schools and of economy our texts are a sort of a hybrid between a good text and a mere handbook. If the teacher is not careful his students finish calculus not knowing the classes of the functions that have been studied. Unless the teacher is diligent, the student has not even found out the fundamental problems of the course, and his attitude toward advanced calculus is similar to that which he had for his calculus when he finished analytics.

(6) Finally, the student's attitude toward learning for learning's sake needs a revision. What student thinks of studying when not in school? Money can

be made with little study, and a position can be secured through channels other than that of scholastic attainment. The question uppermost in every student's mind is, "Of what practical value is this?" The information he seeks is that which he thinks will enable him to make ready money. But so long as the colleges and universities grant diplomas by the wholesale, why should we blame the student if he desires a diploma rather than the possession of that for which the diploma stands.

How can interest in calculus be increased? Clearly by trying to remedy the conditions set forth above. The following suggestions are offered. (1) Take every opportunity available to convince our fellow teachers outside of mathematics that we have a live, growing science. (2) Make more use in our elementary classes of the history of mathematics and the biographies of mathematicians both dead and living. (3) Be sure that our mathematics majors and advanced students know the various divisions of our science and its importance in the world. (4) Study carefully the problem of the dependence of all the sciences upon mathematics and make these results available. (5) Rearrange our courses to meet changed conditions, giving more credit where credit is due. (6) Prepare a series of charts, on various topics, that can be used in the classroom, especially one showing divisions of our science. (7) Assemble all our students by classes for general lectures along broad and unifying lines. (8) Consider some means on the part of the Iowa Section of the Association of recognizing the ability of any outstanding student who has finished the calculus. (9) Finally, let the Mathematical Association of America, the mathematics sections of the State teachers' associations, and the National Council of Teachers of Mathematics bend every effort to keep mathematical instruction in the hands of those prepared to give it.

GENERALIZATIONS OF THE THEOREM OF PYTHAGORAS AND EUCLID'S THEOREM OF THE GNOMON[1]

WILLIAM J. HAZARD, University of Colorado

Theorem 1, *Fig.* 1:

If any parallelogram P be drawn with its vertices $abcd$ in the sides of another parallelogram Q which has vertices $efgh$, the area of P equals the sum of parallelograms $afij$ and $djkh$ which are formed by drawing di and ak parallel to the respective sides of Q.

Outline Proof:

Move cb along itself to position mn. Move dm along itself to position jo. Then parallelogram $abcd=anmd=anoj=afij+fnoi$. Triangle $cbg=mno$ and mbi is common to them. Hence trapezoid $bnoi=migc$. But triangle $fnb=pmc$, hence parallelogram $fnoi=pigc=djkh$. Hence $P=afij+djkh$.

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[1] Presented to the Rocky Mountain Section of the Mathematical Association of America, April, 1927.

This theorem generalizes the theorem of Pythagoras in three ways, for if we first assume Q to be a rectangle, then a square, and finally make $cg = dh$, we shall have Fig. 2 which represents the Hindu proof[2] of the theorem of Pythagoras as an extremely special case of our theorem.

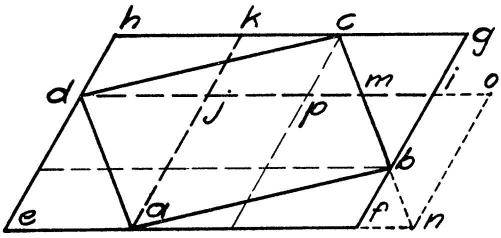

Fig. 1

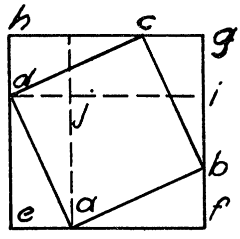

Fig. 2

Theorem 2, *Fig*. 3:

If parallelogram P be drawn inside parallelogram Q with a common diagonal ac, the area of P will equal the difference of parallelograms *ofid* and *jdlh* which are formed by drawing *ji* and *ol* through the vertex d of P, parallel to the respective sides of Q.

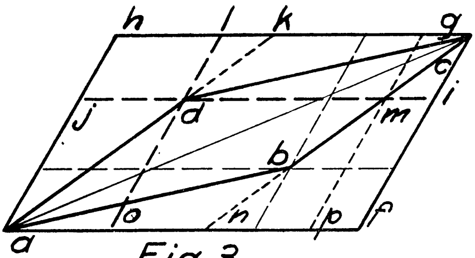

Fig. 3

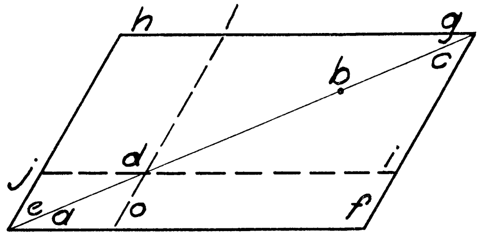

Fig. 4

Outline Proof:

Move *cb* along itself to position *mn*. Move *an* along itself to position *op*. Then parallelogram $abcd = anmd = opmd = ofid - pfim$. But triangle $nfg = kha$; $npm = dja$ and $mig = kld$, hence parallelogram $pfim = jdlh$ and $P = ofij - jdlh$.

Assume the vertices b and d in Fig. 3 to be moved parallel to the ends of Q (or otherwise) until they fall in the diagonal *ge* as in Fig. 4. The area of P has been thus reduced to zero and $ofid = jdlh$. Then our theorem No. 2 is seen to be a generalization of Euclid's theorem of the gnomon, Book I, Prop. 43, "In any parallelogram the complements of the parallelograms about the diagonal are equal to one another."

[2] See Rao, *Paper Folding*, or Heath's *Euclid*, I, 47, notes.

In Fig. 4, *eodj* and *digl* are the "parallelograms about the diagonal," and the other two are the "complements."[3]

The transition from Fig. 1 to Fig. 3 may be observed by noting that in Fig. 1, when c approaches g, a approaches e. At the limiting position *djkh* is reduced to zero and P equals *efid* which now coincides with *afij*. When vertices b and d are now moved in parallel to *ef*, the area of P decreases more rapidly than the area of *ofid* in Fig. 3, and thus necessitates the subtraction of the parallelogram *jdlh* as shown.

The theorem of the gnomon may be extended to space by considering any parallelopiped as in Fig. 5, with three planes passed through a point p of the diagonal *bh*, parallel to the respective faces. The diagonal *bh* is projected as the diagonal *bd* in the parallelogram *abcd*, and it is projected as the diagonal *be* in the parallelogram *abfe*. The shorter segment *bp* of the diagonal *bh* is the diagonal of the parallelopiped *ilp*. The longer segment *ph* is the diagonal of the parallelopiped *psh*. Following Euclid's language, these two are the "parallelopipeds about the diagonal." The "complements" of *ilp* are *ikm*, *jln*, and *prf*. These three complements are equal.

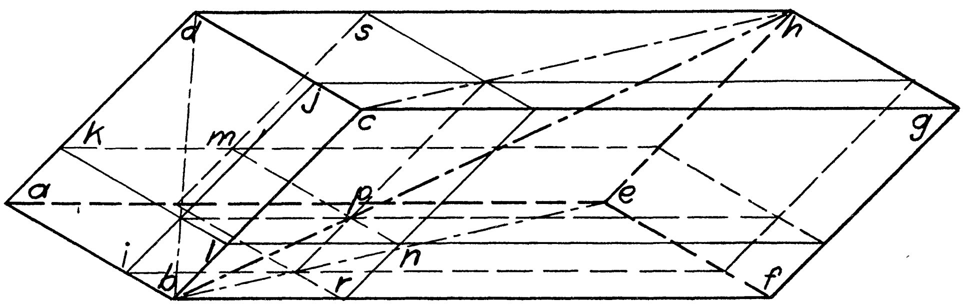

Fig. 5

Theorem 3, Fig. 5:

Outline Proof:

By the plane theorem of the gnomon, face *ki* = face *jl*. Hence parallelopiped *ikm* = *jln* for *km* = *ln*. Similarly, face *cn* equals face *nf* and *jln* = *prf* as they have equal thickness, *pn*. It will be observed that these three "complements" have each one face in common with the parallelopiped *ilp*. Similarly, the three parallelopipeds having each one face in common with *psh* are equal.

[3] See Geo. W. Evans, *Some of Euclid's Algebra*, The Mathematics Teacher, Mar., 1927.

That part of the black disc showing through the slots is equivalent to the projections of T upon the axes and hence for every possible value of the angle A, shows the sine and cosine with the proper algebraic sign. A slow and uniform rotation of T around the circle shows the relative rate of variation of the functions as well as their values. The axes are divided decimally and readings may be taken to show numerically that the sum of the squares of sine and cosine is unity. When discussing the functions of any angle A plus or minus ninety degrees, it is only necessary to give the whole model a quarter turn to show the actual interchange of ordinate and abscissa. This makes very evident, the reason for using the cofunctions of A.

II. AN EXTENSION OF THE BERNOULLI SUMMATION FORMULA

By H. S. VANDIVER, University of Texas

The Bernoulli summation formula,[1]

$$1^n + 2^n + \cdots + (k-1)^n = \left[(b+k)^{n+1} - b^{n+1}\right]/n+1,$$

where in the expansion of $(b+k)^{n+1}$ we replace b^i by b_i, and where also $b_0=1$, $b_1=-1/2$, $b_{2i}=(-1)^{i-1}B_i$, $b_{2i+1}=0$, $i>0$, the numbers $B_i=1/6$, $B_2=1/30$, etc. being the numbers of Bernoulli, is well known. It is not so well known, however, that this summation formula admits of a simple extension to integers in arithmetical progression which is apparently due to Glaisher.[2]

I shall give here a proof which is a bit different from Glaisher's. Consider the known relation, using the symbolic method of Blissard and Lucas,[3]

$$(x+1+b)^{n+1} - (x+b)^{n+1} = (n+1)x^n,$$

which holds for any positive integer n as well as for $n=0$, where as before b^i in the binomial expansion is to be replaced by b_i. In this relation we put x/q in lieu of x and obtain:

$$(x+q+qb)^{n+1} - (x+qb)^{n+1} = (n+1)qx^n.$$

In this relation we replace x by $x+q$, $x+2q \cdots$, $x+(k-1)q$, in turn and add the results. We obtain:

$$\frac{(x+kq+qb)^{n+1} - (x+qb)^{n+1}}{(n+1)q} = x^n + (x+q)^n + \cdots + \{x+(k+1)q\}^n.$$

This formula obviously gives the summation of the n-th powers of any set of integers in arithmetical progression in terms of x, n, q, and the Bernoulli numbers.

[1] P. Bachmann, *Niedere Zahlentheorie*, vol. 2, p. 22.

[2] Quarterly Journal of Pure and Applied Mathematics, vol. 31 (1900) pp. 193–99.

[3] Lucas, *Theorie des Nombres*, Paris (1891) p. 240.

For the case $x=2$, $q=2$, as well as the case $x=1$, $q=2$, the formula was derived by Bachmann[4] by the use of two different expansions of

$$\sin nx \cos x/\sin x.$$

III. A CONVENIENT CHECK ON THE ACCURACY OF THE PRODUCT OF TWO MATRICES

By WILLIAM E. ROTH, Madison, Wisconsin

Let $A=(a_{ij})$ be a square matrix of order n, and designate the sum of the elements of the i-th row of A by a_i and the sum of the elements of the j-th column of A by α_j. Then

$$(1)\qquad a_i = \sum_{j=1}^{n} a_{ij}, \quad \alpha_j = \sum_{i=1}^{n} a_{ij}, \quad (i,j = 1,2, \cdots, n).$$

Now we introduce the notation, $S(A)$, to designate the sum of all the elements of A; then

$$(2)\qquad S(A) = \sum_{i=1}^{n} a_i = \sum_{j=1}^{n} \alpha_j,$$

according as we sum the elements of A by rows at a time or by columns at a time. With this definition of $S(\quad)$ we can readily show that

$$(3)\qquad S(A+B) = S(A) + S(B),$$

and, if m is a scalar multiplier of A, that

$$(4)\qquad S(mA) = mS(A).$$

Now the element p_{ij} of the product $P=AB$ is given by

$$p_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \qquad (i,j = 1,2, \cdots, n),$$

and if we let the sum of the elements of the i-th row of B be given by b_i and that of the j-th column of B by β_j, to correspond to the notation employed for A, then

$$(5)\qquad S(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{n} \alpha_k b_k.$$

Similarly

$$S(BA) = \sum_{k=1}^{n} \beta_k a_k.$$

In particular, $S(AB)=S(BA)$, when A and B are commutative; however this property is not a necessary one in order that the equality hold. From (5) we have at once

$$(6)\qquad S(A^2) = \sum_{k=1}^{n} \alpha_k a_k.$$

[4] Loc., cit., pp. 292, 293.

If R is the inverse of A, and if r_i and ρ_j are the sums of the elements of its i-th row and of the j-th column respectively, then

$$S(AR) = S(RA) = S(I) = \sum_{k=1}^{n} \alpha_k r_k = \sum_{k=1}^{n} a_k \rho_k = n. \tag{7}$$

Formulae (5), (6), and (7) are very convenient, though not absolute, checks on the accuracy of the products involved.

The idea above may be extended to products of three matrices and the extension is of interest but is not convenient for checking the accuracy of such products. The element p_{ij} of the product $P=ABC$ is given by

$$p_{ij} = \sum_{h=1}^{n} \sum_{k=1}^{n} a_{ih} b_{hk} c_{kj} ;$$

and if c_i and γ_j denote the sums of the elements of the i-th row and of the j-th column, respectively, of C, then we find that

$$S(ABC) = \sum_{h=1}^{n} \sum_{k=1}^{n} b_{hk} \alpha_h c_k .$$

This is a bilinear form whose matrix is B, the middle factor of the product. For A^r, where r is a positive integer, we have at once

$$S(A^r) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{r-2} \alpha_i a_j, \qquad (r \geqq 2),$$

where $a_{ij}^{(r-2)}$ is the element in the i-th row and the j-th column of A^{r-2}. This formula reduces to (6) if we agree to designate A^0 by I, the unit matrix. By virtue of (3) and (4) the operation here studied may be extended to polynominals in matrices. Special forms such as symmetric and skew-symmetric matrices have interesting properties when regarded in light of the above.

RECENT PUBLICATIONS

Edited by Roger A. Johnson, Hunter College, New York, N. Y., to whom books and communications should be sent.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

A Debate on the Theory of Relativity. By R. D. Carmichael, W. D. MacMillan, M. E. Hufford, and H. T. Davis. The Open Court Publishing Co., 1927. 154 pages.

The volume under review contains a report of the debate on the theory of relativity held at Indiana University in May, 1926 under the auspices of the

Indiana chapter of Sigma Xi. The most considerable direct attempt made by any of the speakers to define what a theory is or what properties a theory should have is contained in Professor Carmichael's opening speech. He points out (p. 36) that a theory "must be in suitable agreement with the facts of nature, it must have those aesthetic qualities which render it pleasing to the human spirit, and it must furnish what is to us the most agreeable theory from the point of view of convenience."

It seems advisable to state here, in somewhat explicit form, a view of the nature of a physical theory.[1] The statement is made because only in this way does it seem possible to estimate what this debate accomplished and what it did not. The remarks will refer to an ideal[2] rather than to any actual physical theory; and will be so brief as to contain, of necessity, certain crudities and inaccuracies.

An ideal physical theory starts with a set of undefined elements and certain primary unproved propositions concerning these elements. By the use of the canons of logic, there is then deduced a further set of propositions concerning the elements. These deduced propositions may be called the secondary propositions or the theorems of the theory. The undefined elements are then identified with certain physical entities, part of which at least are experimentally measurable. The secondary propositions or theorems thereby become proposed laws of nature, since they state quantitative relationships of uniformity in the external world which may be tested by experiment. The "theory" is then this entire logical structure,—undefined elements, undefined or primary propositions, the deduced theorems, and the proposed laws of nature which result from the theorem due to the physical identification of the elements.

One may then note three qualities which a theory may or may not possess. It may or may not possess the quality of *self-consistency*, according as the primary propositions are or are not consistent and according as there has or has not been an error in the purely logical structure of the theory.

Secondly, the theory may or may not possess the property of *correctness* or *truth*, according as the proposed natural laws do or do not turn out to be actual natural laws. It is doubtful whether the validity of a law can ever be rigorously established by experiment. Experimental evidence has, however, shown that some proposed laws must be discarded, and it has permitted others to be (temporarily, at least) retained. There will, moreover, always be border-line cases concerning which scientific men will disagree; and this disagreement is an inevitable and desirable result of the subjective judgments involved.

Thirdly, the theory may or may not possess the property of *value*. A little thought will convince anyone that a theory may be self-consistent and true,

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[1] This statement is drawn, in part, from the penetrating analysis found in Chapter VI of N. R. Campbell's *Physics, The Elements*, Cambridge University Press (1920).

[2] Without this qualification the statement would give the false impression that actual physical theories have cleaned house, logically speaking, to the same extent as have certain branches of pure mathematics.

in the senses just discussed, but still be utterly valueless. The value of the physical theory seems to lie in its ability to "explain" the natural laws which are associated with it, and which are its fruition. This "explanation" is chiefly of two sorts, explanation by simplification and explanation by analogy; and the theory is, in the case of these two types of explanation, likely to be called a "mathematical theory" or a "mechanical theory" respectively.

The value which attaches to a theory which explains laws by simplification seems to arise from the subjective sense of satisfaction with which one recognizes that the several laws to which the theory leads appear as united and hence simplified through the central logical structure of the theory. In the case of explanation by analogy, the value of the explanation results partially from the same, and partially from other no less subjective considerations. There are important instances of theories in which there is possible a double identification of the undefined elements. One of these identifications leads to the theory itself, and the other to the analogy. Thus in the kinetic theory of gases certain elements are identified with the molecules of the gas and these elements are also thought of as (that is, identified with) minute perfectly elastic spheres. This second identification furnishes a mechanical model, and the double theory, considered as a whole, is a mechanical theory which explains by analogy. The satisfaction that any one person feels as a result of such an explanation is clearly a subjective matter. A great school of physicists has enjoyed mechanical theories and their explanations by analogy. The tendencies of modern theoretical physics have convinced a goodly number of persons that we should be very skeptical of most explanations by analogy. Thus, one who really believes in the electrical constitution of matter recognizes that the laws of ordinary mechanics are statistical consequences of the laws for individual electrodynamic action; so that a mechanical explanation of electrodynamics is without meaning or importance.

These remarks concerning the nature of a physical theory lead to three conclusions, each important for our present purpose. First, unless one is raising very fundamental questions concerning the validity of the canons of logic, there is a reasonably definite way of determining whether or not a theory is self-consistent.[3] Secondly, the question as to the truth of a theory must be attacked through experiment. After the experimental evidence is obtained, each person must decide for himself as to whether this evidence convinces him that the laws of the theory have or have not been established, this judgment being of an obviously subjective nature. Thirdly, the question as to whether a theory is valuable is, to a still greater extent, a subjective matter. It depends upon one's personal decision as to whether or not the theory explains its laws. Thus a theory can be true for one and untrue for another, and it may certainly be valuable to one and of little value to another. If one judges a theory to be valuable, he is the more likely to also judge it true, and vice

[3] The self-consistency of the theory of relativity was not questioned by any one of the debaters, although it has been seriously questioned by others.

versa. It has even been seriously suggested that some theories are doubtless valuable even though we judge them to be definitely untrue.

These remarks seem to the reviewer to make clear what such a debate could and could not accomplish, and they make it easy to report what actually did occur. In support of the theory, Professor Carmichael, opening for the affirmative, explained in an exceedingly clear and interesting way the nature of the restricted and the general theory of relativity. Only penetrating familiarity and long experience with the doctrines of relativity could produce so illuminating a review of its most important features. He called attention to the naturalness of the ideas, to the diverse laws which are obtained without the introduction of *ad hoc* hypotheses, and to the primitive simplicity of a space-time event as compared with the artificial separation into space and time. In the light of the remarks above, we should thus say that the first speaker expounded the theory and expressed and justified his subjective judgment that it has great value. He also affirmed his belief that the theory is true, but left arguments on this point to the other member of the affirmative team.

Professor MacMillan, opening for the negative, confessed that he is relatively indifferent as to whether or not the theory is true because, in his estimation, it is without value. It lacks value to him because its explanations do not explain. His intuitions are outraged by the abandonment of Euclidean space and Newtonian time, and the result is to him confusion rather than simplification. As one of the great thinkers on cosmological problems, he has on many happy occasions helped to enlarge our vision of those vast stretches of time during which our solar system has been evolving, and such considerations have helped him to gain a sort of cosmic patience. He is willing that science wait a few score or a few hundred years, if necessary, for a theory which will meet all our new experimental facts, but which will, at the same time, hold to the "postulates of normal intuition." He is unwilling to argue about the extremely delicate experimental tests because, true or not true, the theory makes him scientifically unhappy. It is a valuable thing that sound and sincere ideas of this sort be expressed. To seek to modify them through argument is idle and improper. The only way to change these opinions is to make Professor MacMillan over into some other person, and that is a project which American mathematicians will not support.

Following these two opening addresses, Professor M. E. Hufford, for the negative, and Professor H. T. Davis, for the affirmative, considered the experimental evidence for the truth of the theory of relativity. The Kaufman-Bucherer experiment, the Trouton-Noble experiment, the Michelson-Morley-Miller experiment, the Michelson-Gale experiment, the star deflection test, the spectrum shift test, the advance in perihelion test, and the evidence from spectral line structure are all discussed, pro and con. These delicate and subtle experiments cannot, in the reviewer's opinion, be profitably discussed before a popular audience and in an hour's time. It is not scientifically sound to represent that a discrepancy of one per cent or twenty per cent, or even fifty per cent

between theory and experiment is decisive evidence for or against a theory; nor is it safe to make the somewhat naive assumption that the minute one begins to talk of experimental evidence he is on "solid ground." Only those two great winnowers for the truth, time and the theory of probability, will be able to ultimately convince scientists that the first, second, . . . order effects of the theory of relativity are or are not confirmed by experiment. Professors Hufford and Davis had a difficult if not a hopeless task. Their discussion is interesting but it is, of necessity, not very enlightening or convincing.

The debate closed with short rebuttals, by Professors MacMillan and Carmichael, in which are restated and amplified their estimates of the value of the theory and its probable philosophical influence. The book, as a whole, is interesting and stimulating. It contains a useful summary of a good bit of experimental evidence. It contains illuminating and suggestive confessions of scientific faith. But if one wants to know whether the theory of relativity is, for him, a true and a valuable theory, he must steep himself in its doctrines and then search his own mind.

Warren Weaver

Operational Methods in Mathematical Physics. By Harold Jeffreys. London, Cambridge University Press, 1927. vi+99 pages.

Invariants of Quadratic Differential Forms. By Oswald Veblen. London, Cambridge University Press, 1927. vi+102 pages.

These two works constitute numbers 23 and 24, respectively, of the important series of monographs known as the "Cambridge Tracts in Mathematics and Mathematical Physics," and they maintain the high standard set by the previous numbers of that series.

A good idea of the scope of Jeffreys's work is conveyed by the chapter headings which follow: Fundamental Notions; Complex Theory; Physical Applications; One Independent Variable; Wave Motion in One Dimension; Conduction of Heat in One Dimension; Problems with Spherical or Cylindrical Symmetry; Dispersion; Bessel Functions. The main result of the first section of the work is the derivation of Heaviside's expansion theorem which gives in a compact and useful form the solution of a system of linear differential equations with constant coefficients satisfying assigned initial conditions. Since this theorem is implicit, and almost explicit, in the fundamental work of Cauchy[1] on linear differential equations, the especial merit of the operational methods in this connection would seem to be their compactness rather than their fruitfulness. However, the author proceeds to a discussion of the dynamics of continuous systems, which are dominated by partial differential equations rather than ordinary, and here the power of the operational method cannot be denied.

[1] See the following articles by the reviewer: *The Cauchy-Heaviside expansion formula and the Boltzmann-Hopkinson principle of superposition.* Bulletin of the American Mathematical Society, vol. 33, (1927), pp. 81–89; *The duty of exposition with special reference to the Cauchy-Heaviside expansion theorem,* this Monthly, vol. 34 (1927), pp. 234–241.

The statement by the author (p. 53) that a "general proof that the results given by the operational method, when applied to the vibrations of continuous systems, are actually correct, has not yet been constructed" should act as a challenge to mathematicians. The book is concise, almost terse, and will not prove easy reading for anyone but an expert, but we feel that many a reader will return to consult it often with profit.

Veblen's book is a very important contribution and it merits a somewhat detailed analysis. In Chapter 1 the algebraic part of the theory is discussed and use is made of the generalized Kronecker symbol (called by the author Kronecker delta) to develop the theory of determinants. In Chapter 2 invariants and tensors are introduced and the reader must be prepared for a terminology which is as yet not the general one. An invariant is an *object* not changed by transformations of coordinates; thus a tensor is a particular invariant, namely one whose components are point functions; in the formulæ of transformation for the components of a tensor a certain integral power N of the Jacobian is permitted, and N is called the *weight* of the tensor. When $N=0$ we have the classical tensor, and when $N=1$ we have a tensor density. Similarly, relative scalars are introduced. In Chapter 3 the quadratic differential form $ds^2=g_{\alpha\beta}dx^\alpha dx^\beta$ is introduced and the Christoffel three-index symbols are defined; it is shown that these satisfy, under a transformation of coordinates from x to $\bar{x}$, the familiar equations,

$$\bar{\Gamma}_{pq}^{r} = \Gamma_{\alpha\beta}^{\delta}\frac{\partial x^\alpha}{\partial \bar{x}^p}\frac{\partial x^\beta}{\partial \bar{x}^q}\frac{\partial \bar{x}^r}{\partial x^\delta}+\frac{\partial \bar{x}^r}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial \bar{x}^p\partial \bar{x}^q},$$

and the $\Gamma_{pq}{}^r$ are called the components of the invariant of affine connection. The formulæ for covariant differentiation follow; particularly interesting is that for the covariant derivative of a relative scalar. Geodesic coordinates simplify very much the derivation of fundamental formulæ, particularly the Bianchi identity. In Chapter 4 Euclidean geometry, i.e. the geometry for which the curvature tensor vanishes, and Euclidean affine geometry, i.e. the geometry of an affine connection whose components vanish in a particular coordinate system, are discussed; associated tensors (usually called reciprocal) are introduced and an interesting formula for the area of a 2-cell is given. The familiar concepts of curl and divergence are generalized. Chapter 5 is devoted to the equivalence problem. This is the problem of determining, if possible, a transformation from n x's to N u's such that $g_{\alpha\beta}dx^\alpha dx^\beta=\bar{g}_{\lambda\rho}\, du^\lambda du^\rho$, where g_{rs} and $\bar{g}_{pq}$ are given functions of x and u respectively. An instance of importance is the problem of imbedding a given curved space in a Euclidean space of more dimensions. Chapter 6 deals with the affine geometry of paths, a theory developed by the author and his colleagues at Princeton. Good use is made of affine normal coordinate systems which may be regarded as the natural analogue of Riemann's geodesic coordinates. A generalization of covariant differentiation, known as affine extension, follows and affine normal tensors, which are the extensions of the affine connection, are discussed.

Such is the book; and even a casual reader may well judge that only a master and creator of the subject could present so much intelligibly in 102 pages. Having given it the unnecessary tribute of our admiration we must now, if only to escape being insipid, call attention to some matters we regard as lapses from grace. In the historical remarks to Chapter 2, the term tensor is attributed to Einstein (1916); it is really due to W. Voigt, *Die fundamentalen Eigenschaften der Krystalle* (1898). The curl of a covariant tensor of rank one is not a metrical concept nor one of affine connection, but rather of analysis situs, and so its derivation by means of covariant differentiation is unfortunate. Similarly for generalized divergence and curl and the Stokes'-Poincaré tensor. The introduction of two concepts of divergence is apt to be confusing; in physical applications the divergence of the product of a covariant tensor density by a contravariant vector is what appears. This product ρu^r is an alternating covariant tensor of rank $n-1$ and from this can be formed in the general geometry of analysis situs, an alternating covariant tensor of rank n (a scalar density), namely, $(\partial/\partial x^\alpha)(\rho u^\alpha)$. The absolute scalar formed from this by scalar multiplication by the alternating contravariant tensor of rank n, $\pm g^{-1/2}$ is called the divergence. It is not the divergence of the contravariant vector u^r but of the contravariant vector density ρu^r. The only place where the metrical character of the space appears at all is in the step from the *relative* scalar $(\partial/\partial x^\alpha)(\rho u^\alpha)$ to the *absolute* scalar $g^{-1/2}(\partial/\partial x^\alpha)(\rho u^\alpha)$ or in replacing ρ by $\mu g^{1/2}$, where μ is an *absolute* scalar. The point raised here is rather essential since it concerns the foundations of electromagnetic theory. Maxwell's equations concern themselves with the ideas of curl and divergence; as such they have no concern with the metrical character of the space. It is only when we wish to generalize properly the equations $\mathbf{D}=\mathbf{E}, \mathbf{B}=\mathbf{H}$ that the metrics play any rôle.

F. D. Murnaghan

Non-Riemannian Geometry. By L. P. Eisenhart. American Mathematical Society Colloquium Publications, Volume VIII. 184+viii pp.

This book contains the material presented by Professor Eisenhart to the American Mathematical Society at its Ithaca Colloquium, September 1925, under the title, "The New Differential Geometry." The choice of subject-matter is well made from the standpoint of both the general reader and the specialist; and the exposition is in the author's usual lucid compact style.

The book divides itself into four chapters. Chapters I and II deal with affinely connected manifolds, the former with asymmetric, the latter with symmetric connections. The definition of an affinely connected manifold is an n-dimensional space to which a set of functions L^i_{jk} of the coordinates x has been assigned so as to define the parallel transportation of a contravariant vector ξ^i along an arbitrary curve by means of the differential equations

$$\frac{d\xi^i}{dt} = L_{jk}^{\;i}\frac{dx^i}{dt}\xi^k,$$

dimensionality $n-1$, the latter part to sub-spaces of any dimensionality, $m<n$. An affine geometry is induced in a hypersurface of an affinely connected manifold as soon as we choose arbitrarily at each point of the hypersurface a contravariant vector, called the pseudonormal, to play a rôle like that of the unit normal to a hypersurface in a Riemannian space. This pseudonormal chosen, a geometry quite like that of a hypersurface embedded in a Riemannian space ensues. Thus we have asymptotic lines; conjugate directions; lines of curvature; an analogue to the geodesic curvature of a curve on a surface; generalized Gauss-Codazzi equations, which here as in the classic theory of surfaces appear as integrability conditions of a certain system of covariant partial differential equations; etc.

The above sketch may serve as a somewhat rough indication of what the book contains; for more definite ideas and details the text itself must be consulted. As appendix there is given an extensive bibliography arranged in chronological order, which will help the reader in furthering his acquaintance with the subject.

The proof-reading appears to have been especially well-done; in several readings of the text the reviewer did not notice any typographical error.

J. Douglas

Plane Trigonometry. By Harding and Mullins. The Macmillan Company, 1928. 118 pages.

What, another textbook on trigonometry? Yes, and one would almost say there is no need of any more since there are already too many. Then, why this one? Probably this question is best answered by the authors in their preface to this new book: "This text contains the material that is usually required in a semester's course in trigonometry. Since the present tendency is in the direction of a concise treatment, an effort has been made to present clearly and without too much detail the essentials of the subject."

The material is the same as that which is included in most texts on trigonometry. The arrangement of this material and the concise treatment of it are the distinctive features of the book.

The book begins with an introductory chapter on theorems in plane geometry which are most useful in the development of trigonometry. The authors omit, however, two theorems which are used later, the one that acute angles are equal if their sides are perpendicular, and the one that an inscribed angle in a circle is measured by one-half its intercepted arc.

Chapter II is a chapter on coordinates. The presentation is concise but clear.

Chapter III takes up "Angles and Trigonometric Functions." After an angle of any magnitude has been defined, the definitions of the trigonometric functions are given for an angle terminating in any quadrant. This makes for a clear understanding of the subject.

Chapter IV deals with the solution of the right triangle. The natural trigonometric tables are explained and then the tables are used in solving right triangles.

Right triangles are also solved by the use of logarithms but it is evident that the authors presuppose a knowledge of logarithms on the part of the student as no explanation of this theory is given.

In Chapter V come the various laws used in the solution of oblique triangles; and following these the solution of oblique triangles is taken up. This arrangement of the solution of oblique triangles immediately following the solution of right triangles before taking up the addition and subtraction formulas is made possible by the material given in the introductory chapter.

These first five chapters, consisting of 62 pages, cover all the work in so-called "Numerical Trigonometry." The remaining six chapters cover the work usually given in "Trigonometric Analysis."

In Chapter VI, "Reduction Formulas", the functions of angles of the second, third, and fourth quadrants are reduced to functions of a first quadrant angle in two ways. An angle of the second quadrant is thought of as being equal to 90° plus an angle of the first quadrant and also as 180° minus an angle of the first quadrant. Then follow the general laws of reduction.

Chapter VII is entitled "Relations Between the Functions." It first derives the fundamental formulas and then applies these formulas to the proof of identities and the solution of equations. The work in trigonometric identities is brief, almost too brief.

Chapter VIII, "Trigonometric Analysis," takes up the addition and subtraction formulas, double-angle formulas, half-angle formulas, and sum and difference formulas.

Chapter IX, "Unit of Measure," introduces the student for the first time to radian measure and applies it to the length of arc, and to angular and linear velocity.

Chapter X, "Inverse Trigonometric Functions," and Chapter XI, "Line Values and Graphical Representation," are the two final chapters.

The authors succeed in their effort "to present clearly and without too much detail the essentials of the subject." The treatment is concise but clear. It is not merely brief. The authors do not omit essentials nor do they put important material that should be explained in the text as problems to be worked out by the student. The flexibility of the text is such that a teacher desiring a short course covering the minimum essentials can omit the last three chapters without loss of continuity.

The book is a very readable one with neat appearing pages and many well labeled figures. The book contains a large number of carefully chosen problems illustrating the various topics. The problems illustrating the right triangle and oblique triangle are especially well-chosen.

T. E. Mergendahl

College Algebra. By C. I. Palmer and W. L. Miser. New York, McGraw-Hill Book Company, Inc., 1928. 377 pages, including answers. $2.50.

College Algebra by Palmer and Miser is conventional in its development and

treatment of the subject. The material has been selected and arranged, however, so that the text can easily be adapted to classes that vary considerably in preparation and for courses that differ in extent from one of three hours to one of five hours.

There is a wealth of well-graded exercises and problems, a large number of the latter being of a practical nature. The paragraphs on "equivalency" should be suggestive to the better grade of student. The chapter on "inequalities" includes work that will be of assistance in the discussion of equations and the plotting of their loci in Analytics.

In the preface, the authors state: "In the development of ideas and in the discussions, the average student has been kept constantly in mind; and it is to him the authors have addressed themselves rather than to the trained mathematician." The average student certainly ought to be able to read this text with understanding.

The publishers have put out the book in attractive form: the paper, type, and figures are all good.

On the whole, it appears to be a teachable text and one that ought to be very satisfactory.

James E. Donahue

MATHEMATICS CLUBS

All reports of club activities should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES

The Mathematics Section of the Science Club of Hanover College, Hanover, Indiana.

The officers for 1927–28, first semester, were: Frank Bard, president; Margot Lambertson, secretary; second semester, Jesse Harmon, president; Mary E. Holderman, secretary.

Officers for 1928–29, first semester, are George Bishop, president; Louise Plummer, secretary; Professor H. A. Zinszer, faculty adviser.

The program for the year 1927–28 included the following papers:

October 12, 1927. "Trigonometric functions," by Margaret Bellamy, '29.

November 9. "The function concept in geometry," by Eugene Crouch, '28; "The Pythagorean theorem," by Hope Rankin, '29.

December 14. "Matter, space, and time," by Professor H. T. Davis (Indiana University).

February 8, 1928. "Fresnel's mathematical theory of diffraction," by Professor Harvey A. Zinszer.

March 14. "The mathematics of chemistry," by George Balas, '28.

April 11. "Cubic equations," by Harland Harris, '28.

May 9. Picnic at Butler Falls.

May 23. "Summation of series," by Raymond Park, '28.

The club meets semi-monthly; one meeting is sponsored by the mathematics and physics sections, the other by the chemistry, biology and geology sections. The club sent four of its members as delegates to the state mathematical meeting at Butler University.

(Report by Miss Louise Plummer)

Mathematics Club of Washington Square College, Evening Organization, Washington Square, New York.

Professor Frederick John, Adviser.

The program of the Washington Square College Mathematics Club for the year 1927–1928 was the following:

October 19, 1927. Plans for year discussed.

November 16. "Determination of π," by Mr. Henry Salkind, '29.
"Squaring a circle," by Mr. Henry Salkind, '29.

December 14. "Zeno's paradoxes," by Mr. Louis Kalinkowitz, '29.

January 4, 1928. "Perpetual motion," by Mr. Henry Salkind, '29. "Mathematical fallacies," by Miss Theresa Zwerling, '30.

February 29. "Proofs of the Pythagorean theorem," by Mr. William Schutzman, '30.

March 28. "Life of Sir Isaac Newton," by Mr. Nathan Sirota, '30.

April 11. "Relativity," by Mr. Charles K. Payne, member of the faculty.

April 25. "Relativity," (concluding lecture) by Mr. Charles K. Payne, member of the faculty.

(Report by Morris Kline, Chairman)

The Undergraduate Mathematics Club of the State University of Iowa.

The officers for the year 1927–1928 were: Dr. N. B. Conkwright, faculty adviser; H. A. Meyer, president; Dorothy McCoy, secretary.

October 20, 1927. "Some remarkable points of the plane triangle," by Professor Roscoe Woods.

November 3. "Problem of packing," by Professor R. P. Baker.

November 17. "Isotropic lines," by Professor L. E. Ward.

December 1. "Cissoids," by Mr. Watson Davis.

January 12, 1928. "Logarithms of complex numbers," by Mr. C. S. Carlson.

February 9. "Some astronomical magnitudes," by Mr. C. C. Sherman.

February 23. "Trends of present day mathematics," by Mr. S. H. Huffman.

March 8. "Mathematical recreations," by Mr. Fred Webber.

March 22. "Sir Isaac Newton and contemporary mathematicians," by Mr. H. A. Wright.

April 19. "Convergence tests," by Dr. N. B. Conkwright.

(Report by Miss Dorothy McCoy, Secretary)

The Mathematics Club of Mount Holyoke College, South Hadley, Massachusetts.

The following programs were given in the year 1927–28:

October 15, 1927. "Summer work with the Western Electric Company," by Margaret Crierson, '29. "Foreign coins," by Stephanie Locke, '28.

November 12. "Maps," by Professor James Pierpont of Yale University.

January 21, 1928. "Fiedler's cyclography," by Professor B. H. Brown, of Dartmouth College.

February 18. "Christopher Rudolph's *Die Casse*," by Margaret Weeber, '28. "Stifel's *Arithmetica Integra*, by Mabel Miller, '28. "Early mathematical texts in the United States," by Stephanie Locke, '28.

March 9. "Explanation of models of algebraic surfaces," by Professor Emilie N. Martin, of Mount Holyoke College. "Blaise Pascal," by Harriet Smith, '28.

April 7. "The logistic curve," by Professor Lowell J. Reed, of Johns Hopkins University.

The Mathematics Club of the University of Kansas, Lawrence, Kansas.

The officers of the Mathematics Club of the University of Kansas for the year 1927–28 were: President, Leslie McKeehen, '28; Vice-president, Byron Rexroth, '28; Secretary-treasurer, Winnona Venard, '28.

The following topics were presented at the meetings:

October 3, 1927. Business meeting.

October 17. "Mathematics in Italy," by Professor E. B. Stouffer.
November 7. "Pythagorean numbers," by Mabel Penrod, Gr.
November 21. "Life of Leibnitz," by Mary Bates, '28. "Figurate numbers," by J. P. Jenison, Gr.
December 5. "Trilinear coordinates," by Corinne Hattan, Gr.
January 6, 1928. "Clocks and timepieces," by Edwin Titt, Gr.
February 6. "Fundamental notions of relativity," by Professor J. J. Wheeler.
February 20. "Curve fitting," by James Edson, '28.
March 5. "Mexico," by J. M. Gonzalez, '28. "Codes and ciphers," by Ruth Swonger '29.
March 19. "Linkages," by George Heald, Gr. "Newton's pasturage problem," by Florence McClure, '29.
April 2. "Non-Euclidean geometry," by Josephine Braucher, Gr.
April 16. "Russian peasant method of multiplication," by Helen Trotter, '29. "Card tricks," by Edna Dobson, '29.
May 7. "Practical applications of complex numbers," by Millard Smith, '28.
May 14. Mathematics Club Annual Picnic.

(Report by Winnona Venard)

Newtonian Society of the State College of Washington, Pullman, Washington.

The program of the Newtonian Society of the State College of Washington for the year 1927–28 was the following:

October 11, 1927. Election of officers for year 1927–28: President Miss Arlene Perry, '28: Secretary-treasurer, Miss Pansy Swannack, '28; Reporter, Mr. Clarence Ross, '29.
"History of the Newtonian Society," by Florence Johnson, '27. "Conformal representation," by Professor C. A. Isaacs. "Mapping," by Miss Arlene Perry, '28.
October 26. "Perfect and amicable numbers," by Miss Pansy Swannack, '28. "Theory of numbers," by Professor E. C. Colpitts.
November 8. "Development of number systems," by Miss Audrey Graber, '28. "Significance of the theory of functions of a real variable," by Professor H. H. Irwin.
November 22. "Four harmonic points," by Miss Merna Pell, '30. "Projective geometry," by Professor J. R. Vatnsdal.
December 6. "A peculiar sort of algebra," by Mrs. Lorna Herman,'29. "The theory of groups," by Dr. S. E. Shelkunoff.
January 10, 1928. "Astronomy," by Professor C. D. Calogeris and Mr. Ross, '29.
January 24. "The great pyramid of Egypt," by Mrs. A. T. Mills and Miss Uarda Davis, '28.
February 21. "Classification of numbers," by Miss Marguerite Feix, '29. "Development of numbers," by Miss Florence Johnson, '28.
February 28. "Peano," by Mr. Leland Smith, '29. "Mathematical logic," by Professor C. A. Isaacs.
March 9. Annual Banquet.
March 20. "Chinese numbers," by Mr. Kwang Chang, '28. "Fun fest," by the Seniors.

(Report by Professor C. A. Isaacs)

The Mathematics Club of Columbia College, New York, N. Y.

The program of the Mathematics Club of Columbia College for the year 1927-1928 was the following:

October 25, 1927. Meeting for organization conducted by Professor Fite, adviser of the Club. At the request of the members, the faculty appointed the following committee to take charge of the affairs of the Club: Carl Boyer, '28, Chairman; Isidore Kagno, '30; and Aaron B. Muravchek, '30.
November 9. "The trisection of an angle," by Mr. Abraham Gansler, '28.
December 1. "Transfinite numbers," by Mr. H. W. Raudenbush.
December 15. Informal discussion on topics of interest to the members.

January 12, 1928. "Elimination," by Mr. Samuel Borofsky.
February 14. "The theory of substitutions," by Professor Siceloff.
February 28. "Mascheroni constructions," by Mr. Martin Rosenman, '28.
March 14. "Geometry," by Professor Pfeiffer.
March 21. Continuation of the address by Professor Pfeiffer.
April 3. "Interpretation of the Euclidean postulates," by Mr. Edgar R. Lorch.
April 26. "The theory of equations," by Mr. Herbert Hinman, '28.
May 9. "The significance of existence theorems," by Professor Fite.

(Report by Carl Boyer, '28, Chairman of the Executive Committee)

The Mathematics Club of Brown University, Providence, R. I.

The following program is to be given during 1928–1929:

October 30, 1928. "Neighboring worlds," by Gerald Maurice Clemence, '29. "The astrologer's quest," by Myrtle Congdon Ryder, '31.
November 20. "Soap bubbles," by Mabel Louise Blaney, '29. "The modern mariner," by Charles Henry Vehse, Gr.
January 22. "Some geometrical maxima," by Professor Jacob David Tamarkin, of Brown University.
February 26, 1929. "The game of nim and its theory," by Helen Anna Sparrow, '30. "Skeleton division, magic squares," by David Joseph Colbert, '29.
March 19. "The four color problem," by Allen Fuller Pomeroy, '29. "One-sided surfaces," by Edward Mason Read, 3rd, '31.
April 23. "Maps," by Professor James Pierpont, of Yale University.
May. Picnic.
Committee on Program: Professor Bennett; Doctor Hickson; Mabel Louise Blaney, '29; Helen Anna Sparrow, '30; David Joseph Colbert, '29; Peter Shahdan, '30.
Committee on Arrangements: Charles Hill Wallace Sedgewick, Gr.; Mary Honor Cummings, '29; Myrtle Congdon Ryder, '31; Allen Fuller Pomeroy, '29; Edward Mason Read, 3rd, '31.

(Report by Professor R. C. Archibald).

The Mathematics Club of the College of the City of Detroit, Detroit, Michigan.

The Mathematics Club of the College of the City of Detroit held five meetings during the year 1927–1928 at which papers were presented by students and members of the faculty.

October, 1927. "The mathematical theory of statistical correlation," by Mr. William Borgman.
November 23. "The applications of the theory of equations to organic chemistry," by Mr. Donald Murphy.
January 18, 1928. "Line values of trigonometric functions, and their use in constructing curves," by Miss Jean Persons.
March 20. "Magic squares," by Miss Evelyn Raney.
April 17. "Quadrature of the circle, History of π," by Miss Lura Green.

Officers for the first half of the year were: Mr. W. Herbert Bixby, President; Miss Elinor Batie, Secretary.

For the last half of the year officers were: Miss Evelyn Raney, President; Miss Elinor Batie, Secretary.

(Report by Elinor M. Batie, Secretary)

The Mathematics Club of Cooper Union, New York, N. Y.

The officers for the year 1927–1928 were: Peter Douglas, '28, president; James McConaghy, '30, vice-president; Edward Trapani, '29, secretary-treasurer. Meetings were held at intervals of about three weeks throughout the year with the following program:

November 9, 1927. "Curious properties of numbers," by A. H. Beiler, '25, of the American Gas and Electric Company.

November 30. "Mechanical paradoxes," by Abraham Berkowitz, '28.
December 21. "Stadia surveying," by Kaskel Kallman, '31.
January 11, 1928. "Infinity," by W. G. Findley, Instructor in the Department of Psychology.
February 1. Visit to the Museum of Peaceful Arts. Explanation of mathematical and scientific instruments by Edwin Schwarz, '27.
February 29. "Mental calculations," by Abraham Rosenbaum, '29.
March 31. Joint meeting of the student branches of the A. S. C. E., A. S. M. E., A. I. E. E., the Mathematics Club, and the Chemistry Club. "Human relations," by W. Chevalier, general manager of the Engineering News-Record.
April 11. "Projective geometry," by Professor H. W. Reddick. Election of officers.

A copy of Ball's *Mathematical Recreations and Essays* was awarded by the Club to James J. Murphy, '31, for excellence in mathematics in the first year class.

(Report by Professor H. W. Reddick)

The Mathematical Society of Rutgers University, New Brunswick, N. J.

The officers for 1927–28 were: President, Irving H. Worden; Vice-president, Joseph Ensley Clayton; Secretary-treasurer, Charles R. Eason. Regular monthly meetings, besides two joint meetings with the Society at New Jersey College for Women, were held and papers were read as follows:

Professor S. E. Brasefield, "Harmonic motions."
Professor E. P. Starke, "Trilinear coordinates."
Professor C. M. Huber, "Mathematics of æronautics."
Professor W. E. Breazeale, "The path to the house of light."
Professor Richard Morris, "Two circles of the latter part of the 19th century."
Mr. H. H. Pixley, "Exponentials."
Mr. Chas. R. Wilson, "The logarithms of negative numbers."

Student Papers

Irving H. Worden: "A geometric construction."
Reuben McDaniel: "Theory of congruences" and "Functions of multiple angles."
J. E. Clayton: "Concurrent diagonals of a quadrilateral."
Charles Eason: "Continued fractions."
R. J. Miejdak: "Indeterminate equations."
Charles Higgins: "The Delian problem and the trisection of an angle" and "Vandermonde's theorem."
N. C. Giordano: "A geometrical construction."
Jacob Neuss: "Theorems of Menelaus and Ceva."
Edward Green: "Inverting a series."
C. P. Booraem: "Summations of series by the method of differences."

As an appropriate ending to a very successful year, there was an attendance of twenty two at the annual banquet in May.

(Report by Professor Richard Morris)

The Mathematical Society of the New Jersey College for Women, Rutgers University, New Brunswick, N. J.

During the year 1927–28 the regular meetings were changed from monthly to twice monthly; the usual social events of a reception for new members, a Christmas party, the annual banquet, a subscription dance, a farewell to the seniors, and a picnic in May tendered by Professor and Mrs. C. A. Nelson were given. At a joint meeting with the Men's Club from the University, Professor Solomon Lefschetz of Princeton University gave a lecture on analysis situs. At this meeting there was present a number of teachers from high schools nearby.

At the regular meetings, a member of the faculty had a main topic and he associated with him members of the student body who were interested in related topics.

November 15, 1927. Professor A. A. Titsworth: "Linear perspective." Florence Buckley and Gladys Francis: "The relation of linear perspective to art."

December 13. Professor Richard Morris: "Mid-points of segments as vertices of polygons." Josephine Pokorney: "History of geometry." Marguerite Heyer: "Proof of Desargues's theorem."

January 8, 1928. Professor W. E. Breazeale: "Waves of light." Dora Thorpe: "Theories of light propagation." Leonora Gross: "The velocity of light."

February 14 and 28. Professor C. A. Nelson: "Quadratic equations." Ruth Horsefield and Ruth Donner: "Methods of solution." Margaret Banta, Anna Hobbs, and Osie Labaw: "Solutions of problems."

March 27. Professor Richard Morris, Ruth Bump, Margaret Frahme, and Helena Doane: "The nine-point circle and sums of certain ratios." H. H. Pixley: "Gauss's definitions." Edward A. Green: "A problem in analysis." Chas. P. Borraem: "Methods of differences."

April 24. A. E. Meder, Jr.: "Introductory remarks on order." Virginia Young, Gladys Francis, Mildred Bennett, and Gertrude Gard: "Types of serial order."

May 8. Helen Haerter: "Definition of determinants and development by minors." Ruth Nixon: "Properties of determinants." Katherine Roelker: "Evaluation by minors." Ruth Thomspon: "Summary."

(Report by Professor Richard Morris)

PROBLEMS AND SOLUTIONS

Edited by B. F. Finkel, Otto Dunkel, and H. L. Olson

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

SOLUTIONS

3089 [1924, 353]. *Proposed by Norman Anning, University of Michigan.*

Given four points, O, A, B, C, on a straight line, to construct with straight edge only, the point P on the line such that OP shall be the harmonic mean of OA, OB, OC.

Solution by Otto Dunkel, Washington University.

It will be convenient to denote the four points on the straight line by the letters O, U, V, W; it is desired to locate a point P on the same straight line so that $3/OP=1/OU+1/OV+1/OW$.

Through U, V, W draw the sides of a triangle BC, CA, AB, and let AO, BO, CO cut the opposite sides of this triangle in A', B', C' thus forming a new triangle $A'B'C'$. Let the corresponding sides of the two triangles meet in C'', A'', B''. Then by Desargues's theorem A'', B'', C'' lie in a straight line. It will be shown that the intersection of this line with the given line is the desired point P.

From the quadrilateral $AC'OB'$ we see that $BA'CA''$ is an harmonic set; and the same is true of $AC'BC''$ and $CB'AB''$, for similar reasons. We shall use the following theorem which is easily proved: Given any cubic and a fixed point O, let the variable secant line OP cut the cubic in the points U, V, W.

Then, if k is a constant, the locus of the point P such that $1/OU+1/OV+1/OW=k/OP$ is a straight line. If $k=3$ this locus is called the polar line of O with respect to the cubic. Regarding ABC as a cubic, we shall consider the polar line o of O with respect to this cubic. Let L be the intersection of o with BC, and draw the secant OL cutting CA in M and AB in N. Then $3/OL=1/OL+1/OM+1/ON$, or $2/OL=1/OM+1/ON$, and hence $OMLN$ is an harmonic set. But, since the pencil $A(MONA'')$ is harmonic, L must lie on AA'' as well as on BC. Therefore L and A'' coincide. It may be shown similarly that o goes through B'' and C'', and this completes the proof.

The polar conic of O may be determined by the fact that it must pass through the double points A, B, C, and by the general property that the polar of O with respect to the conic must be the polar line of O with respect to the cubic, that is $A''B''C''$. Now the polar of A'' with respect to the polar conic must be $A'O$, and thus $A''A$ is the tangent at A. Likewise $B''B$ is the tangent at B. Now these five conditions suffice to determine the conic, and this conic must necessarily be tangent to CC'' at C.

If we have five points O, U, V, W, X on a straight line, it is obvious that there is a straight edge construction for the point P such that OP is the harmonic mean of OU, OV, OW, OX.

3291 [1927, 491]. *Proposed by C. N. Mills, Normal, Illinois.*

Given a circle of radius R; AB the side of an inscribed octagon; ACE three consecutive points of the inscribed hexagon; AD the side of an inscribed square. BD intersects CE at O. With O as a center and the chord BC as a radius, describe a circle which intersects AD at P. Show that AP is the side of the inscribed heptagon, with an error of $.00105R$.

Solution by Theodore Bennett, University of Illinois.

By a suitable choice of rectangular coordinate axes the given points may be made to have the coordinates

$$A = (R,0),\ B = (\tfrac{1}{2}R\cdot 2^{1/2},\ \tfrac{1}{2}R\cdot 2^{1/2}),$$
$$C = (\tfrac{1}{2}R,\ \tfrac{1}{2}R\cdot 3^{1/2}),\ D = (0,R),\quad E = (-\tfrac{1}{2}R,\tfrac{1}{2}R\cdot 3^{1/2}).$$

Then we find that

$$O = [\tfrac{1}{2}R(2 + 2^{3/2} - 3^{1/2} - 6^{1/2}),\tfrac{1}{2}R\cdot 3^{1/2}] \quad \text{and}$$
$$BC = R[2 - (1 + 3^{1/2})\cdot 2^{-1/2}]^{1/2}.$$

Then the point P can be determined by the standard methods of analytic geometry. Eventually we find that

$$AP = \tfrac{1}{2}R[-2 + 3^{1/2} + 6^{1/2} - (1 - 2\cdot 2^{1/2} + 4\cdot 3^{1/2} - 2\cdot 6^{1/2})^{1/2}] = .866719R.$$

The side of the inscribed heptagon is $2R\sin(\pi/7)=.867767R$, which differs from AP by $.00105R$.

Also solved by C. J. Stowell.

3293 [1927, 491]. *Proposed by R. E. Gaines, University of Richmond.*

From the foci F and F' of a conic, lines FP and $F'P'$ are drawn parallel to each other and cutting the conic in P and P'; find the envelope of PP'.

Solution by Roscoe Woods, State University of Iowa.

It suffices to determine the envelope for the ellipse $b^2x^2+a^2y^2=a^2b^2$, for we have merely to replace b^2 by $-b^2$ in the resulting equation in order to obtain the result for the hyperbola. Instead of the foci two points $D(d,0)$ and $D'(-d, 0)$ will be used. The two parallel lines PDQ and $P'D'Q'$ have the equations $ny-x\pm d=0$, where n has any value; and we suppose that they cut the conic in P, Q, P', Q'. The lines PQ' and $P'Q$ always pass through the origin for all values of n, and do not therefore envelop a curve in the proper sense. The lines PP' and QQ' are parallel and determine with the given conic a system of conics through the four points P, Q, P', Q' having the equations

$$b^2x^2 + a^2y^2 - a^2b^2 + t(n^2y^2 - 2nxy + x^2 - d^2) = 0,$$

where t is the parameter of the system. Certain values of t determine the degenerate members of the system. It is easily found that $t=-a^2b^2/(a^2+n^2b^2)$ gives the member consisting of the pair of parallel lines PP' and QQ', and that the equations of these lines are

$$b^2nx + a^2y = \pm ab(a^2 - d^2 + b^2n^2)^{1/2}.$$

The envelope, which is found in the usual way, is a conic with the equation

$$b^2(a^2 - d^2)x^2 + a^4y^2 = a^2b^2(a^2 - d^2).$$

The case of $d=a$ will be disregarded as trivial. The envelope has the same x-intercepts, $\pm a$, as the original conic. If the original conic is an ellipse, the envelope is an ellipse or hyperbola according as $a>d$ or $a<d$; but if an hyperbola, the envelope is ellipse or hyperbola according as $a<d$ or $a>d$.

If $d^2=a^2-b^2$ the points D, D' become the foci, F, F' of the problem, and the above equation of the envelope is now $b^4x^2+a^4y^2=a^2b^4$, and the envelope is an ellipse. Thus both the ellipse and the hyperbola having the same value for a and the same absolute value for b yield the same envelope. The y-intercepts for the envelope are equal to the semi-latus rectum of the given conic. By finding the similar envelope for this new ellipse and repeating this process indefinitely, we determine an interesting system of ellipses through the points $(\pm a, 0)$.

A similar method could be used for the parabola by taking two points on the axis, but the lines forming the degenerate conics are not equally inclined to the axis, and the work of isolating the envelope is therefore more complicated.

Note by the Editors. The generalized form of the problem may be stated in a way which suggests a method of finding the equations of PP' and QQ' in a manner different from the above.

Let D be a fixed point on one of the principal axes of a conic and let PDQ

be a variable chord. Find the envelope of the chords through P and Q supplementary to PQ.

Also solved by E. F. Allen, William Hoover, H. R. Kingston, and the Proposer.

3295 [1927, 537]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Given a complete quadrilateral it is possible to describe three circles having their centers at the vertices of the diagonal triangle so that the vertices of the quadrilateral shall be the centers of similitude of the three circles taken in pairs. Moreover one of the three radii may be taken arbitrarily.

NOTE. This is the converse of the known proposition: The centers of similitude of three given circles taken in pairs are the vertices of a complete quadrilateral. See, for instance, Nathan Altshiller-Court, *College Geometry*, p. 160, Johnson Pub. Co., Richmond, Va., 1925.

Solution by A. Pelletier, Montreal, Canada

Let $ABCD-EF$ be a complete quadrilateral where AB and CD meet in F, BC and AD meet in E, AC and BD meet in O, and these last two lines meet FE in O'' and O'. Then $BODO'$, $AOCO''$, $FO''EO'$ are harmonic sets.

Let r (any value) be the radius of a circle (O) with center O, and take $r' = rBO'/BO$ as the radius of circle (O'), $r'' = rAO''/AO$ as radius of (O''). Then B and D are centers of similitude for (O) and (O'), and A and C are similar centers for (O) and (O''). Menelaus' theorem applied to the triangle $OO'O''$ and the secant ABF gives the equality

$$(AO/AO'')(BO'/BO)(FO''/FO') = 1.$$

Hence $r'/r'' = FO'/FO''$; and, since F and E divide $O'O''$ harmonically, they are centers of similitude for (O') and (O'').

3296 [1927, 537]. *Proposed by J. Rosenbaum, Milford, Connecticut.*

It is well known that the radius of the inscribed circle of a right triangle is equal to half the difference between the sum of the legs and the hypotenuse. Derive an analogous expression for the radius of the inscribed sphere of a right tetrahedron.

Solution by Otto J. Ramler, Catholic University, Washington, D. C.

Let the edges of the right trihedral angle of the right tetrahedron $O\text{-}ABC$ be a, b, c. Let the areas of the triangles AOB, BOC, COA, ABC be A_z, A_x, A_y, A, respectively. Then it is well known that $A_x^2+A_y^2+A_z^2=A^2$. (See Osgood and Graustein, *Plane and Solid Analytic Geometry*, p. 517, Th. 2.)

If r is the radius of the inscribed sphere, we have

$$6\cdot\text{Volume } O\text{-}ABC = abc = 2r(A_x + A_y + A_z + A).$$

Solving for r, we have

$$r = \frac{abc}{2(A_x + A_y + A_z + A)} = \frac{abc(A_x + A_y + A_z - A)}{2[(A_x + A_y + A_z)^2 - A^2]} = \frac{A_x + A_y + A_z - A}{a + b + c},$$

when proper substitutions are made in the denominator.

Also solved by Elizabeth E. Nixon, A. Pelletier, G. A. Yanosik, and Paul Wernicke.

3297 [1927, 537]. *Proposed by C. N. Mills, Normal, Illinois.*

Suppose BOD to be a quadrant of a circle of radius R; find the radius of a circle inscribed therein. Also find the radius of a circle which will touch both circles and the line OB.

Solution by J. H. Neelley, Carnegie Institute of Technology.

The center Q of the inscribed circle must lie on the bisector of the angle BOD. Let OQ cut the given quadrant of a circle in P, and let C be the foot of the perpendicular from Q to OB. Then the radius of the inscribed circle is $x = OC = 2^{-1/2} \cdot OQ$. Since $OQ = R - x$, we have at once $x = (2^{1/2} - 1)R$.

The second part of the problem has the trivial solution of a circle of radius x inscribed in the second quadrant tangent to OB at C. This fact will be utilized later. Let the second required circle be tangent to OB at F and have center E, and let N be the foot of the perpendicular from E to CQ. Then, since OE passes through the point of tangency of this circle and the quadrant of a circle $OF^2 = R(R - 2y)$, where y is the radius of the circle. In the right triangle ENQ, $EN = OF - x$, $NQ = x - y$, $EQ = x + y$. Hence $(OF - x)^2 = (x + y)^2 - (x - y)^2 = 4xy$. Eliminating OF by means of the previous equation and inserting the value of x, we obtain the equation

$$[2^{3/2} - 1]^2 y^2 - [14(2)^{1/2} - 18]Ry + [2^{1/2} - 1]^2 R^2 = 0.$$

This gives, on rejecting the trivial solution, $(2^{1/2} - 1)R$,

$$y = \frac{(2^{1/2} - 1)R}{(2^{3/2} - 1)^2} = \frac{[5(2)^{1/2} - 1]R}{49}.$$

Also solved by E. C. Kennedy, H. M. Lufkin, Elizabeth E. Nixon, A. Pelletier, O. J. Ramler, and F. L. Wilmer.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

During the first two years of its operation, the Information Bureau to which reference is made on page 1 of this issue, under the direction of Professor H. W. Kuhn, has received 115 registrations of candidates for appointment, and has answered inquiries from 18 institutions in search of candidates. It is hoped that the use of its services will be largely increased as the nature of its work becomes better known.

Professor Hermann Weyl, of Princeton University, delivered an address on the "Law of conservation and rules of intensity in quantum theory" at the University of Minnesota at the dedicatory exercises for the new physics building on November 30, 1928 in conjunction with the November meeting of the American Physical Society.

Professor W. B. Ford, of the University of Michigan, has been given leave of absence for the current academic year. He is traveling in Europe and, as representative of the "Carnegie Endowment for International Peace," he will lecture at the Universities of Leiden and Utrecht, the University of Brussels, the Universities of Lille and Grenoble, and the University of Pisa.

Dr. Charles G. Crooks, for twenty-eight years professor of mathematics in Centre College and for a number of years dean of the college, has retired as dean emeritus because of ill health.

Assistant Professor J. R. Kline has been promoted to a full professorship at the University of Pennsylvania.

Dr. John Williamson has been appointed associate in mathematics at Johns Hopkins University.

Announcement is made of the death of William Theodore Gauss on November 14, 1928, at his home in Colorado Springs, Colorado, at the age of seventy-seven years. Mr. Gauss was a grandson of the illustrious German mathematician, Carl Friedrich Gauss, and through his mother was a nephew of the noted German astronomer, Friedrich Wilhelm Bessel. For many years he spent much time and money in assembling a large and valuable collection of Gauss memorabilia, which is now being used by Mr. G. Waldo Dunnington, of Washington and Lee University, in a biography of C. F. Gauss. This death reduces the number of Gauss's grandchildren to four, one of whom resides in California; the others reside in Missouri. His brother, the Reverend Dr. Joseph Gauss, is superintendent of the Brooks Bible Institute in St. Louis, Missouri.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Thirteenth Summer Meeting of the Association, Boulder, Colorado, August, 1929.

Fourteenth Annual Meeting, Des Moines, Iowa, December 31, 1929, January 1, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled.

ILLINOIS, Carthage, Ill., May 3-4.
INDIANA, Culver Military Academy, May 3-4.
IOWA.
KANSAS.
KENTUCKY.
LOUISIANA-MISSISSIPPI.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA.
MICHIGAN.
MINNESOTA, St. Paul, May.
MISSOURI, Kansas City, Mo., November.
NEBRASKA.
OHIO, Columbus, Ohio, April 4.
PHILADELPHIA.
ROCKY MOUNTAIN, Greeley, Colo., April.
SOUTHEASTERN.
SOUTHERN CALIFORNIA, University of Redlands, March 9.
TEXAS.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE LA.-MISS. BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teachers' agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

ON STERN'S DIATOMIC SERIES

By D. H. LEHMER, Brown University

The purpose of the present paper is to extend the investigation of the following "Diatomic Series" studied by Stern.[1]

0	1																1
1	1								2								1
2	1				3				2				3				1
3	1		4		3		5		2		5		3		4		1
4	1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1
	.	.	.		.	.	.	.	.		.	.	.	.	.	.	.
	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Each line of this table is formed from the preceeding one by inserting between consecutive elements their sum. The lines of this table, of which there are infinitely many, are numbered as indicated on the left.

Stern has proved a number of interesting facts concerning this series among which are the following:

1. The number of terms in the n-th line is 2^n+1, and their sum is 3^n+1.

2. The number of terms in the table down to and including the n-th line is $2^{n+1}+n$ and their sum is $\frac{1}{2}(3^{n+1}+1)+n$.

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[1] Journal für Mathematik, vol. 55, page 193.

3. The average value of the terms in the n-th line is nearly $(3/2)^n$ which is twice the average value of the terms in the whole table down to and including the n-th line.

4. The table is symmetric: the k-th term on the nth line is equal to the (2^n+2-k)th term.

5. The terms which appear in the n-th line as the sums of their two adjacent terms are called *dyads* of the n-th order. There are 2^{n-1} dyads of the n-th order and $2^{n-1}+1$ non-dyads. The dyads occupy the positions of even rank in the line.

6. In the sequence of terms (a, b, c), $(a+c)/b$ is an integer.

7. If the sequence (a, b, c) appears in the n-th line the dyad b occurs in the $(n-k)$th line where $k=(a+c-b)/2b$.

8. Two consecutive terms have no common factor.

9. The sequence (a, b) can occur but once in the table.

10. If a and b are relatively prime the sequence (a, b) appears in the line whose number is one less than the sum of the quotients appearing in the expansion of a/b in a regular continued fraction.

11. The number k cannot appear as a dyad in the n-th line if $n \geqq k$.

12. The number of times an element k appears in the $(k-1)$st and all succeeding lines is Euler's $\phi(k)$.

13. The number p is a prime if and only if it appears $(p-1)$ times in the $(p-1)$st line.

We proceed to a closer study of the series. A term of the series is determined by its line and its rank in that line. We shall exhibit an algorithm for finding the value of the term in the k-th line and of rank R.

Theorem 1. *If a number has the rank R_n in the n-th line it appears directly below in the $(n+k)$th line with the rank:*

$$R_{n+k} = 2^k(R_n - 1) + 1. \tag{1}$$

In fact this formula is easily seen to satisfy the necessary recurrence: $R_{n+k} = 2R_{n+k-1}-1$, and to have the proper initial value for $k=0$. If R_n is equal to one, we see that the rank of one is always one. It is easy to see that the number m appears in the $(m-1)$st line as a dyad of rank 2. Applying the preceding theorem with $R=2$ we see that the rank of m in the n-th line is $2^{n-m+1}+1$. This being true for all values of $m \leqq n+1$, it follows that the $(m-1)$st line contains all the natural numbers from 1 to m in descending order, the integer l appearing at the rank $2^{m-l}+1$. Thus the 4-th line contains the numbers 5, 4, 3, 2, 1 with the ranks of 2, 3, 5, 9, 17 respectively.

If r_1 and r_2 are consecutive entries in the n-th line and $r_1 > r_2$, then r_1 is a dyad. In the line above there are the two entries (r_1-r_2, r_2). If $r_1-r_2 > r_2$, r_2 is again a non-dyad and in the $(n-2)$nd line there will appear the sequence (r_1-2r_2, r_2). But if on the other hand $r_1-r_2 < r_2$, then r_2 is a dyad and the $(n-2)$nd line will reveal the sequence $(r_1-r_2, 2r_2-r_1)$.

In fact if

$$(2)\qquad \begin{aligned} r_1 &= q_1 r_2 + r_3, \\ r_2 &= q_2 r_3 + r_4, \\ r_3 &= q_3 r_4 + r_5, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ r_{m-2} &= q_{m-2} r_{m-1} + r_m, \end{aligned}$$

we pass upwards q_1 lines to the top of the column of r_2 where the sequence (r_3, r_2) occurs. Taking the column of non-dyads r_3 we move upwards q_2 lines to the sequence (r_3, r_4) and so on until we at last reach the sequence (r_{m-1}, r_m) or else (r_m, r_{m-1}) according as m is even or odd. But r_1 and r_2, being consecutive terms, are prime to each other so that we at length run into the edge of the table at $(r_{m-1}, 1)$ or $(1, r_{m-1})$.

From the above equations we can write r_1/r_2 as a continued fraction:

$$r_1/r_2 = [q_1, q_2, q_3, \cdots, q_{m-2}, r_{m-1}].$$

Since the sequences $(r_{m-1}, 1)$ or $(1, r_{m-1})$ are on the $(r_{m-1}-1)$st line, it follows that (r_1, r_2) occurs on the n-th line where

$$n = q_1 + q_2 + q_3 + \cdots r_{m-1} - 1,$$

which is in fact Stern's result 10.

If instead of choosing a sequence (r_1, r_2) we select values of $q_1, q_2, q_3 \cdots r_{m-1}$ such that their sum is $n+1$, and calculate the corresponding continued fraction we will get a sequence (r_1, r_2) on the n-th line; for the equations (2) may be solved backwards for the r_i knowing the q_i.

Since there are 2^n+1 terms in the n-th line we can form 2^n fractions r_1/r_2. Since the table is symmetric for every fraction r_1/r_2 we have a corresponding fraction r_2/r_1. The quotients in the expansion of these two fractions are identical, except that in the case of the proper fraction, the set of quotients is preceded by zero. If we change the complete quotient $1/(r_{m-2})$, which is never unity, to $1/[r_{m-1}-1+(1/1)]$ in the expansion of the proper fraction and disregard the zero quotient q_0, the number of quotients and their sum will remain unchanged. Since the sequence (r_1, r_2) occurs but once in the table we have by this device a set of quotients corresponding to each fraction in the line with no two sets identical. The number of these sets is 2^n which gives at once the proof of the theorem in the theory of partitions that the number of ways of expressing $n+1$ as the sum of positive integers is 2^n, sums differing only in the order of their terms being counted as distinct.

If r_1/r_2 expands with an odd (even) number of quotients, r_2/r_1 has an even (odd) number of quotients. In other words for every expression of $n+1$ as a sum of an odd number of integers, there corresponds one and only one set of even number of integers whose sum is $n+1$. This establishes the theorem that the numbers of ways of expressing $n+1$ as a sum of an odd or even number of integers are the same and are equal to 2^{n-1}.

By finding all the representations of $n+1$ as the sum of integers we can calculate all the sequences (r_1, r_2) on the n-th line. The question now arises how to distribute these sequences on the line. In a particular case the distribution can be effected by tentative methods using the facts 4, 5, 6 and others. But we shall develop a formula which assigns to any number (r_1) in the sequence (r_1, r_2) a definite rank. A given sequence (r_1, r_2) appears on the left or right side of the table according as the number of quotients in the expansion of r_1/r_2 is odd or even. We can always suppose that the given sequence (r_1, r_2) is such that the continued fraction for r_1/r_2 has an even number of quotients and is such that $r_1 > r_2$.

For if r_1/r_2 has an odd number of quotients and is thus on the right side of the table, the fraction r_2/r_1 on the left has an even number. If $r_2 > r_1$ then the adjacent sequence $(r_2, r_2 - r_1)$ can be taken in lieu of the given sequence. When we find the rank of r_1 on the left side, the rank of r_1 on the right can be easily calculated. For example if we wish to find the position of 85 in the sequence (85, 16) we expand the continued fraction $85/16 = [5, 3, 5]$. Thus (85, 16) is on the 12th line on the right hand side. If we know the rank of 85 in (16, 85) which is on the left side of the table we can answer our question by subtracting this rank from $2^{12}+2$. But (16, 85) is followed by $(85, 85-16)$ or (85, 69). We find that[1] $85/69 = [1, 4, 3, 5]$ which has in fact an even number of quotients. Referring to equations (2), the rank of a dyad r_{m-1} in the $(r_{m-1}-1)$st line is 2. According to our Theorem 1, the rank of r_{m-1}, q_{m-2} lines down is $2^{q_{m-2}}+1$ and that of its right hand neighbour r_{m-2} is $2^{q_{m-2}}$. Then q_{m-3} lines farther down the number r_{m-2} has the rank $2^{q_{m-3}}(2^{q_{m-2}}-1)+1$, and its right neighbour r_{m-3} has the rank

$$2^{q_{m-3}+q_{m-2}} - 2^{q_{m-3}} + 2.$$

Similarily the rank of r_{m-4}, q_{m-4} lines farther down is

$$2^{q_{m-4}+q_{m-3}+q_{m-2}} - 2^{q_{m-4}+q_{m-3}} + 2^{q_{m-4}}.$$

Finally the rank of r_1 on the right side of the line n is

$$R = 2^{q_1+q_2+q_3+\cdots+q_{m-2}} - 2^{q_1+q_2+\cdots+q_{m-3}} + \cdots - 2^{q_1+q_2} + 2^{q_1}. \tag{3}$$

We shall next consider the problem of finding the number r_1 which has a given rank R in a given line n. If $R > 2^{n-1}$ we can consider instead the corresponding rank 2^n+2-R which is less than 2^{n-1}. If R is odd, the number r_1 is a non-dyad in the n-th line and we can follow r_1 back to the $(n-k)$th line where it becomes a dyad, and hence has an even rank. By Theorem 1, k is the largest power of 2 in $R-1$ and the rank of r_1 in the $(n-k)$th line is $R_1 = (R-1)2^{-k}+1$. Thus we have only to consider the cases in which R is even and less than 2^{n-1}. Let now r_2 be the left neighbour of r_1 and let

$$r_1/r_2 = [q_1, q_2\ q_3 \cdots q_{n-2}, r_{m-1}]$$

[1] The labor of making the second expansion is obviated by noting that if $r_1/r_2 = [q_1 q_2 q_3 \cdots]$, then $r_1/(r_1-r_2) = [1, q_1-1, q_2 \cdots]$ or $[q_2+1, q_3, q_4 \cdots]$, according as $r_1 > 2r_2$ or $r_1 < 2r_2$.

The rank of r_1 is given by equation (3). We have only to determine the quotients q_i. The first quotient q_1 is simply the highest power of 2 in the given R, q_2 is the highest power of 2 in $R\cdot 2^{-q_1}-1$, q_3 is the highest power of 2 in $(R\cdot 2^{-q_1}-1)2^{-q_2}+1$, and so on.

In this way we can determine the successive partial quotients in the expansion of r_1/r_2. Thus far we have not taken into account the number of the line n. The complete quotient r_{m-1} is determined as the difference between $n+1$ and the sum of the partial quotients. Thus we see that the complete quotient r_{m-1} is a function of the line n for a given rank R. It follows from the theory of continued fractions that the numbers which occupy the same rank in successive lines are in arithmetical progression whose common difference is the numerator of the penultimate convergent. The value of r_1 for a given line can now be readily computed from the continued fraction expansion of r_1/r_2.

Example: What number on the hundredth line has the rank of one million?

$$
\begin{array}{llll}
2^{q_1}(2^{q_2+q_3+\cdots q_{m-2}} & -\cdots+1) = 1000000 & & q_1 = 6\\
2^{q_2}(2^{q_3+\cdots q_{m-2}} & -\cdots-1) = 15624 & & q_2 = 3\\
2^{q_3}(2^{q_4+\cdots q_{m-2}} & -\cdots+1) = 1954 & & q_3 = 1\\
2^{q_4}(2^{q_5+\cdots q_{m-2}} & -\cdots-1) = 976 & & q_4 = 4\\
2^{q_5}(2^{q_6+\cdots q_{m-2}} & -\cdots+1) = 62 & & q_5 = 1\\
2^{q_6}(2^{q_7+\cdots q_{m-2}} & -\cdots-1) = 30 & & q_6 = 1\\
2^{q_7}(& \cdots\cdots\cdots+1) = 16 & & \underline{q_7 = 4}\\
 & & & \sum_{i=1}^{7} q_i = 20
\end{array}
$$

$$r_{m-1} = r_8 = 101 - 20 = 81, \qquad r_1/r_2 = [6,3,1,4,1,1,4,81].$$

Calculating the value of r_1/r_2, we have $r_1 = 97139$.

In the preceding discussion we have found answers to our problems in terms of certain algorithms. We shall pass on to the consideration of series of dyads whose values are given by a definite formula.

Theorem 2. *If the sequence (r_1, r_2) occurs in the n-th line with $r_1 > r_2$ and the rank of r_1 being R, the smallest dyad occuring between the terms r_1 and r_2 on the line $n+k$ is r_1+kr_2 and its rank is 2^kR.*

Consider the portion of the table in which we are interested.

n	r_1								r_2
$n+1$	r_1				r_1+r_2				r_2
$n+2$	r_1		$2r_1+r_2$		r_1+r_2		r_1+2r_2		r_2
$n+3$	r_1	$3r_1+r_2$	$2r_1+r_2$	$3r_1+2r_2$	r_1+r_2	$2r_1+3r_2$	r_1+2r_2	r_1+3r_2	r_2

In the line $n+2$, the smallest dyad between r_1 and r_2 is r_1+2r_2, since $r_1>r_2$.

Applying the same reasoning to the sequence (r_1+r_2, r_2), the smallest dyad on the $(n+3)$rd line is r_1+3r_2, etc. The rank of each minimum dyad at each step is one less than that of r_2. By Theorem 1, after k moves, the rank of r_1+kr_2 is 2^kR, which is the theorem.

In general if (r_1, r_1+r_2, r_2) be any three consecutive numbers in the $(n+1)$st line, the largest dyad in the $(n+2)$nd line between r_1 and r_2 is obtained by starting from the dyad r_1+r_2 and moving down one line and towards its largest neighbour. Thus if $r_1>r_2$, we would move to the dyad $2r_1+r_2$. Since $r+r_2>r_1$ we would next move down to the right to the dyad $3r_1+2r_2$.

We shall define by a zig-zag move one which is continually descending and changing its direction from left to right and from right to left etc. at each line. A right (left) zig-zag move starts down towards the right (left). Continually applying the above reasoning we have the theorem:

Theorem 3. *If the sequence (r_1, r_2) for $r_1>r_2$ appears in the n-th line the right zig-zag move passes over dyads which are greater than any other elements between r_1 and r_2 in any line.*

Corollary. *In any line $n>1$ there are two equal terms which are larger than all the other terms on the line.*

This follows at once from considering the sequence (1, 3, 2) and the symmetry of the table. We shall return to these maximum dyads later. We next consider the rank of any dyad after k steps of a right zig-zag move starting with a term of rank R_0. The first step brings us down towards the right to r_1+r_2, a dyad of rank $R_1=2R_0$ by Theorem 1. The next move takes us down to the left to a dyad of rank $R_2=2R_1-2$, and after k steps we stop on the dyad in question of rank:

$$R_k = 2R_{k-1} - [1 + (-1)^k]. \tag{4}$$

The solution of this difference equation gives us:

$$R_k = \tfrac{1}{3}\{2^k + 2 + [1 + (-1)^{k+1}]\} + 2^k(R_0 - 1).$$

It is easy to verify that this solution satisfies the required recurrence (4) and that for $k=0$ it has the proper value R_0. For a left zig-zag move the recurrence is:

$$R_k = 2R_{k-1} - [1 + (-1)^{k+1}],$$

the solution of which is seen to be

$$R_k = \tfrac{1}{3}\{-2^k + 4 - [1 + (-1)^{k+1}]\} + 2^k(R_0 - 1).$$

These zig-zag moves have another important property namely: the dyads passed over are such that any one is equal to the sum of the preceding two dyads. For in the above diagram

$$3r_1 + 2r_2 = (2r_1 + r_2) + (r_1 + r_2) \text{ for the left move,}$$

and

$$2r_1 + 3r_2 = (r_1 + 2r_2) + (r_1 + r_2) \text{ for the right move.}$$

The property is proved by induction.

We have then to deal with sets of dyads which satisfy the difference equation:

$$W_{n+2} = W_{n+1} + W_n.$$

The general theory of the recurring series of the second order has been considered at length by Lucas.[1] The series W_n are determined when one assigns definite values to W_0 and W_1. If $W_0=0$. $W_1=1$ the series W_n is the celebrated Fibonacci or Pisano series:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \cdots$$

and the value of the n-th term is:

$$U_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

The series is fundamental in our discussion. If W_0 and W_1 assume other values than (0, 1) it can be shown that

$$W_n = W_1U_n + W_0U_{n-1},$$

where U_n is the n-th term of the Fibonacci series. Making use of this fact we can write down the value of the dyad occuring at the end of k steps of a right or left zig-zag move. It is only necessary to know the first two dyads W_0 and W_1.

The maximum dyad on each line deserves special attention. Taking the lines $n=0$, 1 we have for $W_0=1$, $W_1=2$,

$$W_n = 2U_n + U_{n-1} = U_{n+2}.$$

Hence we can state:

Theorem 4: *In any line n the largest dyads have the common value*

$$U_{n+2} = [(1+\sqrt{5})^{n+2} - (1-\sqrt{5})^{n+2}]/2^{n+2}\sqrt{5}\ ;$$

and their ranks are

$$R_n = \tfrac{1}{3}(2^r + 2 + [1 + (-1)^{n+1}])$$

and

$$2^n + 2 - R_n = \tfrac{1}{3}(2^{n+1} + 4 - [1 + (-1)^{n+1}]).$$

Again if we start with one in the second line and make a right zig-zag move we pass over the dyads 1, 3, 4, 7, 11, 18, $\cdots$. Here $W_0=1$, $W_1=3$. $W_n=3U_n + U_{n-1}$. Substituting the expressions derived for U_n and U_{n-1} we have

$$\begin{aligned} W_n &= 3\left[\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}\right] + \frac{(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1}}{2^{n-1}\sqrt{5}} \\ &= \frac{(1+\sqrt{5})^{n+1} + (1-\sqrt{5})^{n+1}}{2^{n+1}}. \end{aligned}$$

[1] American Journal of Mathematics, vol. 1, pp. 184–240, 289–321.

These numbers are what Lucas terms V_n. We can state the following theorem:

Theorem 5: *In any line $n>2$ the largest dyads in the first and last quarters of the line n have the common value;*

$$\frac{(1+\sqrt{5})^{n+1}+(1-\sqrt{5})^{n+1}}{2^{n+1}}$$

and the ranks

$$R=\tfrac{1}{3}\{2^{n-1}+2+[1+(-1)^n]\} \text{ and } \tfrac{1}{3}\{5\cdot 2^{n-1}+4-[1+(-1)^n].$$

Any number of such theorems may be written out.

In general, if the values W_0 and W_1 are the first two dyads in a right zig-zag move where W_0 is in the n-th line with the rank R, there is on the $(n+k)$th line the dyad

$$(5)\quad W_k=W_1\left(\frac{(1+\sqrt{5})^k-(1-\sqrt{5})^k}{2^k\sqrt{5}}\right)+W_0\left(\frac{(1+\sqrt{5})^{k-1}-(1-\sqrt{5})^{k-1}}{2^{k-1}\sqrt{5}}\right)$$

whose rank is

$$R_k=\tfrac{1}{3}\{2^k+2+[1+(-1)^{k+1}]\}+2^k(R_0-1).$$

The same value is found l lines farther down with the rank

$$(6)\qquad R_{k+l}=\frac{2^l}{3}\{2^k(3R_0-2)-1+[1+(-1)^{k+1}]\}+1.$$

Let m be any line with $m>n$. Let $m-n$ be represented as the sum of two integers $k+l$ in all the $m-n$ ways. Then there exist $(m-n+1)$ elements on the m-th line whose values are obtained by putting the values of k in (5) and whose ranks are obtained by putting the values of l and k in (6).

It may be remarked that

$$W_k/W_{k-1}=[1,1,1,1,1,\cdots 1,q_{k+1},q_{k+2}\cdots q_{m-2}].$$

It is hopeless to try to account for every dyad in the table by the zig-zag moves, since there exist sequences (r_1, r_2) in the n-th line with all possible arrangements of quotients whose sum is $n+1$. We might define other moves to give other types of continued fraction expansions such as 1, 2, 1, 2, 1, 2, $\cdots$ or 1, 2, 3, 1, 2, 3, $\cdots$ and an enormous variety of other simple types. But we would still be at a loss to account for the infinitude of non periodic expansions occuring in the majority of cases.

Example: The numbers $r_1=2960276935825111$, $r_2=1679421121698828$ being selected at random, to determine whether the sequence (r_1, r_2) appears in the array and, if so, where. We find that

$$r_1/r_2=[1,1,3,4,1,2,7,1,7,7,1,5,10,7,1,3,1,2,1,10,1,3,1,4,5,9,3,1,6,16].$$

Therefore since r_1 and r_2 are prime to each other and the sum of the quotients

is 124, the sequence appears in the 123rd line and on the left side of the middle since there is an even number of quotients. In fact the rank is

$$R = 2^{108} - 2^{102} + 2^{101} - 2^{98} + 2^{89} + \cdots + 2^5 - 2^2 + 2$$
$$= 321666940077382478983219549565470.$$

The number of terms on this line is

$$2^{123} + 1 = 10633823966279326983230456482242756609.$$

So the sequence (r_1, r_2) is about 1/33058 of the way across the line. The average value of the terms on this line is

$$(3/2)^{123} = 4562730984784777544048,$$

which is 15 million times larger than the number r_1. The largest term on this line is

$$U_{125} = 594251147575126432128751 25$$

whose least rank is

$$R = 35446079887597756610768188274142522 04.$$

SIMILAR-PERSPECTIVE TRIANGLES

By FRANK EDWIN WOOD, Northwestern University

1. *Introduction.* The properties of projectively defined configurations are often known for special metric cases before the properties of the general case are considered; such however is not always the case, and in particular the metric properties of the configuration arising from two similar-perspective triangles have only been studied since the properties of the general Desargues configuration were obtained. Moreover in the majority of papers upon this subject, the idea of sense has not been considered, so that many of the theorems obtained are either incompletely proved or are incorrectly stated. Various new theorems are stated in this paper, reference being made to any known to be in the literature.

Two triangles will be said to be *similar-perspective* when (a) they are similar, (b) they are in perspective with concurrent lines joining vertices at which are equal angles, (c) all ten points of the Desargues configurations arising from them are finite.[1] In this paper two triangles are regarded as similar-perspective only in case (d) they have the same relative sense, in addition to the other properties.

One readily sees that any point of a Desargues configuration can be taken as the center of two perspective triangles made up from points of the configuration, so that we can speak of *two corresponding triangles*, and of *a vertex* (*or point*) *corresponding to a triangle*.

[1] This condition excludes the consideration of two triangles with corresponding sides parallel.

Cominotto[1] seems to have been the first to consider triangles which are at the same time similar and perspective. He ignores the relative sense of the two triangles, and his theorems are incompletely proved.

Speckman[2] has discussed in considerable detail the relations between two similar and perspective triangles with the opposite sense. His papers naturally do not overlap this one, but rather form a complement to it.

Jerabek,[3] however, has considered the field investigated here, and some of his theorems are restated in this paper.

Rohn[4] has considered congruent-perspective triangles, and other figures.

2. *The Desargues configuration arising from two similar-perspective triangles.* Let A, B, C be arbitrary distinct points not in a straight line, and let P be a point distinct[5] from them, lying upon the circumcircle of the triangle ABC. Let PA, PB, PC meet a second circle through P in the points A', B', C' respectively. Then the triangle $A'B'C'$ is perspective to the triangle ABC, P being the center of perspectivity. Moreover, the two triangles are similar, corresponding angles being equal. Let us use the symbol $\backsim$ to mean supplementary, $\angle A \leqq\!\!\backsim \angle B$ meaning that the angle A is equal or supplementary to the angle B. Then we have at once $\angle A \leqq\!\!\backsim \angle CPB \leqq\!\!\backsim \angle C'PB' \leqq\!\!\backsim \angle A'$, etc., and it can be shown in each case (given the order relations upon each of the lines of the configuration) that $\angle A = \angle A'$ etc.

Let AB and $A'B'$ meet at C'', BC and $B'C'$ at A'', CA and $C'A'$ at B'', so that A'', B'', C'' are collinear. Depending upon the order of the points C'', A, B and of B'', C, A, one has $\angle B''AC'' \leqq\!\!\backsim \angle BAC$, similarly $\angle B''A'C'' \leqq\!\!\backsim \angle B'A'C'$, and since $\angle BAC = \angle B'A'C'$, then $\angle B''A'C'' \leqq\!\!\backsim B''AC''$ so that the four points B'', C'', A, A' lie upon a circle. Similarly one shows that there is a circle through A'', B'', C, C', and a circle through C'', A'', B, B'. We also have a circle through A, B, C, P and a circle through A', B', C', P. It follows then that each point of the Desargues configuration arising from the triangles ABC and $A'B'C'$ lies upon the circumcircle of both its corresponding triangles, and so is exterior to them, and therefore each pair of corresponding triangles

[1] E. Cominotto, *Una disposizione particolare dei triangoli simili*, Periodico di Matematica, vol. 10 (1894), pp. 103–104; vol. 11 (1895), pp. 59–61.

[2] H. A. W. Speckman, *Eien niewe cirkel in den moderen driehoek*, Nieuw Archief (2) vol. 5 (1902), pp. 367–373; *Over omgekeerd gelijkormige driehoeken, perspectief gelegen* Nieuw Archief (2), vol. 6 (1905), pp. 179–188.

[3] V. Jerabek, *Sur les triangles à la fois semblables et homologiques*, Mathesis (2) vol. 6 (1896), pp. 81–83; *Concerning the theorem of Desargues-Weyr* (written in Bohemian), Casopis, vol. 41, pp. 30–32; this second article was not available to the author.

[4] K. Rohn, *Kongruente Dreiecke, Dreikante, Vierkante und Tetraeder in perspektiver Lage;* Leipzig, Berichte, vol. 71, (1919), pp. 160–192. This article was not available to the author, but in the review of it, in Jahrbuch über die Fortschritte der Mathematik, vol. 47 (1919–20), p. 560, the first theorem attributed to Rohn is incorrect as stated; the paper does not seem to overlap the present article.

[5] If P coincides with one of the three points, say A, the line PA, thought of as a limiting line as P approaches A along the circumcircle of ABC, would be tangent to the circumcircle at A. With this understanding, the restriction that P be distinct from A, B and C might be removed.

has the same sense. In particular the triangles ABC and $A'B'C'$ have the same sense.

If A', B', C' be the vertices of a triangle perspective to the given triangle, with the center of perspectivity P lying on the circumcircles of ABC and of $A'B'C'$, the configuration just considered would arise, giving

Theorem 1. *If the circumcircles of two perspective triangles meet at the center of perspectivity, the triangles are similar and have the same sense.*

Suppose that A_1BC and $A_1'B'C'$ are perspective triangles with the angles at corresponding vertices equal, and that they have the same sense. Let P be the center of perspectivity. Draw circles circumscribing CBP and $C'B'P$ and let CA_1 meet the first circle in A, and let AP meet the second circle in A'. By theorem 1, the triangles ABC and $A'B'C'$ are similar-perspective. Since $\angle A_1' C'B' = \angle A_1CB = \angle ACB = \angle A'C'B'$, it follows that A_1', A', and C' are collinear. Since the triangles A_1BC and $A_1' B'C'$ are similar, $AC/A'C' = BC/B'C'$, whence $A_1C/A_2' C' = AC/A'C'$. Then either A_1 and A_1' coincide with A and A' respectively, or else the lines AC and $A'C'$ are parallel.[1] In the latter case the other corresponding sides of the triangles A_1BC and $A_1' B'C'$ are also parallel. Thus we have proved

Theorem 2. *If two similar triangles with the same sense are so placed that the lines joining corresponding vertices (at which angles are equal) are concurrent, then either the corresponding sides of the two triangles are parallel, or the center of perspectivity is upon the circumcircle of each of the two triangles.*[2]

If the parts of one triangle are arranged in the reverse order to the parts of the other, then Theorem 2 (as others to follow) is not valid.

The triangles ABC and $A'B'C'$ considered in the argument preliminary to Theorem 1 are the most general similar-perspective triangles, from Theorem 2. Using the results obtained in that discussion, we have

Theorem 3. *In the Desargues configuration arising from two similar-perspective triangles, each point of the configuration lies upon the circumcircle of each of its corresponding triangles.*

Theorem 4. *The ten points of a Desargues configuration arising from a pair of similar-perspective triangles lie by fours upon five circles with two circles through each point.*[3]

Let us consider the pair of triangles, $B''CC'$ and $C''BB'$ which are perspective from A''; we have noted that the triangles of each pair of corresponding

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[1] Reference theorem: If two given lines ABC and $A'B'C'$ are met by three distinct lines PAA', PBB', PCC' so that $AB:A'B' = BC:B'C'$, then the two given lines are parallel.

[2] This theorem is incorrectly given by Rohn, loc. cit. for a special case.

[3] These two theorems, though not stated by Jerabek (loc. cit.), would follow immediately from relations established by him.

triangles have the same sense, so these triangles $B''CC'$ and $C''BB'$ have the same sense. Moreover the circles circumscribing them pass through their center of perspectivity, A'', so they are similar by Theorem 1. A similar argument applies to the other vertices, whence we have

Theorem 5. *If the triangles of one pair of corresponding triangles of a Desargues configuration are similar-perspective, then all ten pairs of corresponding triangles are pairs of similar-perspective triangles.*

Let the triangle $A'B'C'$ vary so as to be similar-perspective to the fixed triangle ABC with a given line $A''B''C''$ as axis. Then the loci of the points A', B', C' are the circles $B''C''A$, $C''A''B$, $A''B''C$, respectively, while the locus of P, the center of perspectivity, is the circle ABC. Let A' approach O, the second intersection of the circles ABC and $B''C''A$, along the latter circle; then P also must approach O. Now B' is the intersection of the lines $A'C''$ and PB; so when A', and therefore P also, is at O, B' is also; likewise C'. The point O represents a point triangle similar-perspective to the given triangle ABC, and the directions of the three sides are the lines $A''O$, $B''O$, $C''O$, so that the angles between these lines are equal to the angles of the triangle ABC. Moreover since B' and C' approach O with A', then the circles $C''A''B$ and $A''B''C$ also pass through O. Finally one can show by a similar argument that the circle $A'B'C'$ passes through O, whence we have

Theorem 6. *The five circles of theorem 4 have a point in common.*[1]

In the preceding paragraph, the triangle ABC and the line $A''B''C''$ were chosen arbitrarily, and form an arbitrary quadrilateral. If each of the four lines in turn be omitted, we get four triangles; so as a special case of the preceding theorem, we have

Theorem 7. *The circumcircles of the four triangles of a quadrilateral formed by omitting one line at a time have a point in common.*[2]

Calling this point the *Miquel point of the triangle and the line* one has, from theorems 4 and 6,

Theorem 8. *In the Desargues configuration arising from two similar-perspective triangles, the twenty Miquel points, one for each triangle of the configuration and the corresponding axis, coincide.*

There are many theorems which might be stated, expressing the relations which have been established; other theorems might be proved as special cases. A few are added.

Theorem 9. *The center of perspectivity of any two similar-perspective triangles is one of the points of intersection of the two circles circumscribing them. The other point is the Miquel point of either triangle and the axis of perspectivity.*

[1] Jerabek, loc. cit.

[2] Miquel, Liouville's Journal de Mathématiques, vol. 3 (1838), pp. 485–487.

Theorem 10. *Let the vertices of a triangle be joined to a point P upon its circumcircle, and let these joins meet a second circle through P in points of a second triangle; then the second triangle is similar-perspective to the first.*

On the other hand, let ABC and $A'B'C'$ be two similar triangles with the same sense, whose circumcircles meet at a point P upon the line AA'; using Theorem 10, one can prove

Theorem 11. *If the join of one pair of corresponding points of two similar triangles with the same sense passes through one of the intersections of their circumcircles, then the joins of the other pairs of corresponding points also pass through the same intersection, and the triangles are similar-perspective.*

By allowing A' to approach A, we are led to

Theorem 12. *The tangents to the circles $B''C''A$, $C''A''B$, $A''B''C$ at A, B, C, respectively, meet in a point Q upon the circle ABC.*

Now let α, β, γ, denote the circles OAA', OBB', OCC', respectively; and let B' move along β to A''; $B'C'$ becomes tangent to α and $A'C'$ becomes tangent to β, so that the angle between α and β is equal or supplementary to the angle ACB; and similarly for the angles between the other pairs of circles. Denote the centers of the circles α, β, γ by A_0, B_0, C_0, respectively. The line OA'' is perpendicular to the line B_0C_0; and there is a similar result for the other sides of the triangle $A_0B_0C_0$, which is therefore similar to the triangle ABC. Moreover the angle A_0OC_0 is equal or supplementary to the angle between the circles α and γ, and so is equal or supplementary to the angle $A_0B_0C_0$, so that the circle $A_0B_0C_0$ passes through O.[1] By repeating the argument with another pair of similar-perspective triangles, one can show that the centers of the circles ABC and $A'B'C'$ lie upon the circle $A_0B_0C_0$, whence we have

Theorem 13. *The centers of the five circles of theorems* 4 *and* 6 *lie upon a circle which passes through the point common to all five circles.*

As a special case, we have

Theorem 14. *The centers of the circumcircles of the four triangles of a quadrilateral lie upon a circle which passes through the Miquel point of the quadrilateral.*[2]

3. *Orthohomological triangles.* We have shown that $\angle CB''C' = \angle BC''B'$, $\angle AB''A' = \angle BA''B'$, and $\angle AC''A' = \angle CA''C'$. Now the angle between the lines AB and $A'B'$ may be taken either as the angle $AC''A'$ or as the angle $BC''B'$ in case these two angles are different. By a proper choice of the angle between the lines AB and $A'B'$ when a choice is necessary, and similarly for BC and $B'C'$ and for CA and $C'A'$, there follows

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[1] In this proof, the author follows in part the method of Cominotto, loc. cit.

[2] Gallatly, *The Modern Geometry of the Triangle*, pp. 5–7.

Theorem 15. *The angles between corresponding sides of two similar-perspective triangles are equal.*

When the angles of Theorem 15 are $90°$, corresponding sides of the two triangles are perpendicular to each other and the triangles are called orthohomological.[1]

Orthohomological triangles are special cases of similar-perspective triangles and some of the preceding theorems lead to theorems involving orthohomological triangles. If a given pair of similar-perspective triangles are right triangles, then in the Desargues configuration arising from them at least three pairs of corresponding triangles (there are ten pairs in all) are right triangles, and one pair are orthohomological triangles. In a Desargues configuration arising from two orthohomological triangles, at least three pairs of triangles are right triangles.

4. *An application. Theorem of Tafelmacher.*[2] Let ABC be any given triangle and let ABC_1 be a triangle with vertices A and B coinciding with the vertices A and B of the triangle ABC; let a triangle AB_1C be similarly placed upon the side AC of the triangle ABC, the triangle AB_1C being similar to ABC_1 with angles at vertices with the same letter, or with a subscript equal; let A_1BC similarly be placed upon BC. Let the three triangles A_1BC, AB_1C, ABC_1 have the same sense. Using preceding theorems, especially Theorem 10, one can prove the following

Theorem 16. *If similar triangles ABC_1, AB_1C, A_1BC, all directed inward or all directed outward with respect to a given triangle ABC, are placed upon the sides of ABC in such a way that all the angles A are about one point, and likewise for B and C, then the lines AA_1, BB_1, CC_1 are concurrent; also the circumcircles of the triangles A_1BC, AB_1C, ABC_1 pass through the point of concurrence.*

The first half of this theorem is due to Tafelmacher.

5. *The nets of conics associated with a Desargues configuration.* The results of section 2 may be extended in part to a general Desargues configuration. Take two conics, one through the points P, A, B, C and the other through the points P, A', B', C', of a general Desargues configuration in such a way that the conics have four distinct intersections and so that no one of their three intersections (besides P) lies upon a line of the configuration. Let us make a projective transformation taking two of these three points into the circular points at infinity, and the third into any general point of the plane; then the two conics will be transformed into circles, and the Desargues configuration into one arising from two similar-perspective triangles. Therefore we have

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[1] M. J. Neuberg, *Triangles orthohomologiques*, Mathesis (2), vol. 5, (1895), pp. 267–8.

[2] Tafelmacher, *Verallgemeinerung eines von Herrn M. Linnich behandelten Lehrsatzes*, Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht, vol. 44 (1913), pp. 315–6. Tafelmacher's proof is analytic.

Theorem 17. *A general Desargues configuration can be projected into some configuration arising from similar-perspective triangles.*

Conversely any general Desargues configuration can be obtained by projection of some configuration arising from similar-perspective triangles; the circles of theorems 4 and 6 project into conics through three points, and so into conics which belong to a net, giving

Theorem 18. *There are five groups of four points each which can be obtained from a Desargues configuration by taking any point of the configuration and the vertices of either of its corresponding triangles. Five conics can be determined each by a group of four points and an arbitrary point of the plane. These five conics belong to a net which has three basis points.*[1]

THE NUMBER OF REPRESENTATIONS BY CERTAIN POSITIVE TERNARY QUADRATIC FORMS

By BURTON W. JONES, University of Chicago

1. *Introduction.* The number of solutions of $x^2+y^2+z^2=n$ in integers for any positive integer n is a known function of the class number[2] (a transcendental function) of n. More specifically Kronecker[3] proved, in effect: The number of solutions of the equation $x^2+y^2+z^2=n$, where n is neither an exact square nor three times an exact square, is equal to 12 times the difference between the number of reduced non-equivalent forms, $ax^2+2bxy+cy^2$ with a or c odd and the number of such forms with both a and c even, where a, b and c are integers, $a>0$ and $ac-b^2=n$. If n is an even square, then to 12 times the difference mentioned we must add 6; if n is an odd square we must subtract 6; if n is 3 times an exact square we add 8.

However, the number of solutions of $ax^2+by^2+cz^2=n$, where a, b and c are positive integers, is not, in general, known. The number of solutions of $x^2+2y^2+3z^2=n$, when n is of the form $6m\pm 1$, was expressed by Liouville[4] as a function of the class number of n, but he gave no proof. Nazimoff[5] proved Liouville's statement for n of the form $6m\pm 1$ by methods similar to those below. It should be noted that there is more difficulty when n is a multiple of

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[1] This theorem can be obtained from the work of W. van der Wonde, e.g. *De kubische involutic van den eersten rang in het platte vlak*, Amsterdam Akademie, Verslag, vol. 18 (1910), pp. 842–851.

[2] A set of forms of determinant n with certain restrictions on the coefficients and such that any form of determinant n is equivalent to one of the set is called a set of reduced forms. Thus each reduced form defines a "class" of equivalent forms, the "class number" being the number of such distinct classes. For more detailed explanation see Mathews' *Theory of Numbers.*

[3] Journal für Mathematik, vol. 57 (1860), p. 253.

[4] Journal de Mathématiques, (2), vol. 14 (1869), p. 359.

[5] *Applications of the theory of elliptic functions to the theory of numbers*, translated by Chaimovitch, p. 97.

3. Uspensky[1] has dealt with some such relationships among the above and other forms incidentally, being chiefly concerned with relationships among certain class number functions, and has only very fragmentary results for this problem.

This paper gives the complete solution of the problem: to express the number of solutions of $\beta=n$ in terms of class number functions, where β is the form $x^2+2y^2+3z^2$, this being done by giving the number of solutions of $\beta=n$ in terms of the number of solutions of $x^2+y^2+z^2=n'$, where n' is a certain multiple or factor of n. Incidentally the same problem is solved for the forms $\alpha=x^2+y^2+2z^2$ and $\gamma=x^2+3y^2+6z^2$.

The methods used here are applicable not only to these forms but to a wide range of forms including many which, on the surface, seem to have no relation to the form $\sigma=x^2+y^2+z^2$.[2]

We shall find the following notations useful:

$N\ (n)$ = the number of integral solutions of $\sigma=n$. $(\sigma=x^2+y^2+z^2)$

$A\ (n)$ = the number of integral solutions of $\alpha=n$. $(\alpha=x^2+y^2+2z^2)$

$B\ (n)$ = the number of integral solutions of $\beta=n$. $(\beta=x^2+2y^2+3z^2)$

$C\ (n)$ = the number of integral solutions of $\gamma=n$. $(\gamma=x^2+3y^2+6z^2)$

$A'(n)$ = the number of integral solutions of $\alpha=n$ with not all three of the variables divisible by 3.

We have the identity:

$$A(9n) = A'(9n) + A(n), \tag{1}$$

since $9n=\xi^2+\eta^2+2\zeta^2$ with $\xi=3\xi'$, $\eta=3\eta'$, $\zeta=3\zeta'$ implies $n=\xi'^2+\eta'^2+2\zeta'^2$ and conversely.

2. *$A(n)$ must first be considered.* If σ represents $2n$, there exists a solution ξ, η, ζ such that $\xi^2+\eta^2+\zeta^2=2n$.

If $n\equiv 1 \pmod 2$, one of ξ, η, ζ is even and the other two both odd. From symmetry one third of the solutions ξ, η, ζ of $\sigma=2n$ will thus have $\zeta=2\zeta'$ and $\xi^2+\eta^2\equiv 0 \pmod 2$. Then $\xi+\eta=2\xi'$, $\xi-\eta=2\eta'$ are solvable for ξ' and η' and, substituting in $\sigma=2n$, we get $\xi'^2+\eta'^2+2\zeta'^2=n$. Thus, to every solution $(\xi, \eta, 2\zeta')$ of $\sigma=2n$ there corresponds uniquely a solution (ξ', η', ζ') of $\alpha=n$ and conversely, to every solution (ξ', η', ζ') of $\alpha=n$ corresponds uniquely a solution $(\xi, \eta, 2\zeta')$ of $\sigma=2n$. Thus

$$A(n) = \tfrac{1}{3}N(2n) \quad \text{when} \quad n \equiv 1 \pmod 2. \tag{2}$$

If $n\equiv 0 \pmod 2$, $\xi\equiv\eta\equiv\zeta\equiv 0 \pmod 2$. Taking $\zeta=2\zeta'$ and making the above substitution, we have by the same process

$$A(2n) = N(4n). \tag{3}$$

[1] *Bulletin de L'Académie des Sciences de L'Union des Républiques Soviétiques Socialistes*, vols. 19, 20 (1925, 1926): a series of five memoirs. See also the American Journal of Mathematics, vol. 50 (1928), p. 93.

[2] This may be seen from transformations employed in the writer's dissertation, *Representation by positive ternary quadratic forms*, The University of Chicago, June, 1928.

3. *$B(n)$ when $n \not\equiv 0$ (mod 3).* Consider a solution ξ, η, ζ of $\alpha = 3n$, that is $\xi^2+\eta^2+2\zeta^2=3n$. Then either ξ or $\eta \equiv 0$ (mod 3) but not both. From symmetry, we reduce the number of solutions of $\alpha=3n$ by one-half if we take $\xi = 3\zeta'$. Then $\eta^2 \equiv \zeta^2$ (mod 3), $\eta \pm \zeta \equiv 0$ (mod 3) for one and only one value of the sign and

$$\xi = 3\zeta', \quad \eta \pm \zeta = \pm\, 3\eta', \quad \xi' = \eta' - \zeta' \tag{4}$$

are solvable for ζ', η' and ξ' uniquely. Substituting in $\alpha = 3n$, we get $\xi'^2+2\eta'^2+3\zeta'^2=n$. Now, noting the inverse of the transformation (4)

$$\xi/3 = \zeta', \quad \zeta = \eta' - \xi', \quad \eta = \pm\,(2\eta' + \xi'), \tag{4'}$$

we see that every solution $(3\zeta', \eta, \zeta)$ of $\alpha = 3n$ determines, by (4), a solution (ξ', η', ζ') of $\beta = n$ which, in turn, determines, by (4′), a pair of solutions $(3\zeta', \pm\eta, \zeta)$ of $\alpha = 3n$. These are distinct since $\eta \not\equiv 0$ (mod 3). Thus with each pair $(3\zeta', \pm\eta, \zeta)$ of solutions of $\alpha=3n$ is associated by (4) uniquely one solution (ξ', η', ζ') of $\beta = n$ and conversely. Thus we have

$$B(n) = \frac{1}{4}A(3n) \quad \text{for} \quad n \not\equiv 0 \pmod 3, \tag{5}$$

and, using (2) and (3),

$$\begin{aligned} B(n) &= \frac{1}{4}N(6n) \quad \text{for} \quad n \equiv 2 \quad \text{or} \quad 4 \pmod 6 \\ B(n) &= \frac{1}{12}N(6n) \quad \text{for} \quad n \equiv 1 \quad \text{or} \quad 5 \pmod 6. \end{aligned} \tag{6}$$

4. *$B(n)$ when $n = 3n'$.* As in the previous section, every pair $(3\zeta', \pm\eta, \zeta)$ of solutions of $\alpha = 3n$ with $\eta\zeta \not\equiv 0$ (mod 3) is associated by (4) with a solution (ξ', η', ζ') of $\beta = n$ with $\xi' \not\equiv \eta'$ (mod 3), since $\xi' = \eta' - \zeta$, and therefore with $\xi' \equiv -\eta' \not\equiv 0$ (mod 3) since $\xi'^2+2\eta'^2+3\zeta'^2=3n'$ and conversely to every solution (ξ', η', ζ') of $\beta = n$ with $\xi' \equiv -\eta' \not\equiv 0$ (mod 3) corresponds uniquely a pair of solutions $(3\zeta', \pm\eta, \zeta)$ of $\alpha = 3n$ with $\eta\zeta \not\equiv 0$ (mod 3). On the other hand, to every pair of solutions (ξ', η'', ζ'), $(\xi', -\eta'', \zeta')$ of $\beta = n$ with $\xi'\eta'' \not\equiv 0$ (mod 3) corresponds one solution with $\xi' \equiv -\eta' \not\equiv 0$ (mod 3) and which, by (4), determines uniquely a pair of solutions $(3\zeta', \pm\eta, \zeta)$ of $\alpha = 3n$ with $\eta\zeta \not\equiv 0$ (mod 3). Thus, the number of solutions of $\alpha = 3n$ with $\eta\zeta \not\equiv 0$ (mod 3) and $\xi \equiv 0$ (mod 3) is equal to the number of solutions of $\beta = n$ with $\xi'\eta' \not\equiv 0$ (mod 3).

Furthermore, if $\eta\zeta \equiv 0$ (mod 3), $\eta \equiv \zeta \equiv 0$ (mod 3), since $\xi = 3\zeta'$, and we have a representation of $9n'$ by α with the variables all divisible by 3. Thus, the number of representations of $3n$ by α with $\eta\zeta \not\equiv 0$ (mod 3) is $\frac{1}{2}A'(9n')$. Also, if $\xi'\eta' \equiv 0$ (mod 3), $\xi' \equiv \eta' \equiv 0$ (mod 3) since $\beta = 3n'$. Suppose we have a solution $\eta' = 3\eta''$, $\xi' = \xi''$, ζ' of $\beta = n$. Then, substituting in $\beta = 3n'$, we have $3\xi''^2+6\eta''^2+\zeta'^2=n'$ and thus for every solution of $\beta = n$ with ξ' and η' both divisible by 3 there is a unique solution of $\gamma = n'$ and conversely. Thus the number of representations of $\beta = n$ with $\xi'\eta' \not\equiv 0$ (mod 3) is $B(3n') - C(n')$ and we have therefore

(7) $$B(3n') - C(n') = \tfrac{1}{2}A'(9n') = \tfrac{1}{2}A(9n') - \tfrac{1}{2}A(n').$$

If $n' \equiv 2 \pmod 3$, $C(n') = 0$ and we have

(8) $$B(3n') = \tfrac{1}{2}A(9n') - \tfrac{1}{2}A(n'), \quad n' \equiv 2 \pmod 3.$$

If $n' = 3n''$, $\gamma = n'$ implies $z \equiv 0 \pmod 3$ and thus $C(n') = B(n'')$ and we have the reduction formula:

(9) $$B(9n'') = (Bn'') + \tfrac{1}{2}A(27n'') - \tfrac{1}{2}A(3n'').$$

For the consideration of the remaining case: $n' \equiv 1 \pmod 3$ we need

5. *$C(n')$ where $n' \equiv 1$ (mod 3).* Suppose $\xi^2 + \eta^2 + 2\zeta^2 = n'$ with $\xi\eta\zeta \not\equiv 0 \pmod 3$. Then $\eta^2 \equiv \zeta^2 \pmod 3$, $\zeta \pm \eta \equiv 0 \pmod 3$ holds for one and only one of the signs and

(10) $$\eta \pm \zeta = \pm 3\eta', \quad \zeta' = \eta' - \zeta$$

are solvable for η' and ζ' with the proper choice of sign. Substituting $\eta = \pm(3\eta' - \zeta)$ in $\alpha = n'$, we get $\xi^2 + 6\eta'^2 + 3\zeta'^2 = n'$. Noting the inverse transformation we see, as in §3, that with every pair of solutions $(\xi, \pm\eta, \zeta)$ of $\alpha = n'$ with $\xi\eta\zeta \not\equiv 0 \pmod 3$ is associated uniquely a solution (ξ, η', ζ') of $\gamma = n'$ with $\eta' \not\equiv \zeta' \pmod 3$ and conversely. Therefore $A(n')$ with $\xi\eta\zeta \not\equiv 0 \pmod 3$ is equal to $2C(n')$ with $\eta' \not\equiv \zeta' \pmod 3$.

Suppose α represents n' with $\xi\eta\zeta \equiv 0 \pmod 3$. Then either $\eta^2 + 2\zeta^2 \equiv 0$ (mod 3) or $\xi^2 + 2\zeta^2 \equiv 0 \pmod 3$ which implies that ζ and either η or ξ respectively are $\equiv 0 \pmod 3$. We reduce the number of solutions by half if we consider only those for which $\xi \not\equiv 0 \pmod 3$. Then, using substitution (10), we have that with every solution (ξ, η, ζ) of $\alpha = n'$ with $\eta \equiv 0 \equiv \zeta \pmod 3$ is associated uniquely a solution (ξ, η', ζ') of $\gamma = n'$ with $\eta' \equiv \zeta' \pmod 3$ and conversely.

This gives

(11) $$C(n') = \tfrac{1}{2}A(n') \quad \text{if } n' \equiv 1 \pmod 3.$$

Then from (7) and (11) we have

(12) $$B(3n') = \tfrac{1}{2}A(9n') \text{ if } n' \equiv 1 \pmod 3.$$

6. *The final formulae.* Using relationships (2) and (3) with (5), (8), (9), (12) we have with (6) the following relationships between N and B:

$$\begin{aligned} B(3n') &= \tfrac{1}{2}\tau\{N(18n') - N(2n')\}, \text{ if } n' \equiv 2 \pmod 3, \\ B(9n'') &= B(n'') + \tfrac{1}{2}\tau\{N(54n'') - N(6n'')\}, \\ B(3n') &= \tfrac{1}{2}\tau N(18n'), \text{ if } n' \equiv 1 \pmod 3, \end{aligned}$$

where $\tau = 1/3$ or 1 according as n is odd or even.

It may be noted that (11), together with the relationships $C(3n) = B(n)$ and $C(n) = 0$ if $n \equiv 2 \pmod 3$ noted in the previous section, give us, in view of the results above, the complete solution of the problem for the form γ.

The reader may also easily deduce relationships between the proper repre-

sentations by the various forms, i.e. where not all three variables have a factor in common.

After this paper was in type, an article by Uspensky appeared in the American Journal of Mathematics, vol. 51 (1929), p. 51, in which he obtained complete results for the forms here dealt with. This paper, however, goes more into detail concerning the explicit relationships involved, and comparison of corresponding results in the two papers gives certain interesting class number relationships.

CONJUGATE LINES ON A SURFACE

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1. *Introduction.* A system of conjugate lines on a surface consists of two families of curves such that the tangents to a curve from each family at their points of intersection are conjugate tangents to the surface at that point; that is, the tangents are parallel to conjugate diameters of the Dupin indicatrix of the surface at that point. Conjugate lines possess the property that the tangents to the curves of one family at their points of intersection with a curve of the other family form a developable surface. These developable surfaces are circumscribed about the surface, the line of tangency of one being a curve of one family and the generators being tangent to the curves of the other family. Obviously, for each conjugate system on a surface there are two families of these circumscribed developables.[1]

If the equations of a surface are in the form

$$x = x(u,v), \qquad y = y(u,v), \qquad z = z(u,v), \tag{1}$$

the equations of the tangents to the parametric curves are

$$\frac{(X - x)}{\left(\dfrac{\partial x}{\partial u}\right)} = \frac{(Y - y)}{\left(\dfrac{\partial y}{\partial u}\right)} = \frac{(Z - z)}{\left(\dfrac{\partial z}{\partial u}\right)}, \tag{2}$$

$$\frac{(X - x)}{\left(\dfrac{\partial x}{\partial v}\right)} = \frac{(Y - y)}{\left(\dfrac{\partial y}{\partial v}\right)} = \frac{(Z - z)}{\left(\dfrac{\partial z}{\partial v}\right)}, \tag{3}$$

with X, Y, Z as running coordinates. These are the equations of two one-parameter families of ruled surfaces. The condition that these ruled surfaces be developable is

[1] L. P. Eisenhart, *Differential Geometry*, pp. 126–128.

$$\begin{vmatrix} \dfrac{\partial^2 x}{\partial u \partial v}, & \dfrac{\partial^2 y}{\partial u \partial v}, & \dfrac{\partial^2 z}{\partial u \partial v} \\ \dfrac{\partial x}{\partial u}, & \dfrac{\partial y}{\partial u}, & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v}, & \dfrac{\partial y}{\partial v}, & \dfrac{\partial z}{\partial v} \end{vmatrix} = 0. \tag{4}$$

From the properties of determinants it is seen that x, y, and z each satisfy an equation

$$\frac{\partial^2 \theta}{\partial u \partial v} + a\frac{\partial \theta}{\partial u} + b\frac{\partial \theta}{\partial v} = 0, \quad \theta = x, y, z, \tag{5}$$

where a and b are functions of u and v. Either (4) or (5) is the necessary and sufficient condition that the parametric lines on a surface be conjugate. If x, y, and z, functions of u and v, are three linearly independent solutions of a partial differential equation of the type (5), they define a surface upon which the parametric lines form a conjugate system.[1]

Equation (5) is called the point equation of the surface and its invariants are

$$h = \frac{\partial a}{\partial u} + ab, \quad k = \frac{\partial b}{\partial v} + ab. \tag{6}$$

2. The purpose of this paper is to find the equations of those surfaces for which the circumscribing developables are cylinders or cones.

If the developables (2) are cylinders, the direction cosines of the generators for each cylinder must be constant. In (2), u is constant for each generator and hence $\partial x/\partial u$, $\partial y/\partial u$ and $\partial z/\partial u$ must, save for a common factor, be independent of v; that is

$$\frac{\partial x}{\partial u} = f(u,v)U_1, \quad \frac{\partial y}{\partial u} = f(u,v)U_2, \quad \frac{\partial z}{\partial u} = f(u,v)U_3, \tag{7}$$

where U_1, U_2 and U_3 are functions of u alone. Integration of (7) shows that the equations of the surface are of the form

$$x = \int U_1 f(u,v)du + V_1, \; y = \int U_2 f(u,v)du + V_2, \; z = \int U_3 f(u,v)du + V_3, \tag{8}$$

where V_1, V_2 and V_3 are functions of v alone. The circumscribing developables of (8) whose lines of contact are the curves $v = \text{constant}$ are cylinders.

Differentiation of (7) with respect to v gives

$$\frac{\partial^2 x}{\partial u \partial v} = \frac{\partial f}{\partial v}U_1, \quad \frac{\partial^2 y}{\partial u \partial v} = \frac{\partial f}{\partial v}U_2, \quad \frac{\partial^2 z}{\partial u \partial v} = \frac{\partial f}{\partial v}U_3. \tag{9}$$

The elimination of U_1, U_2 and U_3 between (7) and (9) gives the three equations

[1] Eisenhart, pp. 195–198.

$$\frac{\partial^2\theta}{\partial u\partial v} - \frac{1}{f}\frac{\partial f}{\partial v}\frac{\partial \theta}{\partial u} = 0, \quad \theta = x,y,z. \tag{10}$$

This equation is of the type (5); it is seen that $b=0$, and that $a = -(\partial f/\partial v)/f$.

Conversely, if b in (5) be taken as zero this equation may be written

$$(\partial/\partial v)\log(\partial\theta/\partial u) = -a.$$

This can be integrated directly, giving

$$x = \int U_1 e^{-\int a dv} du + V_1, \quad y = \int U_2 e^{-\int a dv} du + V_2, \quad z = \int U_3 e^{-\int a dv} du + V_3,$$

which are equivalent to (8).

If the developables (3) are cylinders, it can be shown in the same manner that $a=0$. Hence: *A necessary and sufficient condition that, if the parametric curves are conjugate, one family of circumscribing developables be a family of cylinders is that either a or b is zero.*

Since one family of a conjugate system may be chosen arbitrarily it can always be chosen so that its circumscribing developables are cylinders; the surface is still general. It is not until conditions are imposed upon both families that the surface becomes restricted.

3. If both families of developables are cylinders, a and b are both zero; equation (5) becomes

$$(\partial^2\theta/\partial u\partial v) = 0, \tag{11}$$

and the surface is the surface of translation

$$x = U_1 + V_1, \quad y = U_2 + V_2, \quad z = U_3 + V_3. \tag{12}$$

Hence the theorem: *Surfaces of translation are the only surfaces that have a system of conjugate lines such that the circumscribing developables to the curves of both families of the system are cylinders.*

4. Equation (2) may be written

$$X = x + w\frac{\partial x}{\partial u}, \quad Y = y + w\frac{\partial y}{\partial u}, \quad Z = z + w\frac{\partial z}{\partial u}\cdot \tag{13}$$

If these developables are cones the generators of each cone must pass through a point, the vertex of the cone. The coordinates of this point are constant for each developable and hence must be independent of v. Let w_0 be the value of w at this point; then,

$$\frac{\partial X}{\partial v} = \frac{\partial x}{\partial v} + w_0\frac{\partial^2 x}{\partial u\partial v} + \frac{\partial w_0}{\partial v}\frac{\partial x}{\partial u} = 0,$$

$$\frac{\partial Y}{\partial v} = \frac{\partial y}{\partial v} + w_0\frac{\partial^2 y}{\partial u\partial v} + \frac{\partial w_0}{\partial v}\frac{\partial y}{\partial u} = 0,$$

$$\frac{\partial Z}{\partial v} = \frac{\partial z}{\partial v} + w_0\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial w_0}{\partial v}\frac{\partial z}{\partial u} = 0.$$

Or

$$\frac{\partial^2\theta}{\partial u\partial v}+\frac{1}{w_0}\frac{\partial w_0}{\partial v}\frac{\partial\theta}{\partial u}+\frac{1}{w_0}\frac{\partial\theta}{\partial v}=0,\quad \theta=x,y,z. \tag{14}$$

On comparing this with (5) it is seen that

$$a=\frac{1}{w_0}\frac{\partial w_0}{\partial v},\quad b=\frac{1}{w_0}\cdot \tag{15}$$

The equations

$$X=x+\frac{1}{b}\frac{\partial x}{\partial u},\quad Y=y+\frac{1}{b}\frac{\partial y}{\partial u},\quad Z=z+\frac{1}{b}\frac{\partial z}{\partial u}, \tag{16}$$

are, therefore, the equations of the locus of the apices of the family of circumscribed cones. This curve is the degenerate focal surface of the congruence of tangents to the curves $v=\text{constant}$.[1]

From (15) it is evident that

$$k=(\partial b/\partial v)+ab=0. \tag{17}$$

In view of (17), equation (5) may be written[2]

$$(\partial\theta_1/\partial v)+a\theta_1=0,\qquad \theta_1=(\partial\theta/\partial u)+b\theta,$$

which can be integrated directly, giving the equations of the surface.

It can be shown in a similar manner that, if the other family of developables are cones,

$$h=(\partial a/\partial u)+ab=0. \tag{18}$$

Hence the theorem: *A necessary and sufficient condition that, if the parametric lines of a surface are conjugate, one family of circumscribing developables be a family of cones is that either one of the two invariants of the point equation of the surface be zero.*

5. If one family of circumscribing developables consists of cylinders and the other of cones, $a=0$, $\partial b/\partial v=0$; $b=f(u)$. Equation (5) becomes

$$(\partial^2\theta/\partial u\partial v)+f(u)(\partial\theta/\partial v)=0, \tag{19}$$

and the equations of the surface are of the form

$$x=\phi(u)[U_1+V_1],\quad y=\phi(u)[U_2+V_2],\quad z=\phi(u)[U_3+V_3],\quad \phi(u)=e^{-\int b d\mu}. \tag{20}$$

Surfaces of this type are discussed by Darboux.[3]

6. If both families of circumscribing developables are cones, h and k are both zero,

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[1] Eisenhart, pp. 404–406.

[2] Darboux, G., *Théorie Générale des Surfaces*, Vol. II, (2nd ed., 1915), pp. 23, 24.

[3] Darboux, Vol. I, (2nd ed., 1914), pp. 181–184.

$$(21)\qquad h = (\partial a/\partial u) + ab = 0, \quad k = (\partial b/\partial v) + ab = 0\,;$$

and hence

$$(22)\qquad (\partial a/\partial u) = (\partial b/\partial v).$$

Equation (22) is the condition that a and b are the partial derivatives of a function λ with respect to v and u respectively. Let $a=(\partial\lambda/\partial v)$, $b=(\partial\lambda/\partial u$; then each of equations (21) reduces to

$$(\partial^2\lambda/\partial u\partial v) + (\partial\lambda/\partial u)(\partial\lambda/\partial v) = 0.$$

This may be written

$$(\partial/\partial u)\log(\partial\lambda/\partial v) = -(\partial\lambda/\partial u),$$

whose integral is $\lambda=\log(U+V)$, where U is a function of u alone and V of v alone. Hence

$$(23)\qquad a = V'/(U+V), \quad b = U'/(U+V).$$

Equation (5) now becomes

$$(24)\qquad \frac{\partial^2\theta}{\partial u\partial v} + \frac{V'}{U+V}\frac{\partial\theta}{\partial u} + \frac{U'}{U+V}\frac{\partial\theta}{\partial v} = 0, \quad \theta = x,y,z.$$

This equation can be integrated readily, giving

$$(25)\qquad x = \frac{U_1+V_1}{U+V}, \qquad y = \frac{U_2+V_2}{U+V}, \qquad z = \frac{U_3+V_3}{U+V},$$

as the equations of the surface having its parametric lines conjugate and having both families of circumscribing developables cones.

The loci of the vertices of the cones are the two curves

$$(26)\qquad \begin{aligned} x_1 &= (U_1'/U'), & y_1 &= (U_2'/U'), & z_1 &= (U_3'/U'),\\ x_2 &= (V_1'/V'), & y_2 &= (V_2'/V'), & z_2 &= (V_3'/V'), \end{aligned}$$

whose equations are obtained from (16).

7. Three types of surfaces, the surfaces of translation (12), the surfaces (20), and the surfaces (25), having conjugate systems whose circumscribing developables are cylinders and cones have been derived. In the first both families consist of cylinders, in the second one of cylinders and one of cones, and in the third both consist of cones. The third class of these surfaces obviously includes the other two, the second being the special case when either U or V in (25) is constant and the first the special case when both U and V are constant.

A simple example of the general type of surface will now be given. If

$$\begin{aligned} U_1 &= a(1-n^2)^{1/2}\sin u, & U_2 &= 0, & U_3 &= c\cos u,\\ V_1 &= 0, & V_2 &= b(1-n^2)^{1/2}\sinh v, & V_3 &= cn\cosh v,\\ U &= n\cos u, & V &= \cosh v, & (n &\neq 0), \end{aligned}$$

equations (25) become

$$(27)\quad x=a\frac{(1-n^2)^{1/2}\sin u}{n\cos u+\cosh v},\qquad y=b\frac{(1-n^2)^{1/2}\sinh v}{n\cos u+\cosh v},\qquad z=c\frac{\cos u+n\cosh v}{n\cos u+\cosh v}.$$

These are parametric equations of

$$(28)\qquad (x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1.$$

Since n is an arbitrary constant that is eliminated when (27) are reduced to (28), a surface of the second degree is a surface of the type (25) in an infinite number of ways. The curves (26) are the two straight lines

$$x_1=-a(1-n^2)^{1/2}n^{-1}\cot u,\qquad y_1=0,\qquad z_1=c/n;$$
$$x_2=0,\qquad y_2=b(1-n^2)^{1/2}\coth v,\qquad z_2=cn;$$

which are at right angles to each other and at a distance $c(n-n^{-1})$ apart.

Incidentally, it is to be noted that the parametric lines of (27) are plane curves, which shows that surfaces of the second degree have an infinite number of plane conjugate systems of curves.

8.[1] The surfaces (25) may be used as the starting point in the discussion of plane conjugate systems of curves on a surface. Consider a surface S with conjugate lines C and their circumscribing developables D. If S is subjected to a transformation by reciprocal polars, S goes into a surface S_1, the developables D into the conjugate lines C_1 on S_1 and the lines C go into the developables D_1 of S_1.[2] For a surface of the type (25) the developables are cones and, since cones go into plane curves, the polar reciprocal of a surface (25) is a surface with a system of plane conjugate lines.

The equations of the polar reciprocal of (25) are

$$(29)\quad\begin{aligned} Dx&=(U+V)(U_2'V_3'-U_3'V_2')+U'[V_3'(U_2+V_2)-V_2'(U_3+V_3)]\\ &\qquad+V'[U_2'(U_3+V_3)-U_3(U_2+V_2)],\\ Dy&=(U+V)(U_3'V_1'-U_1'V_3')+U'[V_1'(U_3+V_3)-V_3'(U_1+V_1)]\\ &\qquad+V'[U_3'(U_1+V_1)-U_1(U_3+V_3)],\\ Dz&=(U+V)(U_1'V_2'-U_2'V_1')+U'[V_2'(U_1+V_1)-V_1'(U_2+V_2)]\\ &\qquad+V'[U_1'(U_2+V_2)-U_2(U_1+V_1)],\end{aligned}$$

where

$$D=\begin{vmatrix} U_1+V_1, & U_1', & V_1'\\ U_2+V_2, & U_2', & V_2'\\ U_3+V_3, & U_3, & V_3'\end{vmatrix}.$$

These are the general equations of a surface with plane parametric conjugate lines.

The partial derivatives of (29) with respect to u and v are:

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[1] Compare Darboux, Vol. I, pp. 176–180.

[2] Eisenhart, pp. 202, 203.

$$\partial x/\partial u = A_2V_3' - A_3V_2', \qquad \partial x/\partial v = B_2U_3' - B_3U_2',$$
$$\partial y/\partial u = A_3V_1' - A_1V_3', \qquad \partial y/\partial v = B_3U_1' - B_1U_3',$$
$$\partial z/\partial u = A_1V_2' - A_2V_1', \qquad \partial z/\partial v = B_1U_2' - B_2U_1',$$

where the A's and B's have been written for the respective coefficients. If the second and third derivatives are computed it is seen that

$$\begin{vmatrix} x_i', & y_i', & z_i' \\ x_i'', & y_i'', & z_i'' \\ x_i''', & y_i''', & z_i''' \end{vmatrix} = 0, \quad i = u,v,$$

which proves that the parametric lines of (29) are plane.

The curves (26) associated with the surface (25) go into the families of planes

$$U_1'x + U_2'y + U_3'z + U' = 0, \tag{30}$$

$$V_1'x + V_2'y + V_3'z + V' = 0, \tag{31}$$

by a polar reciprocal transformation. These are the planes that contain the parametric lines of (29), the planes (30) contain the curves v = constant and the planes (31) the curves u = constant. The same result may be obtained by multiplying x, y, and z in (29) by U_1', U_2', and U_3' respectively and adding, and by V_1', V_2', and V_3' and adding.

Let the equation of the tangent plane of (29) be $ax+by+cz+d=0$. Each point (u, v) of (29) lies on a plane of (30) and on a plane of (31) and also on the tangent plane. Hence the surface is the locus of the intersections of these three planes. This intersection is given by

$$x = \frac{d(U_2'V_3' - V_2'U_3') + U'(cV_2' - bV_3') + V'(bV_3' - cU_2')}{|\, a, U_2', V_3' \,|}$$

and similar expressions for y and z. On comparing these equations with (29), it is seen that, within a common factor,

$$a = U_1 + V_1, \ b = U_2 + V_2, \ c = U_3 + V_3, \ d = U + V;$$

and consequently the equation of the tangent plane is

$$(U_1 + V_1)x + (U_2 + V_2)y + (U_3 + V_3)z + (U + V) = 0.$$

ALGORITHMS FOR THE SOLUTION OF THE QUADRATIC CONGRUENCE[1]

By H. S. VANDIVER, University of Texas

The subject of this paper is an elementary problem of long standing in the history of the theory of numbers. If we consider the congruence

$$x^2 \equiv q \pmod{p}, \tag{1}$$

[1] Presented to the American Mathematical Society at Ithaca, N. Y., Sept. 1925.

where q and p are rational integers, p a prime, and q prime to p, then by the use of the law of quadratic reciprocity we have a definite algorithm not involving trial for ascertaining if integral values of x exist satisfying the congruence. If we consider, however, the problem of determining the value of x in this relation no such convenient procedure has as yet been proposed.

In the elementary theory of binary quadratic forms a method is described for finding the values of x and y in the relation

$$ax^2 + bxy + cy^2 = m,$$

all these symbols representing rational integers. *In this connection it should be noted that in order to carry out this method it is always necessary to know the solution of a congruence of the form*

$$x^2 \equiv k \pmod{m}.$$

We shall here discuss a method for the determination of x in (1) which, however, is not devoid of a certain amount of trial. Its relation to former methods which have been published will also be considered.

As there is a certain analogy between known methods of solving the linear congruence

$$ax \equiv 1 \pmod{p}, \tag{2}$$

where a is an integer prime to p and the solutions of (1), we shall describe briefly the former methods.

1. A root of (2) is given by a^{p-2}, using Fermat's theorem.

2. A solution of (1) is obtained by expanding p/a as a continued fraction and using the relation $pk+al=\pm 1$, where l/k is the penultimate convergant to 0 to p/a.

3. In a previous paper[1] I proposed a method for the solution of (2) which, in the case of prime modulus, may be put in the following form.

Set $k_1p=aq_1+r_1$, where $|r_1|$ is less than a and k_1 is a convenient integer chosen to make $|r_1|$ relatively small.

Proceed in a similar way with p and r_1 so that $k_2p=r_1q_2+r_2$, where $|r_2|$ is less than $|r_1|$.

We obtain ultimately $k_sp=r_{s-1}q_s+r_s$, where $r_s=\pm 1$. Considering these expressions modulo p we may write $r_{i-1}q_i\equiv -r_i \pmod{p}$. Setting $i=1, 2, \cdots, s$, and multiplying all the congruences together we find

$$x = (-1)^s \prod_i q_i/r_s.$$

A solution of (1) which has to a certain extent analogy with the first method described above is the solution of (2) which was given by Cipolla.[2] In this a formula for x is given as follows:

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[1] This Monthly, vol. 31 (1924), pp. 137–140.

[2] Mathematische Annalen, vol. 63 (1907), pp. 54–61. A somewhat different formula for the case $p=8n+1$ (n odd) was given by Caris, in this Monthly, vol. 32 (1925), pp. 294–7.

$$x = \frac{-1}{2^{r-2}} q^{\lambda-1} \sum_{s=0} q^{\mu}/(\omega^{\nu} - 1), \tag{3}$$

where

$$\lambda = (p + 2^r - 1)/2^{r+1}\,; \quad \mu = s(p-1)/2^r\,; \quad \nu = (2s+1)(p-1)/2^r\,;$$

and where ω is a quadratic non-residue of p, and 2^r is the highest power of 2 dividing $p-1$.

For $p \equiv 3 \pmod 4$, this reduces to the well known solution $x \equiv q^{(p+1)/4}$, and if $p \equiv 5 \pmod 8$, we may take $\omega = 2$ and obtain

$$x \equiv \tfrac{1}{2} q^{(p+3)/8}\{2^t + 1 - (2^t - 1)q^t\}, \quad t = (p-1)/4.$$

Note that this solution involves the element of trial in the selection of ω and that (3) becomes very complicated when r is large. Kummer[1] gave a proof of the law of quadratic reciprocity (involving, however, an unproved assumption; see the top of p. 14 of his article). His argument may be modified in such a way as to give an algorithm for the determination of x in (1), if q is prime.

If we examine the Pellian equation

$$t^2 - Du^2 = 1, \tag{4}$$

where D is a positive integer of the form $4n+1$, then t is odd and u is even, and we have integers m and m' so that

$$t + 1 = 2m\delta^2, \quad t - 1 = 2m'\lambda^2,$$

in which $mm' = D$ and $2\delta y = u$. Elimination of t gives

$$m\delta^2 - m'\lambda^2 = 1. \tag{5}$$

Assume now that $D = pq$, and p and q are primes of the form $4n+3$ and that (t, u) is a solution of (4) in which x and y are the smallest possible positive integers. In view of the latter assumption we can not have $1 = \delta^2 - pq\lambda^2$. Also since p is a prime of the form $4n+3$ the relation $1 = pq\delta^2 - \lambda^2$ does not exist. Hence if $(q/p) = 1$, then (5) gives $1 = q\delta^2 - p\lambda^2$, or $q\delta^2 \equiv 1 \pmod p$, and $x \equiv \pm 1/\delta \pmod p$, for the solution of (1).

Example: If $x^2 \equiv 3 \pmod{71}$, we obtain by expanding square root of 213 as a continued fraction, or from existing tables, $194399^2 - 213 \cdot 13320^2 = 1$, whence (5) becomes $1 = 3 \cdot 180^2 - 71 \cdot 37^2$, and $28^2 \equiv 3 \pmod{71}$.

The above method is devoid of the element of trial, but this is not the case if we use an analogous method for p a prime of the form $4n+1$ and q is a prime of the form $4n+3$. Here we have to select an auxiliary prime p' and put $D = pp'q$, p' being selected by trial to satisfy the conditions $(p'/p) = -1$, $(p'/q) = -1$. However, after p' has been found, the method is devoid of trial although the work will be very long in many cases since it is known that the values of t and u in (4) are quite large even for small values of D. Obviously

[1] Journal für Mathematik, vol. 100 (1887), pp. 10–14.

we are determining integers δ and λ which satisfy the relation $1=q\delta^2-p\lambda^2$, a relation which is more special than (1).

In particular cases the computation may become relatively short as it sometimes happens that there are only four or five terms in a period in the development of $\sqrt{D}$ as a continued fraction. For example, consider $x^2 \equiv -4019 \pmod{9043}$.

The method which is described for the solution of (1) obviously has a type or analogy with the second solution of (2). *This leads us to inquire as to the possibility of there being a method for the solution of* (1) *which is analogous to the third method described for the solution of* (2). So far this investigation has yielded nothing.

QUESTIONS AND DISCUSSIONS

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The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

DISCUSSIONS

I. A NOTE ON THE HYDRODYNAMICAL THEOREM ON THE RELATION BETWEEN IMPULSE AND ENERGY

By NRIPENDRANATH SEN, Calcutta University

The well-known hydrodynamical theorem on the relation between impulse and energy viz.,

$$\xi,\eta,\zeta,\lambda,\mu,\nu = \frac{\partial T}{\partial u}, \frac{\partial T}{\partial v}, \frac{\partial T}{\partial w}, \frac{\partial T}{\partial p}, \frac{\partial T}{\partial q}, \frac{\partial T}{\partial r}, \text{ respectively}$$

where u, v, w, p, q, r are six components of the motion of the solid, ξ, η, ζ, λ, μ, ν are six components of the impulse, and T is the kinetic energy of the solid and liquid, has been established in many standard books on the subject, including that of Professor Lamb,[1] on the supposition that the components of motion satisfy the relations

$$\frac{\delta u}{u} = \frac{\delta v}{v} = \frac{\delta w}{w} = \frac{\delta p}{p} = \frac{\delta q}{q} = \frac{\delta r}{r}\,.$$

The object of the present note is to supply the following rigorous proof of the above theorem.

Since ξ, η, ζ, λ, μ, ν are linear in u, v, w, p, q, r the latter may be expressed as linear functions of the former (See Lamb, §123, equation 6). Hence, T may be

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[1] Horace Lamb, *Hydrodynamics*, (5th edition), p. 157.

regarded as a homogeneous quadratic function of ξ, η, ζ, λ, μ, ν. Denoting it by T' when thus expressed, we have by the principle of energy,

$$
\begin{aligned}
(1)\qquad & u\delta\xi + v\delta\eta + w\delta\zeta + p\delta\lambda + q\delta\mu + r\delta\nu = \delta T \\
& = \delta T' = \frac{\partial T'}{\partial\xi}\delta\xi + \frac{\partial T'}{\partial\eta}\delta\eta + \frac{\partial T'}{\partial\zeta}\delta\zeta + \frac{\partial T'}{\partial\lambda}\delta\lambda + \frac{\partial T'}{\partial\mu}\delta\mu + \frac{\partial T'}{\partial\nu}\delta\nu.
\end{aligned}
$$

Therefore

$$
(2)\qquad u, v, w, p, q, r = \frac{\partial T'}{\partial\xi}, \frac{\partial T'}{\partial\eta}, \frac{\partial T'}{\partial\zeta}, \frac{\partial T'}{\partial\lambda}, \frac{\partial T'}{\partial\mu}, \frac{\partial T'}{\partial\nu}.
$$

Also, since T' is a homogeneous quadratic function of ξ, η, ζ, λ, μ, ν,

$$
\begin{aligned}
2T = 2T' &= \xi\frac{\partial T'}{\partial\xi} + \eta\frac{\partial T'}{\partial\eta} + \zeta\frac{\partial T'}{\partial\zeta} + \lambda\frac{\partial T'}{\partial\lambda} + \mu\frac{\partial T'}{\partial\mu} + \nu\frac{\partial T'}{\partial\nu} \\
&= \xi u + \eta v + \zeta w + \lambda p + \mu q + \nu r, \ by\ (2).
\end{aligned}
$$

Performing an arbitrary variation δ and omitting terms which cancel by (1), we find

$$
\begin{aligned}
& \xi\delta u + \eta\delta v + \zeta\delta w + \lambda\delta p + \mu\delta q + \nu\delta r \\
& = \delta T = \frac{\partial T}{\partial u}\delta u + \frac{\partial T}{\partial v}\delta v + \frac{\partial T}{\partial w}\delta w + \frac{\partial T}{\partial p}\delta p + \frac{\partial T}{\partial q}\delta q + \frac{\partial T}{\partial r}\delta r,
\end{aligned}
$$

whence

$$
(3)\qquad \xi, \eta, \zeta, \lambda, \mu, \nu = \frac{\partial T}{\partial u}, \frac{\partial T}{\partial v}, \frac{\partial T}{\partial w}, \frac{\partial T}{\partial p}, \frac{\partial T}{\partial q}, \frac{\partial T}{\partial r}.
$$

II. Rationalizing Factors and the Method of Undetermined Coefficients

By L. J. Paradiso, Cornell University

In elementary algebra the question is sometimes raised by a student as to how a rationalizing factor can be obtained for an expression which contains a root higher than a square root. For example, to rationalize the denominator of $2/((2)^{1/3}+(3)^{1/2})$. In many cases a rationalizing factor is easily obtained by the use of the conjugate, as in the example $(a^{1/2}+b^{1/2})(a^{1/2}-b^{1/2})=a-b$ or by using a factor of an expression such as x^n+y^n. For example, a rationalizing factor of $2(3)^{1/3}-5(5)^{1/3}$ is $(2(3)^{1/3})^2+10(15)^{1/3}+(5(5)^{1/3})^2$; it is obtained by using $x=2(3)^{1/3}$, $y=5(5)^{1/3}$ and $x^3-y^3=(x-y)(x^2+xy+y^2)$. But in rationalizing expressions of the form

$$
a_0 + a_1(p)^{1/n} + a_2(p)^{2/n} + \cdots + a_{n-1}(p)^{(n-1)/n},
$$

where the a_i are rational coefficients, the process usually given is not readily remembered nor easy to teach without a background of the general theory.[1]

[1] See G. Chrystal, Algebra, Part 1, (1878), p. 197.

It is the object of this note to show that theoretically in all cases and practically in many cases a rationalizing factor may be found by the method of undetermined coefficients. Although this method is taught to freshmen in college in the case of partial fractions, it is not commonly employed to find rationalizing factors.

The theorem which we shall use to justify the method of undetermined coefficients is: *If ϕ is an algebraic number of degree m, that is, the root of a uniquely determined equation $f(x)=0$ of degree m with rational coefficients, and if $R(\phi)$ is a rational function of the algebraic number that is, $R(\phi)=g(\phi)/h(\phi)$, then*

$$R(\phi) = r_0 + r_1\phi + r_2\phi^2 + \cdots + r_{m-1}\phi^{m-1},$$

where the r_i are rational coefficients.

Suppose we wish to find a rationalizing factor for

$$a_0 + a_1\phi + \cdots + a_{m-1}\phi^{m-1},$$

where the a_i are rational numbers. The problem amounts to finding rational numbers A_i such that

$$(A_0 + A_1\phi + A_2\phi^2 + \cdots + A_{m-1}\phi^{m-1})(a_0 + a_1\phi + \cdots + a_{m-1}\phi^{m-1}) = 1.$$

Multiplying the factors in the left hand member of the above and arranging the terms, we can equate the constant term and the coefficients of $\phi, \phi^2, \cdots, \phi^{m-1}$ respectively to $1, 0, 0, \cdots$. In this way we obtain m simultaneous equations in the A_i from which we can find the required coefficients.

Example 1: To find the rationalizing factor of $2(2)^{1/3}-(4)^{1/3}-2$, we let $\phi=(2)^{1/3}$ so that $\phi^3=2$. We then have $(A_0+A_1\phi+A_2\phi^2)(2\phi-\phi^2-2)=1$. We group the terms

$$(4A_2 - 2A_1 - 2A_0) + (-2A_2 - 2A_1 + 2A_0)\phi + (2A_1 - 2A_2 - A_0)\phi^2 = 1,$$

from which we get

$$4A_2 - 2A_1 - 2A_0 = 1,\ 2A_0 - 2A_1 - 2A_0 = 0,\ 2A_1 - 2A_2 - A_0 = 0,$$

from which we find the rationalizing factor to be

$$-(2/5) - (3/10)\cdot 2^{1/3} - (1/10)\cdot 4^{1/3}.$$

By this same method we can of course rationalize numerators as well as denominators.

We can generalize this method to cases where more than one algebraic number is involved in the expression to be rationalized; say ϕ of degree m and θ of degree n. In this case in forming the rationalizing factor with undetermined coefficients one must not only put undetermined coefficients with powers of ϕ up to $m-1$ and powers of θ up to $n-1$, but also one must attach undetermined coefficients to all possible different cross products which can be formed from the various powers of ϕ and θ using the restriction,[1] of course, that $\phi^m=\phi$ and

[1] Loc. cit., p. 192, II.

$\theta^n = \theta$. This can be further generalized to any number of algebraic numbers, but in most cases where the index of the root is greater than 3 and the number of different algebraic numbers is more than three it becomes impractical to solve the simultaneous equations obtained.

Example 2, illustrating a more general case: Rationalize the denominator of $1/((a)^{1/2}+(b)^{1/3})$ where a and b are rational. Let $x=(a)^{1/2}$, $x^2=a$, and $y=(b)^{1/3}$, $y^3=b$. Then

$$1/(x + y) = A + Bx + Cy + Dy^2 + Exy + Fxy^2.$$

Expanding, arranging the terms, and equating the coefficients we have

$$aB + bD = 1,\ B + C = 0,\ D + E = 0,\ C + aF = 0,\ A + aE = 0,\ A + bF = 0$$

from which we get

$$1/(a^{1/2} + b^{1/3}) = \frac{ab + a^2\cdot a^{1/2} - a^2\cdot b^{1/3} + b\cdot b^{2/3} - b\cdot a^{1/2}\cdot b^{1/3} - a\cdot a^{1/2}\cdot b^{2/3}}{a^3 + b^2}.$$

III. A Note on the Differential Equations of Geodesics

By B. F. Kimball, Cornell University

The differential equations of geodesics on a surface in parametric form with the arc-length s taken as parameter are often given in three ways. Let u and v be the curvilinear coordinates of a point on the surface,

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

and let $H=(EG-F^2)^{1/2}$. We consider only a region on the surface for which $H>0$. Let primes denote differentiation with respect to s and subscripts differentiation with respect to variable indicated by the subscript. Taking

$$\begin{aligned} P &= Eu'' + Fv'' + \tfrac{1}{2}E_u u'^2 + E_v u'v' + (F_v - \tfrac{1}{2}G_u)v'^2, \\ Q &= Fu'' + Gv'' + (F_u - \tfrac{1}{2}E_v)u'^2 + G_u u'v' + \tfrac{1}{2}G_v v'^2, \end{aligned} \tag{1}$$

the three forms in which the differential equations of geodesics are often given are:

$$P = 0, \quad Q = 0, \tag{2}$$

$$\frac{d(Eu' + Fv')}{ds} - (\tfrac{1}{2}E_u u'^2 + F_u u'v' + \tfrac{1}{2}G_u v'^2) = 0 \tag{3a}$$

$$\frac{d(Fu' + Gv')}{ds} - (\tfrac{1}{2}E_v u'^2 + F_v u'v' + \tfrac{1}{2}G_v v'^2) = 0, \tag{3b}$$

and

$$(4a)\qquad u'' + \left\{\begin{matrix}1\ 1\\ 1\end{matrix}\right\} u'^2 + 2\left\{\begin{matrix}1\ 2\\ 1\end{matrix}\right\} u'v' + \left\{\begin{matrix}2\ 2\\ 1\end{matrix}\right\} v'^2 = 0,$$

$$(4b)^1\qquad v'' + \left\{\begin{matrix}1\ 1\\ 2\end{matrix}\right\} u'^2 + 2\left\{\begin{matrix}1\ 2\\ 2\end{matrix}\right\} u'v' + \left\{\begin{matrix}2\ 2\\ 2\end{matrix}\right\} v'^2 = 0.$$

We note before going further that the expression on the left hand side of equation (3a) is another form of the function P and that the expression on the left hand side of (3b) is Q. Also it is true that the expressions on the left hand side of equations (4a) and (4b) are respectively:

$$(5)\qquad (PG - QF)/H^2 \quad \text{and} \quad (QE - PF)/H^2.$$

Most writers on differential geometry speak of the fact that in finding the geodesics on a surface it is, under certain conditions, sufficient to solve only one equation from any one pair of equations (2), (3) or (4). For instance all solutions of $P=0$ for which v is not constant are geodesics, and such solutions give all the geodesics on the surface for which v is not constant. None of the writers, however, seems to bring together the following simple relations between the geodesic curvature and the pairs of equations (2), (3), and (4) which enable one to see at a glance what the nature of the interdependence of any two equations belonging to the same pair is. Let k_g denote the geodesic curvature. Under the usual definition one finds that[2]

$$(6)\qquad k_g = H^{-1}[(Eu' + Fv')Q - (Fu' + Gv')P].$$

Differentiating the identity

$$(7)\qquad Eu'^2 + 2Fu'v' + Gv'^2 \equiv 1$$

with respect to s it is not hard to show that

$$(8)\qquad Pu' + Qv' \equiv 0.$$

Then from (8) and (6) it is seen that

$$(9)\qquad P = -v'Hk_g, \qquad Q = u'Hk_g.$$

Substitute these values of P and Q in (5) and one finds that

$$(10)\qquad \begin{aligned} (PG - QF)/H^2 &= -k_gH^{-1}(Fu' + Gv'),\\ (QE - PF)/H^2 &= k_gH^{-1}(Eu' + Fv'). \end{aligned}$$

Relations (9) indicate the nature of the interdependence of the two equations of either of the pairs (2) or (3), and the relations (10) show to what extent solutions of one of the equations (4) can be trusted to give geodesics on the surface. Since $Fu' + Gv' = 0$ is the condition that a curve be orthogonal to the

[1] See L. P. Eisenhart, *Differential Geometry*, p. 153, for values of these symbols in terms of E, F, and G and their derivatives.

[2] See L. P. Eisenhart, *Differential Geometry*, p. 134.

$u=$const. curves and $Eu'+Fv'=0$ is the condition that it be orthogonal to the $v=$const. curves, we conclude that *any solution of* (4a) *which is not orthogonal to $u=$const. curves is a geodesic, and that any solution of* (4b) *which is not orthogonal to $v=$const. curves is a geodesic.* It is not improbable that the above results are well-known to differential geometers but one does not find the relations (9) and (10) introduced with the differential equations (2), (3), and (4) in the books now available on differential geometry. Bianchi[1] notes the relation (8), and Bolza[2] notes relations similar to (9) taking the calculus of variations point of view. Lilienthal,[3] in discussing geodesic curvature, derives formulae (9) and perhaps will apply them to the discussion of geodesics in the second part of Vol. II which does not seem to be out yet.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College, New York, N. Y.

All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Lennes, N. J. College Algebra. New York, Harper and Brothers, 1928. xiv+301 pages. $2.25.

Harding, A. M. and *Mullins, G. W.* Plane Trigonometry. New York, The Macmillan Company, 1928. viii+118 pages.

Harding, A. M. and *Mullins, G. W.* Analytic Geometry. viii+312 pages +14 pages of answers.

Ford, W. B. College Algebra, Revised Edition. New York, Henry Holt and Company, 1928. vi+278 pages.

Ford, W. B. First Course in the Differential and Integral Calculus. New York, Henry Holt and Company, 1928. viii+372 pages. $3.00.

Townsend, E. J., Functions of Real Variables. New York, Henry Holt and Company, 1928. x+406 pages.

Brasch, F. E., Editor. Sir Isaac Newton, 1727–1927. With an introduction by David Eugene Smith. Baltimore, The Williams and Wilkins Company, 1928. x+352 pages. $5.00.

"A series of essays giving a modern interpretation of the great English physicist through American eyes. Discusses Newton as physicist, mathematician, chemist, astronomer, philosopher, religious thinker, etc."

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[1] Bianchi, *Differentialgeometrie* (1899), p. 153.

[2] Bolza, *Concerning the Geodesic Curvature and the Isometric Problem on a Geometric Surface*, The Decennial Publications of the Univ. of Chicago, First series, vol. IX (1904), p. 13.

[3] Lilienthal, *Vorlesungen über Differentialgeometrie*, vol. II (1913), p. 224.

Turnbull, H. W. The Theory of Determinants, Matrices, and Invariants. London and Glasgow, Blackie and Son, Ltd., 1928. xvi+338 pages. 25 s.

"A general outline of invariant theory and related subjects. Both direct and symbolic methods are used. A special feature is the inclusion of recent researches on higher forms than the binary. Seven chapters are devoted to determinants and matrices."

Knopp, Konrad. Theory and Application of Infinite Series. Translated from the second German edition by Miss R. C. Young. London and Glasgow, Blackie and Son Ltd., 1928. xii+572 pages. 30 s.

The second German edition was reviewed in this Monthly vol. 32 (1925), p. 382, by Tomlinson Fort. In the English edition there has been added a chapter (62 pages) on Euler's summation formula and asymptotic expansions.

Schrödinger, Erwin. Four Lectures on Wave Mechanics. London and Glasgow, Blackie and Son, Ltd., 1928. viii+54 pages. 5 s.

Fisher, R. A. Statistical Methods for Research Workers. Second edition. London, Oliver and Boyd, 1928. xiv+270 pages, with 6 tables. 15 s.

One of a series of Biological Monographs and Manuals, and designed primarily for the use of workers in biological science.

Strachan, Robert C. A Table of Hyperbolic Radians, adapted to the derivation of hyperbolic functions from trigonometric tables. With explanatory text and notes on the catenary. Published by the author, 75 West St., New York N. Y. 1928. $2.00.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Plane Trigonometry. By Raymond W. Brink. The Century Company, 1928. xii+198 pages.

Trigonometry, Plane and Spherical. By David Raymond Curtiss and Elton James Moulton. D. C. Heath & Co., 1927. xi+264 pages.

Plane Trigonometry. By N. J. Lennes and A. S. Merrill, with the editorial cooperation of H. E. Slaught. Harper & Brothers, 1928. x+179 pages.

These recent texts in trigonometry help us to believe that better things are in store for the students as well as for the teachers of this subject. We may ask, in considering any trigonometry, the following questions: Should the subject "be regarded primarily as preparation for calculus rather than surveying"? Is the purpose of the course skill in computation, or a knowledge of trigonometric analysis, or both? What minimum amount will meet a student's actual needs? How much will serve his further mathematical training and development? Is the course in trigonometry to be given to students of high school maturity or to freshmen in college who may have had or are having some advanced work in algebra? An examination of these three texts leads one to feel that there is in

each a forward look to the calculus, and an aim to serve the student not merely in computation but in all further training in science and mathematics.

Professor Brink sets forth his own plan and purpose very definitely at the beginning of the preface of his text, "to keep clearly before the student the values which he may expect to receive in return for his efforts." The book begins consistently with a discussion of practical situations and gives a good variety of applied problems throughout. When the student has seen something of the usefulness of the trigonometric functions, has grown somewhat accustomed to the idea of the general angle, with its unrestricted size and double quality value, a more detailed analytical study is made of the functional relations. The final lists of exercises on completing the chapters furnish excellent problem material.

The general angle is used in defining the trigonometric functions; the notation arc sin $1/2$ is used in place of the notation $\sin^{-1}1/2$ which has tormented students so long with its seemingly meaningless exponential form; and the full discussion of logarithms, as well as the careful explanation of the solution of trigonometric equations, suggests the advantage of such a text for students with a minimum of algebra.

As has been said, the text aims, in true American fashion, to gain the interest of the students through its practical values, believing this interest will carry over into a more analytical and critical study of the background of the subject.

In the text of Curtiss and Moulton, the first chapter of over 30 pages gives one of the most complete presentations of the general angle to be found in our American text-books in trigonometry. The angle is referred to both rectangular and polar coordinates; the six trigonometric functions are defined in terms of ordinate, abscissa, and distance; and the chapter is concluded with a study of some special angles including not only 30°, 45°, 60°, but also those angles whose functions are numerically the same. There is no hint of suggested values in return for efforts expended but perhaps these would come with a broader viewpoint and stimulated mental energy. A student who had mastered the substance of this chapter would begin to "see what it is all about" and would already be far along in trigonometry.

The solution of triangles both right and oblique follows, and the more difficult work, that of reduction of angles and the relations between changes in and processes on the angle and changes in and processes with the functions. Fundamental identities, radian measure, inverse functions and trigonometric equations complete the course. The question arises whether radian measure and inverse functions would not stand a better chance in later mathematical work if started earlier in trigonometry and used consistently in it. A chapter on logarithms is included although the author suggests the possible omission of computation by logarithms as "in accord with the growing tendency to calculate with slide rules, machines, and multiplication tables."

The trigonometry by Lennes and Merrill is an attractive book to read and to

teach. Starting with the simple definition of trigonometric functions of an acute angle in terms of the sides opposite and adjacent to the angle and the hypotenuse, in a right triangle which may be formed to include the angle, the work is consistently developed to include the general angle with its theory. Certain features of the book should be mentioned: the paging that leaves diagram and proof together; several groups of problems covering the same work, which may be given to different sections of the same class; cumulative reviews at the end of the course; a full chapter on logarithms; and a chapter on the history of trigonometry, suggesting that student and teacher may be helped by seeing the part the subject of trigonometry has played in mathematics and science in the past. Radian measure, variation of functions, inverse functions, and solution of trigonometric equations are relegated to the last chapters. Why not introduce them all earlier and use them as occasion suggests "ever after"? And why a "five-part" theorem in reduction of angles instead of a "one-part"? Our students of trigonometry, and even more our students of calculus, are crying out for a good, single, general theorem in reduction of angles.

HARRIET E. GLAZIER

Theory of Probability. By the late William Burnside with a preface and memoir on Burnside by A. R. Forsyth. Cambridge University Press, 1928. xxx +106 pages.

This posthumous work by the distinguished English mathematician William Burnside "is the expression of his views so far as they had been framed into a system," to quote from Forsyth's memoir which precedes Chapter 1. In the preface Forsyth states that Burnside's first paper on the subject was one of 1918, and that the manuscript now published "was written at intervals before the middle of 1925"; in other words, when Burnside was at the ages of sixty-six and seventy-three years, respectively. Long before this his work had placed him among outstanding mathematicians. Therefore we may anticipate that the work reflects the advantages of approaching a difficult subject with mature judgment and a mastery of analysis. On the other hand, the subject could not have received from him the same sustained thought which he gave to the theory of groups and allied subjects.

The outstanding feature of Chapter 1, the Introduction, is the rule given by Burnside for the calculation of probabilities. It ends with the phrase "provided that *each two* of the n results are *assumed* to be equally likely." Burnside contrasts this rule with one given by Poincaré, embodying the phrase "à condition que tous les cas soient également vraisemblables." The distinction is discussed in a note given on page 101 in which two issues are raised:

1. The relative significance of the two phrases "provided that all cases are equally likely" and "provided that all cases are *assumed* to be equally likely."

2. Assuming that *each two* cases are equally likely as against assuming that *all* cases are equally likely.

Regarding the first issue raised, it may be questioned whether Burnside has

interpreted the phrase "à condition que tous les cas soient également vraisemblables" in the sense that Poincaré had in mind when he said "comment reconnaître que tous les cas son également probables? . . . . nous devrons, dans chaque application, faire des *conventions*, dire que nous considérerons tel et tel cas comme également probables," on page 28 of the *Calcul des Probabilités*, second edition.

Burnside gives two groups of examples illustrating the application of his rule and the formulae of the Introduction: those solved by direct methods being placed in Chapter II, while such classic problems as the "duration of play," involving finite difference equations and generating functions, appear in Chapter III. Bridge players should be warned that Burnside's solution of example II, page 16, overlooks the question of mutually exclusive cases. A hand containing a single heart might also contain but a single diamond; therefore, the number of favorable cases is less than four times (13) $(39!)\div(27!)\,(12!)$. Example X, page 22, deals with the distribution of points placed at random on a line; it is typical of an important class of problems which are given further consideration in Chapter VI, entitled "Probabilities connected with geometrical questions."

The major subject of Chapter IV is "Methods of approximation"; but also, page 45, "It is convenient here to introduce two conceptions which prove to be of great value in many appplications of the theory of probability; viz. those of the 'probable value,' and the 'most probable value.'"

In Chapter V, "Probability of causes," the Bayes' formula is developed and illustrated by several examples. Burnside brings out clearly the extent to which the solution of an *a posteriori* problem depends on the information existing with reference to the *a priori* probabilities in favor of the possible causes. When such information is lacking the solution of a problem of this type is possible only on the basis of some assumptions. Different sets of assumptions usually lead to different final results so that the weight assignable to each set of assumptions is a question of major importance when the theory is applied to practical problems.

The book closes with two chapters dealing with errors of observations. Inconsistencies within these chapters and also in the cognate papers published by Burnside in the Cambridge Philosophical Society Proceedings call for careful consideration, particularly on account of their bearing on the question of "small samples."

In Article 27, where a set of observations made on an unknown quantity is dealt with, the *a priori* probability function for the unknown true value of the measured quantity is introduced. Therefore, in Article 36, where not only the observed quantity but the precision constant h in the Gaussian law is unknown, one would expect to find an *a priori* function involving both of these unknowns; but no *a priori* function whatever appears and the analysis proceeds with the totally different problem of observations not yet made on a known quantity with a known precision constant. Burnside arrives at a result that is identical

with the formula published by Student in 1908, a formula which is not the answer to the *a posteriori* problem *unless arbitrary assumptions are made* such as:

1. *A priori* all values of the unknown observed quantity are equally likely.

2. *A priori* the probability that the precision constant has a value in the neighborhood of h is inversely proportional to h.

The *a posteriori* problem here considered constituted the subject matter of two papers by Burnside entitled, "On errors of observation" and a "Note on Dr. Burnside's recent paper on errors of observation" by R. A. Fisher (listed under "Cognate Papers" on page 104 of the book). In his second paper Burnside retracts the conclusions set down in his first paper; in his note R. A. Fisher is, in the opinion of the reviewer, far from giving a legitimate treatment of the problem. However, we are indebted to Burnside for a short rigorous derivation of the Student formula.

EDWARD C. MOLINA

Vorlesungen über Differential- und Integralrechnung. Bd. I. Funktionen einer Veränderlichen. By R. Courant. Berlin, Julius Springer, 1927. xiv+410 pages.

The standpoint and aim of this excellent book are clearly stated in the following freely translated extract from the author's preface.

"Mathematical literature is certainly not poor in good works on the calculus. Nevertheless it is not easy for the beginner to find a book that affords a straight path into the living body of the science and gives an intelligent freedom of motion among its applications. The beginner should not be wearied by diffuseness or lack of content, nor can he support a pedantry which does not recognize the difference between essentials and non-essentials and which—as often in systematized axiomatic treatments—throws a misty veil over the actual driving forces of our science and over its objective kernel.

"It is indeed easier to see and feel defects than to remedy them. I am far from the idea that I can supply the ideal text for the beginner. But I do not believe that the publication of my lectures is superfluous. They differ from current texts in the order and choice of material, in their tendency, and perhaps also in their expository style.

"Most striking, perhaps, is the break with the surviving tradition of separating the differential and integral calculus. This separation, unwarranted by either material or didactic considerations and merely a result of historic accidents, hinders a clear enforcement of the essential point—the connection between definite integral, indefinite integral, and derivative.

"I have tried to give the reader a clear view of the close connection of analysis to its applications and, with all mathematical rigor and precision, to yield to intuition its full prerogative as the original source of mathematical truths. Indeed the exposition of science as a closed system of static truths, without reference to their origin and goal, has an esthetic charm and fulfils a deep philosophical need. But as an exclusive basis for development or as a

opens with the definite integral, treated first as an area, then analytically as the limit of a sum with computed examples. Then comes the derivative, introduced by the problem of tangency before the analytic definition is given. The derivative as a velocity and general rules for differentiation follow. Next the integral is considered as a function of its upper limit and its derivative computed. Thence the central relationship of the calculus. The chapter also contains the mean value theorems for derivatives and integrals, simple methods of graphical integration, and excellent general remarks on the connection between the mathematical integral and derivative and their counterparts in the natural sciences. The supplement deals with the existence of an integral of a continuous function and the connection between theorems of mean value. Chapters III and IV now develop systematic methods for computing derivatives and integrals and Chapter V is concerned with applications to geometery and mechanics.

The following chapters turn to more specific problems: Taylor's formula and polynomial approximations for functions (VI), numerical computation, including Stirling's formula (VII), infinite series and other limiting processes (VIII), Fourier's series and trigonometric interpolation (IX), and finally the differential equations of the simplest oscillatory phenomena (X).

The treatment is uniformly elegant and clear, the exposition growing more concise toward the end of the book. The most interesting chapter, perhaps, is the one on Fourier's series. The development of functions $f(x)$, such that $f(x)$, $f'(x)$, $f''(x)$ are continuous in a finite number of subintervals of $(-\pi, \pi)$ and which approach definite limits as the boundary points are approached from either side, is treated in detail. It is hard to see how the treatment could be improved.

LOUIS BRAND

Fourfold Geometry. By D. B. Mair. D. Van Nostrand Co. viii+183 pp.

This book deals with the geometry of a four-dimensional Euclidean space and the equivalent special theory of relativity. Vector methods and notation are used systematically. The terminology employed seems excessively bizarre in some instances; such terms as "biplane" and "backhand stroke" with their respective aviation and tennis connotations are rather distracting to the mathematical reader. On the whole, though, the exposition is quite clear. The printing and page arrangement are agreeable to the eye.

JESSE DOUGLAS

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3361. *Proposed by B. F. Finkel, Drury College.*

Find the envelope of a system of circles having for diameter a secant of constant length, $2r$, of a conic.

3362. *Proposed by R. E. Gaines, University of Richmond.*

A triangle ABC circumscribes an ellipse of axes $2a$ and $2b$ so that A lies on one directrix and B on the other, while AB, BC, CA touch the ellipse at P, Q, R. Show that the envelope of QR and the locus of C are ellipses with $2a$ as the major axis of one and the minor axis of the other, while $2b$ is the mean proportional between the other axes.

3363. *Proposed by Otto Dunkel, Washington University.*

In an urn there are $k+1$ counters of which one is a blank while the others are numbered from 1 to k. A single counter is withdrawn and then replaced, and this is continued until there are n such drawings. In how many ways may the n drawings be made so that in the first r or more drawings only blanks are obtained and in the rest of the n drawings no numbered counter is followed by as many as r consecutive blanks? Also determine in how many ways n drawings may be made so that r or more consecutive blanks are drawn.

3364. *Proposed by Otto Dunkel, Washington University.*

In an urn there are $k+1$ counters of which one is a blank while the others are numbered from 1 to k. A single counter is withdrawn and then replaced and this is continued until there are n drawings. In how many ways may the n drawings be made so that in the first r drawings only blanks are obtained and the $(r+1)$th drawing is a number, while in the rest of the n drawings no numbered counter is followed by precisely r blanks. Also determine the number of ways the n drawings may be made so that precisely r blanks are drawn in succession.

SOLUTIONS

3162 [3158; 1926, 47]. *Proposed by Otto Dunkel, Washington University.*

Show that the absolute values of the imaginary roots (also of the negative roots when n is even) of the equation

$$x^{n+1} - (k+1)x^n + k = 0, \qquad k > 0, \quad n \geqq 2,$$

lie between $[k/(k+2)]^{1/n}$ and unity; also that the absolute values of the imaginary roots are greater than that of the negative root of (a) the given equation when n is even, of (b) the equation obtained by replacing n by $n-1$ when n is odd.

If $k \geqq 1$ the absolute values of the imaginary roots (and negative roots) lie between .61 and 1.

Solution by the Proposer.

The real roots of the given equation will first be examined, and it will be convenient to write

$$(1) \qquad \begin{aligned} f(x) &= x^{n+1} - (k+1)x^n + k, \qquad k > 0, \quad n > 1, \\ f'(x) &= x^{n-1}[(n+1)x - n(k+1)]. \end{aligned}$$

If n is odd there can be no negative roots. If n is even, $f'(x)$ is positive for $x<0$, $f(-1)=-2$, and $f(0)=k>0$; and therefore there is one and only one negative root and it lies between -1 and 0.

In all cases $x=1$ is a root, and it will be shown that there is one and only one other positive root, which will be denoted by a. Let $\bar{x}=n(k+1)/(n+1)$ so that $f'(\bar{x})=0$; then $\bar{x}<1$, $\bar{x}=1$, or $\bar{x}>1$ according as $k<1/n$, $k=1/n$, or $k>1/n$. If $k<1/n$, then as x takes on the values 0, $\bar{x}$, 1, $f(x)$ takes on values k, $-$, 0, and then increases. Hence a and 1 are the only positive roots and $0<a<\bar{x}<1$. If $k=1/n$, then $x=1$ is a double root and there is no other positive root. If $k>1/n$, we find in a similar manner that 1 and a are the only positive roots, and that $1<\bar{x}<a<1+k$.

For the consideration of the imaginary roots it will be convenient to set $x=1/y$, and to examine the equation

$$(2) \qquad ky^{n+1} - (k+1)y + 1 = 0.$$

This equation has the real positive roots 1 and $A=a^{-1}$, and thus $kA^{n+1}-(k+1)A+1=0$. Suppose that y is an imaginary root. Then by subtracting the two equations in y and A and dividing by $y-A$, which is surely not zero, we have

$$(3) \qquad (k+1)/k = y^n + Ay^{n-1} + \cdots + A^{n-1}y + A^n.$$

We shall prove that $|y|>1$, and then it follows that $|x|<1$, where x is an imaginary root of $f(x)=0$. It will be noticed that the proof applies also if x is a negative root. For suppose that $|y| \leqq 1$ and consider first the case $kn \neq 1$. Then from the above we have

$$(4) \qquad \begin{aligned} (k+1)/k &< |y|^n + A|y|^{n-1} + \cdots + A^{n-1}|y| + A^n, \\ &< 1 + A + \cdots + A^{n-1} + A^n. \end{aligned}$$

The equality sign cannot be used in the first inequality since the different powers of y have different angles. Now (3) above is satisfied by $y=1$. Hence the right side of the last inequality in (4) is equal to $(k+1)/k$, and we have the contradiction $(k+1)/k<(k+1)/k$. If $kn=1$, then $A=1$ and the last inequality becomes the contradiction $n+1<n+1$. Hence $|x|<1$, if x is an imaginary or negative root of $f(x)=0$.

We shall now determine lower bounds for the imaginary and negative roots, and for this purpose it will be convenient to set $z=-x$ and to write

$$z^{n+1}+(k+1)z^n=(-1)^n k. \tag{5}$$

Suppose first that z is an imaginary root, then $k<|z|^n[|z|+(k+1)]$, and hence

$$|z|^n>\frac{k}{[|z|+k+1]}>\frac{k}{(k+2)}, \tag{6'}$$

where the last inequality follows since $|z|<1$. Hence we have

$$1>|x|>(k/k+2)^{1/n}. \tag{6}$$

If x is a negative root, z is positive, and the reasoning is altered merely by replacing the first inequality sign in (6′) by the equality sign, the second and last inequality signs remaining since $|z|<1$, if z corresponds to a negative root. Hence (6) is true also for the negative root if there is one.

If n is even there is a negative root, say $-b$, so that $z=b$ is a root of (5). Hence $b^n[b+k+1]=k$. Suppose then that $x=-z$ is an imaginary root in this case, and that $|x|=|z|\leqq b$. Then

$$k<|z|^n[|z|+k+1]\leqq b^n[b+k+1]=k.$$

Here we have the contradiction $k<k$, and hence $|x|>b$. If n is odd consider the equation $z^n+(k+1)z^{n-1}=k$ and its root b' where $-b'$ is the negative root of the x equation. Also let $x=-z$ be an imaginary root of (5). Suppose that $|z|\leqq b'$; then as before

$$k<|z|^n[|z|+k+1]\leqq b'[b'^n+(k+1)b'^{n-1}] \quad \text{or} \quad k<b'k.$$

The last inequality follows since the bracket with b' is equal to k. But the last inequality is impossible since $b'<1$. Hence the absolute value of any imaginary root of $f(x)=0$ is greater than the absolute value of its negative root if n is even. If n is odd it is greater than the absolute value of the negative root of the corresponding equation in which n is replaced by $n-1$.

It will now be shown that if k is fixed and n increases passing through even values, then b increases. Let b and $\bar{b}$ be the roots corresponding to n and $n+2$. Then $\bar{b}^{n+3}+(k+1)\bar{b}^{n+2}=k$ or $\bar{b}^{n+1}+(k+1)\bar{b}^n=k/\bar{b}^2>k$, since $\bar{b}<1$. But $b^{n+1}+(k+1)b^n=k$, and therefore we must have $\bar{b}>b$. If now n is fixed and even and b increases then k increases. For $k=b^n(b+1)/(1-b^n)$, and as b increases the numerator increases and the denominator decreases. Conversely as k in-

creases b increases. Therefore the absolute values of the imaginary roots are greater than the positive root of $z^3+(k+1)z^2-k=0$, or of $z^2+kz-k=0$. Hence if x is an imaginary root,

$$|x| > \frac{1}{2}[(4k+k^2)^{1/2}-k] > \frac{k}{k+1}\cdot$$

If $k\geqq 1$ the value of the middle term in the above inequalities is greater than 0.6180, and in this case $|x|$ lies between 0.61 and 1.

In the case $k<1/n$ we have $0<a<1$, and in order to complete the discussion it will be shown by the same kind of argument that the absolute values of the imaginary roots are less than a. Using the equation (2) with the root $y=1$ removed we may write the equation

$$(k+1)/k = y^n + y^{n-1} + \cdots + y + 1,$$

which is satisfied by $A=1/a$ and the imaginary roots. Suppose that y is an imaginary root and that $|y|\leqq A$; then

$$\begin{aligned}(k+1)/k &< |y|^n + |y|^{n-1} + \cdots + |y| + 1,\\ &< A^n + A^{n-1} + \cdots + A + 1 = (k+1)/k.\end{aligned}$$

Here we have again the absurdity $(k+1)/k<(k+1)/k$, and therefore $|y|>A$. The same reasoning applies to the negative root. Thus if x is an imaginary root of $f(x)=0$, or a negative root, and if $k<1/n$, $|x|<a$.

It may also be shown by arithmetical reasoning that if n is fixed a increases with k; if k is fixed a increases with n. The equation $f(x)=0$ for $n=4$ has been solved as an illustrative example in this Monthly [1926, 232].

3288 [1927, 491]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

The variable line on which two given circles determine chords of equal length envelops a parabola. The midpoint of the line of centers of the circles is the focus of the parabola, and its tangent at the vertex coincides with the radical axis of the two given circles.

I. *Solution by C. S. Carlson, University of Iowa*

Let us choose the line of centers of the two given circles as the axis of X and the perpendicular bisector of the segment joining the centers as the axis of Y. Then the equations of the two given circles can be written in the form

$$x^2+y^2-2gx+f = 0\cdots, \tag{1}$$

$$x^2+y^2+2gx+f' = 0\cdots. \tag{2}$$

Solving (1) with the line $y=mx+c$ we have the relation

$$x^2(1+m^2)+x(2mc-2g)+f+c^2=0$$

from which the roots x_1 and x_2 give the relations

$$(3) \qquad x_1 + x_2 = -(2mc - 2g)/(1 + m^2)$$

$$(4) \qquad x_1x_2 = (f + c^2)/(1 + m^2).$$

Squaring (3) and subtracting four times (4) we find

$$(5) \quad x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2 = 4(-m^2f - 2mcg + g^2 - f - c^2)(1 + m^2)^{-2}.$$

Now (5) gives the square of the projection of the chord on the axis of X. Using the second circle, a similar procedure gives the square of the projection of the second chord on the axis of X: $4(-m^2f'+2mcg+g^2-f'-c^2)(1+m^2)^{-2}$. Since the chords are equal, by hypothesis, and are projected from the same line upon the axis of X, we have

$$-m^2f - 2mcg + g^2 - f - c^2 = -m^2f' + 2mcg + g^2 - f' - c^2;$$

whence $m^2(f'-f)-4mgc+f'-f=0$. Solving this relation for c and substituting in $y=mx+c$, the equation of the variable line now becomes $y=m(x-a)-am^{-1}$ when $a=(f-f')/4g$. From analytic geometry (See C. Smith, *Conic Sections*, paragraph 94, page 112) we know that this line is always tangent to the parabola $y^2=-4a(x-a)$. From this equation it is now evident that the radical axis of the two circles is the tangent at the vertex and that the focus is the mid-point of the line of centers.

The first part of this problem is found in C. Smith's *Conic Sections* (1910 edition), page 134.

Also solved by Theodore Bennett, Rufus Crane, A. Pelletier, and Paul Wernicke.

II. *Solution by Otto Dunkel, Washington University.*

The envelope may be determined geometrically. Let C_1, C_2 be the centers of the two circles, and F the mid-point of C_1C_2. Let a straight line determine the chords of the two circles A_1B_1, A_2B_2 which are of the equal lengths $2c$. Let the projections of C_1, F, C_2 on this secant be M_1, N, M_2; then NM_1 and NM_2 are of equal lengths m. Hence for either circle $NA \cdot NB=(m-c)(m+c)$, and thus the tangents to the circles from N are of equal length, or, if N lies within a circle, the product of the segments of the chords through N is the same for both circles. Hence N lies on the radical axis of the two circles, which cuts C_1C_2 perpendicularly in V, and also FN is parallel to C_1M_1 and C_2M_2 and therefore perpendicular to M_1M_2. This determines the construction for the secant M_1M_2. Obviously the points may be lettered so that NA_1 and NA_2 are equal in length, and the same thing is then true of NB_1 and NB_2.

Consider two such secants NM_1 and $N'M_1'$ intersecting in P', where FNM_1 and $FN'M_1'$ are right angles. A circle may be passed through F, N, N', P' with center at the mid-point of FP'. As N' approaches N, the point P' approaches a limiting position P on NM_1 and the circle approaches a circle through F and P tangent to VN at N. To construct P we draw the fixed auxiliary line DD' perpendicular to VF at D such that V is the mid-point of

DF. Draw any line FNF' cutting VN and DD' in N and F' respectively. Through F' draw the line $F'P$ perpendicular to DD' at F', and NP perpendicular to FF' at N meeting the first perpendicular in P. Draw FP cutting in C the perpendicular NC to VN at N. Then FC, CP, CN are of equal lengths, and C is the center of the limit circle and P is the limit point on NM_1, and this line is the tangent to the envelope at P. Since $FN=NF'$, $FP=F'P$; hence the point P lies on a parabola with focus at F, tangent to VN at V, and having DD' as directrix.

3292 [1927, 491]. *Proposed by R. E. Gaines, University of Richmond.*

A chord of a given conic is of constant length; find the locus of its pole.

Solution by the Proposer.

The equation of the conic may be written

$$(1) \qquad ax^2 + by^2 + 2gx = 0, \qquad bg \neq 0\,;$$

and, if (α, β) is the pole of the chord, the equation of the chord is

$$(2) \qquad (a\alpha + g)x + b\beta y + g\alpha = 0.$$

If the length of the chord is $2l$ and if its extremities have the coördinates (x', y'), (x'', y''), then

$$(3) \qquad (x'' - x')^2 + (y'' - y')^2 = 4l^2.$$

Since the coördinates of the extremities satisfy (2) we may write (3) in the form

$$(4) \qquad [(x'' + x')^2 - 4x'x''][1 + (a\alpha + g)^2 b^{-2}\beta^{-2}] = 4l^2.$$

Eliminating y from (1) and (2) we have

$$(5) \qquad [ab\beta^2 + (a\alpha + g)^2]x^2 + 2g[a\alpha^2 + b\beta^2 + g\alpha]x + g^2\alpha^2 = 0$$

as the equation for x' and x''. Inserting the sum and the product of the roots of this equation in (4), and replacing (α, β) by (x, y), we obtain the equation of the locus required:

$$g^2[ax^2 + by^2 + 2gx][(ax + g)^2 + b^2y^2] = bl^2[(ax + g)^2 + aby^2]^2.$$

If $a\neq 0$, the conic has the center $(-g/a, 0)$, and the locus possesses this point as an isolated double point.

Also solved by William Hoover, W. J. Patterson, Paul Wernicke and Roscoe Woods.

3299 [1927, 538]. *Proposed by H. W. Reddick, Cooper Institute of Technology, New York City.*

Solve the differential equation $(d^{n+1}y/dx^{n+1}) - (dy/dx) - (ny/x) = 0$, (a generalization of Problem 3227).

Solution by R. H. Sciobereti, Berkeley, California.

In the form

$$xy^{(n+1)} - xy' - ny = 0, \tag{1}$$

the proposed differential equation is a particular case of Laplace's equation and its solution may be expressed by means of a definite integral. Assuming $y = \int_C Ze^{zx}dz$ taken along a contour which will be defined later, where Z is an unknown function of z, let us differentiate the required number of times with respect to the parameter x and substitute in the original equation; then we shall obtain

$$\int_C Ze^{zx}(P + Qx)dz = 0, \quad \text{where} \quad P = -n \quad \text{and} \quad Q = z(z^n - 1).$$

If ZQ be determined by the condition $d(ZQ) = ZPdz$, or $Z = Q^{-1}e^{\nu}$, where $\nu = \int PQ^{-1}dz$, the definite integral becomes equal to the variation of the function

$$U = ZQe^{zx} = e^{zx+\nu} = e^{zx}z^n/(z^n - 1)$$

along the path of integration C. In order to obtain a solution of (1) it will be sufficient to choose the path of integration in such a way that the function U should return to its initial value after the variable z describes the whole contour, provided however that $y = \int_C Ze^{zx}dz$ has a finite value different from 0. The two functions U and Z have the same roots and poles, namely the roots of the polynomial $Q = z(z^n - 1)$. Let us take for the path of integration C a circle whose center will be successively each of the poles $e^{2k\pi i/n} = \alpha_k$ of the function Z, then the residue of $e^{zx}z^{n-1}(z^n - 1)^{-2}$ relative to each one of these poles will be a particular solution of the differential equation. The residue corresponding to the pole α_k is obviously the coefficient of $(z - \alpha_k)$ in the development of

$$f(z) = e^{zx}z^{n-1}(z - \alpha_k)^2(z^n - 1)^{-2}$$

about α_k. It is therefore $[df(z)/dz]_{z=\alpha_k}$; from Cauchy's fundamental formula $2\pi i f'(x) = \int_C f(z)dx/(z - x)^2$, the residues may be readily calculated as follows: denoting by α_k any one of the roots of the equation $z^n - 1 = 0$, and by $\phi(z)$ the product $\prod_i (z - \alpha_i)$ $(i = 0, 1, 2, \cdots, n-1$, but $i \neq k)$ which is equal to the function $(z^n - 1)/(z - \alpha_k)$, we shall have

$$2\pi i \frac{d}{dz}\left[\frac{e^{zx}z^{n-1}}{\phi^2(z)}\right]_{z=\alpha_k} = \int_C \frac{e^{zx}z^{n-1}/\phi^2(z)}{(z - \alpha_k)^2}dz,$$

and after performing all the indicated algebraic operations, the residue nxe^{μ}, where $\mu = \alpha_k x$, relative to the pole α_k is a particular solution of the proposed equation; and since the n particular solutions corresponding to the n roots of unity are linearly independent, it follows that $\sum_k C_k xe^{\mu}$ is also a solution. The general solution may now be obtained by $(n+1)$ quadratures, since n substitutions of the form $y = xe^{\mu}v$, $u = dv/dx$ will reduce the original equation to an

equation of the first order which is integrable by a quadrature. But another particular solution may be obtained by selecting a new and suitable contour of integration. In general, if x is a complex quantity of the form $re^{i\alpha}$ and $z=\rho e^{i\theta}$, the path of integration may be taken as a straight line starting from the origin, not passing through any pole $e^{2k\pi i/n}$ and making with the real axis an angle such that the real part of the product $xz=r\rho e^{i(\theta+\alpha)}$ should be negative, since the function $U=e^{zx}z^n/(z^n-1)$ is to vanish at both extremities. In particular if x is real the definite integral $\int e^{xz}z^{n-1}(z^n-1)^{-2}dz$ will be a solution, when taken along the straight line $\phi=(2k+1)\pi/n$, provided $n/2<2k+1<3n/2$, for $x>0$; and $-n/2<(2k+1)<n/2$, for $x<0$. Hence there is a particular solution

$$C_{n+1}\int_0^\infty e^\sigma \rho^{n-1}(\rho^n+1)^{-2}d\rho, \quad \text{where} \quad \sigma = x\rho e^{(2k+1)\pi i/n}.$$

It takes the form

$$C_{n+1}\int_0^\infty e^{-x\rho}\rho^{n-1}(\rho^n+1)^{-2}d\rho\,,$$

for $x>0$ and n odd.

3300 [1927, 538]. *Proposed by Emma M. Gibson, Central High School, Springfield, Missouri.*

If u and v are particular integrals of the equation

$$(d^3y/dx^3)+P(d^2y/dx^2)+Q(dy/dx)+Ry=0,$$

prove that $y=Au+Bv$ is the complete solution, where

$$A'/v = -B'/u = ce^\mu/(u'v-uv')^2 \qquad \left(\mu = -\int Pdx\right),$$

c being a constant and the accents denoting differentiation with respect to x. (University of Edinburgh Examinations, 1903.)

Solution by Frederic H. Miller, Cooper Union Institute of Technology

Since u and v are two solutions, $y=au+bv$ is a solution, where a and b are constants. Consequently we assume the complete solution in the form $y=Au+Bv$, where now A and B are functions of x. Then $y'=A'u+Au'+B'v+Bv'$, and if we make the parameters A and B satisfy the additional relation

$$A'u+B'v=0, \tag{1}$$

we have

$$y' = Au'+Bv',$$
$$y'' = A'u'+Au''+B'v'+Bv'',$$
$$y''' = A''u'+2A'u''+Au'''+B''v'+2B'v''+Bv'''.$$

Substitution of these values into the differential equation gives us

$$(2) \qquad A''u' + 2A'u'' + B''v' + 2B'v'' + P(A'u' + B'v') = 0.$$

Now from equation (1),

$$B' = -A'u/v, \quad B'' = -(A''uv + A'u'v - A'uv')/v^2;$$

and these values when put into (2) yield

$$A''v(u'v - uv') + A'[2v(u''v - uv'') + (Pv - v')(u'v - uv')] = 0.$$

This is an equation of the first order in A'; its solution is

$$A'/v = ce^{\mu}/(u'v - uv')^2, \quad \left(\mu = -\int P dx\right).$$

Combining this with equation (1) we get the desired relation

$$A'/v = -B'/u = ce^{\mu}/(u'v - uv')^2,$$

and hence the complete solution of the differential equation is

$$y = c\left[u\int\frac{ve^{\mu}}{(u'v - uv')^2}dx - v\int\frac{ue^{\mu}}{(u'v - uv')^2}dx\right].$$

3302 [1928, 41]. *Proposed by R. E. Gaines, University of Richmond.*

Determine the envelope of the chord of a central conic which subtends a right angle at a given focus.

Solution and Generalization by R. Goormaghtigh, La Louvière, Belgium.

Let A and B be two moving points on two plane curves (A) and (B), the angle subtended by AB at a given point O being a constant α, and denote by E_α the envelope of AB. If C is a fixed circle with center O, the angle between the polars of A and B, for the circle C, is $\pi - \alpha$. Therefore, E_α is the polar curve, with respect to C, of the isoptic curve Γ, for the angle $\pi - \alpha$, of the polar curves (A') and (B') of (A) and (B) with respect to C.

When (A) and (B) are two conics having a common focus O and the same corresponding directrix, (A'), (B') are two concentric circles, and E_α is also a conic having the same focus and corresponding directrix as (A) and (B). In Problem 3302, $(A) \equiv (B)$; in this case, the radius of Γ being the radius of (A') multiplied by $\sqrt{2}$, the eccentricity of $E_{\pi/2}$ is the eccentricity of (A) divided by $\sqrt{2}$. When (A) and (B) are the same central conic Σ, and O is any fixed point, the polar curve of Σ is a conic and Γ is a spiral curve of Perseus when $\alpha \neq \pi/2$, and a circle when $\alpha = \pi/2$. Hence the envelope is the polar curve of the spiral curve of Perseus when $\alpha \neq \pi/2$, and a circle when $\alpha = \pi/2$.

When O is a point on the conic Σ and when $\alpha \neq \pi/2$, the polar curve of Σ is a parabola, Γ is a conic, and E_α is also a conic; in this case, if $\alpha = \pi/2$, Γ is a straight line and AB passes through a fixed point (Fregier's theorem). When $\alpha = \pi/2$ and when O is the center of Σ, Γ is a circle, and $E_{\pi/2}$ is also a circle; if K is the radius of C and $2a$, $2b$ the axes of Σ, the axes of the polar curve are

$2K^2/a$, $2K^2/b$ and Γ is a circle with center O and radius $K^2(a^{-2}+b^{-2})^{1/2}$; therefore the radius of $E_{\pi/2}$ is $ab/(a^2+b^2)^{1/2}$. See Problem 3237 [1927, 98].

Also solved by William Hoover, J. H. Neelley, O. J. Ramler, Roscoe Woods, F. L. Wilmer, and the Proposer.

3305 [1928, 92]. *Proposed by Paul Wernicke, Washington, D. C.*

(a) Prove that x^2+y^2, where x and y are integers, cannot be the square of an integer unless x or y is divisible by 3.

(b) Modify the theorem for the case that both x and y are divisible by 3.

(c) Generalize the theorem so as to cover other exponents, thereby proving a part of Fermat's greater theorem.

I. *Solution by R. S. Underwood, Texas Technological College.*

The generalized results of which (a) is a special case [$n=2$, $p=3$ in case (1) below] may be stated thus:

If $x^n+y^n=z^n$ has integral solutions (n an integer),

(1) when $n=(p-1)k (p\geqq 3)$, x or y must be divisible by p (a prime),

(2) when $n=\frac{1}{2}(p-1)k (p\geqq 5)$, x, y, or z must be divisible by p.

Proof of (1): Let $n=(p-1)m$, and let neither x nor y be divisible by p. Then, by a well-known theorem, due to Fermat,

$$x^n \equiv y^n \equiv 1 \pmod{p}.$$

When z is not divisible by p,

$$z^n \equiv 1 \pmod{p}.$$

When z is divisible by p,

$$z^n \equiv 0 \pmod{p}.$$

In either case, $x^n+y^n\equiv 2 \pmod{p} \not\equiv z^n \pmod{p\geqq 3}$,

and the theorem is proved.

Proof of (2): Let $n=\frac{1}{2}(p-1)k$, and let x, y, and z be prime to p. Then, when $p\geqq 5$, $x^n\equiv \pm 1$, $y^n\equiv \pm 1$, $z^n\equiv \pm 1 \pmod{p}$. Therefore

$$x^n + y^n \equiv \pm 2 \quad \text{or} \quad 0 \pmod{p} \not\equiv z^n \pmod{p};$$

and the theorem is proved.

II. *Solution by Laurence Hampton, University of Oklahoma.*

If p is a prime number and p^k is the highest power of p, $k\geqq 0$, which is contained in both x and y, then either x or y must contain the factor p^{k+1} if the equation $x^{p-1}+y^{p-1}=z^{p-1}$ is solvable in integers. The proof is as follows: Since x and y are each divisible by p^k, z must also be so divisible, and we may take out this factor and write $X^{p-1}=Z^{p-1}-Y^{p-1}$. Suppose Y does not contain p as a factor; then $Y^{p-1}\equiv 1 \pmod{p}$. Also $Z^{p-1}\equiv 0$ or $1 \pmod{p}$ according as Z does or does not contain p as a factor. Hence $X^{p-1}\equiv -1$ or $0 \pmod{p}$. But only the

second of the two is possible. Hence x is divisible by p^{k+1} and z is divisible by p^k.

Also solved by A. Pelletier and J. E. Rowe.

3306 [1928, 92]. *Proposed by H. A. Mangan, Vandalia, Mo.*

A man bought a horse for A dollars and agreed to pay interest and principal in n equal monthly installments, interest r per cent per annum. Show how to compute r and n.

As a practical example, a certain loan company will lend \$1200 to be repaid in 120 monthly installments of \$12.50 each. What is the rate of interest? Show how to compute it.

Solution by J. E. Williams, Virginia Polytechnic Institute.

Let A be the original loan, B the amount of each monthly installment, and R the monthly rate of interest, $R>0$.

It is easy to see that the amount remaining after n installments have been paid is

$$A(1+R)^n - BR^{-1}(1+R)^n + BR^{-1}. \tag{1}$$

If the debt is to be fully paid by n installments, then (1) must be zero, and

$$(1+R)^n = 1/(1-AB^{-1}R). \tag{2}$$

In the practical example, $A=\$1200$, $B=\$12.50$ and $n=120$. We then have

$$(1+R)^{120} = 1/(1-96R) \quad \text{or} \quad 120 \log(1+R) + \log(1-96R) = 0.$$

An approximate solution is $R=1/260$. Therefore the annual rate of interest is about 4.6%.

If R is given, n can be found by taking the logarithm of both sides of (2).

Also solved by C. F. Bowler, S. A. Corey, J. A. Duerksen, A. Pelletier, W. J. Patterson, A. W. Richardson, and F. L. Wilmer.

3307 [1928, 93]. *Proposed by C. N. Mills, Normal, Illinois.*

A pyramid whose faces are equilateral triangles and whose base is a square stands on a horizontal plane and faces the cardinal points. If α is the sun's altitude and β the distance from the meridian and θ the vertical angle of the shadow prove that

$$\tan\theta = \tan 2\alpha \cos(\tfrac{1}{4}\pi - \beta).$$

I. *Solution by W. J. Patterson, The University of Western, Ontario.*

Let $ABCD$ be the square base of the pyramid with the edge $2S$ and center O, situated so that DA is directed east and AB, north; also let P be the vertex of the pyramid and V the vertex of its shadow. Then

$$CO = OA = OP = 2^{1/2}S, \quad OV = 2^{1/2}S\cot\alpha, \quad \angle COV = \tfrac{1}{4}\pi + \beta, \quad \angle AVC = \theta.$$

Also

$$2S^2[\csc^2\alpha \pm 2\cot\alpha\cos(\tfrac{1}{4}\pi + \beta)]$$

is the value of $(VC)^2$ for the $+$ sign and $(VA)^2$ for the $-$ sign. The square of the area of ACV is then

$$S^4 \sin^2 \theta[\csc^4 \alpha - 4 \cot^2 \alpha \cos^2 (\tfrac{1}{4}\pi + \beta)].$$

But it is also $4S^4 \cot {}^2\alpha \sin^2(\frac{1}{4}\pi + B)$; and after replacing $\sin^2\theta$ by $\tan^2\theta/(1+\tan^2\theta)$, we find

$$\tan \theta = 2 \cot \alpha(\cot^2\alpha - 1)^{-1} \sin (\tfrac{1}{4}\pi + \beta) = \tan 2\alpha \cos (\tfrac{1}{4}\pi - \beta).$$

In solving for tan θ it should be noticed that for small values of α and β, $\tan \theta$ is positive.

II. *Solution by R. H. Sciobereti, Berkeley, California.*

In this solution the formula of the problem is obtained in a different way, and attention is directed to the fact that it is true only when the vertex V of the shadow lies in certain parts of the plane. Formulae for other positions are derived. The results will be indicated in a condensed form using the above notation. Consider the two regions: I, the angle between the extensions of CD and AD; II, the rectangular strip between the extension of AD, the parallel to it through O, and the part of DC included between these parallels. Then V lies in I when $2^{1/2} \cot \alpha \sin \beta > 1$ and $2^{1/2} \cot \alpha \cos \beta > 1$, and the formula of the problem is true. If $2^{1/2} \cot \alpha \cos \beta > 1$ and $2^{1/2} \cot \alpha \sin \beta < 1$, then V lies in II and $\tan \theta = (2^{1/2} \cos \beta - \tan \alpha)/(\cot \alpha - 2^{1/2} \cos \beta)$. If $2^{1/2} \cot \alpha \sin \beta > 1$, $2^{1/2} \cot \alpha \cos \beta = 1$, then V lies on the extension of CD, and $\tan \theta = 2/(\tan \beta - 1)$. If $2^{1/2} \cot \alpha \sin \beta = 1$, $2^{1/2} \cot \alpha \cos \beta > 1$, then V lies on the extension of AD, and $\tan \theta = 2/(\cot \beta - 1)$.

The remaining cases of the position of V are easily obtained from these results.

Also solved by G. A. Yanosik.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor G. H. Hardy, of Oxford University, who has been lecturing at Princeton University during the first semester of the present academic year on "Chapters in the theory of Functions," lectured at the Ohio State University on January 18, 1929 on the "Theory of Primes," and at the University of Chicago on January 21 on the "Analytical Theory of Numbers."

Professor H. Weyl, of Princeton University, lectured at the University of Iowa on December 2 and 3, 1928 on "Group Theory and Quantum Mechanics."

At the New York Meeting of the American Association for the Advancement of Science, Professor E. T. Bell, of the California Institute of Technology,

was elected vice-president and chairman of Section A, and Professor C. N. Moore, of the University of Cincinnati, was re-elected secretary; Professor H. L. Rietz, of the University of Iowa, was elected vice-president and chairman of Section K, and Professor C. F. Roos, of Cornell University, was elected secretary. At the same meeting Professor E. B. Wilson, professor of vital statistics, School of Public Health, Harvard University, was elected a member of the executive committee.

During the second half of the present acadamic year, beginning about February 1, Professor H.W. Tyler, of the Massachusetts Institute of Technology, will be on leave of absence in Washington, D. C., in order to establish there a permanent office of the American Association of University Professors of which he has acted as secretary for many years. Professor Tyler was re-elected secretary of this association at the recent New York Meeting.

A complete set of the Bulletin of the American Mathematical Society, including the three volumes of the New York Mathematical Society, is available for purchase. Also ten volumes of the Annals of Mathematics, 2nd. Series. Information may be had from the Secretary of the Association.

Mr. Paul K. Rees, of Texas Technological College, has been appointed assistant professor of mathematics at the University of Mississippi. Mr. Thomas A. Bickerstaff has been appointed to a teaching fellowship at the College.

Professor C. H. Richardson, formerly of Georgetown College and now professor of mathematics at Bucknell University, was recently elected by the Faculty of Bucknell University to serve as head of the department of mathematics for the current academic year.

Associate professor Robert Torrey of the University of Mississippi was granted leave of absence for the first semester of the present academic year. Professor Torrey was injured in a train wreck last June.

Professor Charles N. Wunder, head of the department of mathematics at the University of Mississippi, has been appointed the first dean of men at that university. Professor Wunder will continue as head of the department of mathematics.

The following graduate courses in mathematics are announced for the academic year 1929–30:

The University of Chicago—By Professor E. H. Moore: General analysis III, IV, V. By Professor H. E. Slaught: Differential equations; Advanced calculus; Definite integrals. By Professor L. E. Dickson: Theory of numbers; Advanced topics in algebra and the theory of numbers. By Professor G. A. Bliss: Applications of the calculus of variations I, II; Topics in the calculus of variations. By Professor A. C. Lunn: Statistical mechanics; Vector analysis; Dyadics and

crystal physics; Fourier series and Bessel functions; Units and dimensions; Vector analysis in Riemann-Einstein space. By Professor E. P. Lane: Analytic projective geometry; Metric differential geometry; Solid analytic geometry. By Professor W. D. MacMillan: Theoretical mechanics I, II, III; The problem of three bodies; Dynamics of rigid bodies. By Professor M. I. Logsdon: Theory of functions of a complex variable; Algebraic geometry I, II. By Professor L. M. Graves: Vectors and matrices in higher algebra; Theory of functions of a real variable; Integral and functional equations. By Professor R. W. Barnard: General analysis I, II. By Professor W. Bartky: Stellar statistics. By an instructor to be appointed: Theory of equations; Limits and series; Algebraic invariants. Thesis work and directed reading and research are offered in the foundations of mathematics and general analysis by Professors Moore and Barnard; in algebra and the theory of numbers by Professors Dickson and Barnard; in analysis by Professors Bliss and Graves; in applied mathematics by Professors Lunn, MacMillan, and Bartky; in differential geometry by Professor Lane; and in algebraic geometry by Professor Logsdon.

The following courses in mathematics are announced for the summer of 1929:

The University of Chicago, first term, June 17–July 24; second term, July 25–August 30. In addition to the usual courses in plane analytic geometry and differential calculus, the following advanced courses will be offered. By Professor H. E. Slaught: Differential equations; Elliptic integrals. By Professor A. C. Lunn: Relativity; Lattices and crystal groups. By Professor A. B. Coble (The University of Illinois): Analytic projective geometry; Algebraic geometry. By Professor L. M. Graves: Introduction to analysis; Functions of lines. By Professor R. W. Barnard: Theory of equations; Metric differential geometry. By Professor W. Bartky: Theoretical mechanics; Modern theories of analytic differential equations. By Professor C. C. MacDuffee (Ohio State University): Theory of numbers; Theory of algebraic numbers; Linear algebras. By Professor H. S. Everett: Advanced calculus.

Professor H. B. Fine, of Princeton University, was fatally injured by an automobile on the evening of Friday, December 21 and died about one A.M. on December 22, 1928. He was seventy years of age.

SUGGESTIONS TO AUTHORS

Much needless expense and many errors can be avoided. The editors of several mathematical journals have agreed upon the following suggestions.

1. Typewrite words and the very simplest formulas only.
2. *Do not* try to typewrite any complex formulas. Write them.
3. Keep a copy, and send the editors two copies, if you can.
4. *Do not* underline any symbols or any formulas.
5. Underline theorems with blue pencil (avoid ink).
6. Follow our recent styles in abbreviations, footnotes, etc.
7. Write carefully the (often misunderstood) capitals C K P S V W X Z.
8. Write ϵ, not ε. Write very carefully $\gamma\ \eta\ \kappa\ \lambda\ \nu\ \tau\ \upsilon\ \chi\ \omega$.
9. Among Greek capitals, use only $\Gamma\ \Delta\ \Theta\ \Lambda\ \Xi\ \Pi\ \Sigma\ \Phi\ \Psi\ \Omega$.
10. Punctuate carefully, especially in formulas; thus: $1, 2, \cdots, n$.
11. Use the solidus (/) to avoid fractions in solid lines.
12. Use fractional exponents to avoid root signs everywhere.
13. Use extra symbols to avoid complicated exponents.
14. In typewritten formulas, 1 means "one"; to indicate "ell" in formulas, backspace and overprint /; thus: $\not{1}$. Similarly, 0 means "zero"; to indicate "cap O," backspace and overprint period; thus: $\odot$.
15. Avoid a dash over a letter, except for those shown below.
16. Some samples of unusual types available on monotype machines follow. A more complete list of all such types will be sent on request.

Light Face Greek—$\alpha\ \beta\ \gamma\ \cdots$ (all) $\mathrm{A\ B}\ \Gamma\ \cdots$ (all).

★ Light Greek Superiors—$^{\Delta}$ and $^{\alpha\,\beta\,\gamma\,\cdots}$ (all except o).

★ Light Greek Inferiors—$_{\Delta\Lambda\Sigma\Omega}$ and $_{\alpha\,\beta\,\gamma\,\ldots}$ (all except o).

* Boldface Greek—$\boldsymbol{\alpha\ \beta\ \gamma\ \delta\ \varepsilon\ \zeta\ \eta\ \theta\ \kappa\ \lambda\ \mu\ \nu\ \xi\ \pi\ \rho\ \sigma\ \omega}$ and $\boldsymbol{\Omega}$.

* Lightface German—$\mathfrak{a\ b\ c\ d\ k\ p\ q}$ $\mathfrak{A\ B\ C\ D\ E\ F\ G\ H\ J\ K\ L\ M\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$.

* Boldface German—$\boldsymbol{\mathfrak{d\ A\ B\ D}}$.

Script (special font) $\mathcal{A\ B\ C}\ \cdots$ (all). No lower case manufactured.

* Hebrew— $\aleph$ א בּ ב troublesome to handle.

★ Dashed Italics—$\bar{A}\ \bar{a}\ \bar{B}\ \bar{b}\ \bar{C}\ \bar{c}\ \bar{E}\ \bar{e}\ \bar{F}\ \bar{f}\ \bar{G}\ \bar{g}\ \bar{H}\ \bar{I}\ \bar{\imath}\ \bar{\jmath}\ \bar{K}\ \bar{k}\ \bar{M}\ \bar{m}\ \bar{N}\ \bar{n}\ \bar{o}\ \bar{P}\ \bar{p}\ \bar{q}\ \bar{R}\ \bar{r}\ \bar{s}\ \bar{l}\ \bar{u}\ \bar{v}\ \bar{X}\ \bar{x}\ \bar{Y}\ \bar{y}\ \bar{Z}\ \bar{z}$.

★ Tilda Italics—$\tilde{A}\ \tilde{a}\ \tilde{e}\ \tilde{N}\ \tilde{n}\ \tilde{O}\ \tilde{r}\ \tilde{u}\ \tilde{y}$.

★ Tilda Greek—$\tilde{\alpha}\ \tilde{\epsilon}\ \tilde{\eta}\ \tilde{\omega}\ \tilde{\varpi}$.

★ Dashed Greek—$\bar{\alpha}\,\bar{\beta}\,\bar{\gamma}\,\bar{\delta}\,\bar{\eta}\,\bar{\theta}\,\bar{\mu}\,\bar{\nu}\,\bar{\rho}\,\bar{\omega}\,\bar{\Gamma}$.

★ Dotted Italic—$\dot{a}\ \ddot{a}\ \dot{e}\ \ddot{e}\ \dot{g}\ \ddot{\imath}\ \dot{m}\ \dot{n}\ \dot{q}\ \dot{r}\ \ddot{r}\ \dot{u}\ \ddot{u}\ \dot{x}\ \ddot{x}\ \dot{y}\ \ddot{y}\ \dot{z}\ \ddot{z}$.

★ Dotted Greek—$\dot{\eta}\ \ddot{\eta}\ \dot{\theta}\ \ddot{\theta}\ \dot{\xi}\ \dot{\psi}\ \ddot{\psi}\ \ddot{\omega}$ (single dotted $\zeta\ \phi\ \delta\ \beta\ \gamma$; double dotted γ readily available).

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* Additional characters readily available at small cost.

★ Matrices for additional characters are made upon special orders and necessitate **a delay of from four to eight weeks and average expense of $4.50 per matrix.**

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Thirteenth Summer Meeting of the Association, Boulder, Colorado, August, 1929.
Fourteenth Annual Meeting, Des Moines, Iowa, December 31, 1929, January 1, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled.

ILLINOIS, Carthage, Ill., May 3-4.
INDIANA, Culver Military Academy, May 3-4.
IOWA.
KANSAS, Topeka, Kansas, February 2.
KENTUCKY.
LOUISIANA-MISSISSIPPI.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, May 4.
MICHIGAN, Ann Arbor, Mich., March 16.
MINNESOTA, St. Paul, May 8.
MISSOURI, Kansas City, Mo., November.
NEBRASKA.
OHIO, Columbus, Ohio, April 4.
PHILADELPHIA, University of Pennsylvania, November 30.
ROCKY MOUNTAIN, Greeley, Colo., April 12-13.
SOUTHEASTERN, Macon, Ga., April 29.
SOUTHERN CALIFORNIA, University of Redlands, March 9.
TEXAS, Houston, Texas, June 26.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE LA.-MISS. BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teachers' agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

THE THIRTEENTH ANNUAL MEETING OF THE ASSOCIATION

The thirteenth annual meeting of the Mathematical Association of America was held at Columbia University, New York, New York on Friday and Saturday, December 28-29, 1928 in conjunction with the annual meeting of the American Mathematical Society and in affiliation with the American Association for the Advancement of Science. Three hundred forty were in attendance at these meetings, among them the following two hundred twenty-one members of the Association.

C. R. Adams, Brown University
Beatrice Aitchison, Metropolitan Life Ins. Co., New York, N. Y.
Joseph Allen, College of the City of New York
R. C. Archibald, Brown University
C. L. Arnold, Ohio State University
H. E. Arnold, Wesleyan University
T. B. Ashcraft, Colby College
C. S. Atchison, Washington and Jefferson College

Clara L. Bacon, Goucher College
H. C. Barber, Phillips Exeter Academy
I. A. Barnett, University of Cincinnati
Helen Barton, North Carolina College for Women
R. D. Beetle, Dartmouth College
A. A. Bennett, Brown University
Theodore Bennett, Princeton University
C. A. Bergstresser, Jamaica High School, New York, New York
W. J. Berry, Brooklyn Polytechnic Institute
O. F. H. Bert, Washington and Jefferson College
G. D. Birkhoff, Harvard University
L. M. Blumenthal, Johns Hopkins University
Joseph Bowden, Adelphi College
J. W. Bradshaw, University of Michigan
R. S. Burington, Case School of Applied Science
F. J. Burkett, Trinity College

W. D. Cairns, Oberlin College
Minnie W. Caldwell, Chowan College
B. H. Camp, Wesleyan University
A. D. Campbell, Syracuse University
Mildred E. Carlen, Brown University
R. D. Carmichael, University of Illinois
Evelyn T. Carroll, Wells College

A SIMPLE PRINCIPLE OF UNIFICATION IN THE ELEMENTARY THEORY OF NUMBERS[1]

By R. D CARMICHAEL, University of Illinois

1. *Introduction.* The elementary theory of positive integers lacks the unity which is essential to a structure of the most pleasing esthetic quality. This fact is apparent from, and indeed is emphasized by, the record given in the first volume of Dickson's *History of the Theory of Numbers*—a volume devoted largely to elementary aspects of the theory of positive integers. As long as these numbers are dealt with only by the methods hitherto invented, this proposition concerning esthetic quality is likely to remain true; for there is apparent no ground of hope that these methods, by their development and extension, will grow into each other and so lead to the desired unity. You will therefore not expect me to suggest a means of bringing order into all this confusion. At the best only a part of these scattered results can now be given a place in any structure of thought possessing esthetic unity. But I hope to indicate how a significant part of them may be united by a method which is both elementary and rather comprehensive as regards the material brought into unity through its use.

A convenient point of departure is afforded by certain theorems due to Fermat. If the integer a is prime to the prime p then $a^{p-1}-1$ is divisible by p; more generally, if a and m are relatively prime integers and if $\phi(m)$ denotes the totient of m then $a^{\phi(m)}-1$ is divisible by m. One inevitably raises the question as to what is the least positive integer ν such that $a^\nu-1$ is divisible by p or by m. The question also arises concerning the existence of numbers a such that the least value of ν for which $a^\nu-1$ is divisible by p or m is $p-1$ or $\phi(m)$ respectively. The customary propositions arising here you recognize as part of the classic theory of integers. You observe also that they are all intimately associated with the infinite sequence of integers

$$a-1,\quad a^2-1,\quad a^3-1,\quad a^4-1,\cdots.$$

This will afford the point of departure for the method to be presented. If we write

$$u_n = a^n - 1,\quad n = 0,1,2,\cdots,$$

then it is easy to verify that

$$u_{n+2}-(a+1)u_{n+1}+au_n = 0,\quad u_0 = 0,\quad u_1 = a-1.$$

Conversely, the solution of this difference system leads to the given sequence. The indicated propositions may therefore be presented in the form of equivalent theorems pertaining to the named sequence. As soon as they are seen in this light, several possible generalizations come at once to mind. It is these which afford the simple principle of unification to be treated.

[1] This paper was read by invitation before the Mathematical Association of America at New York City on Dec. 28, 1928.

2. *Recurrent Sequences of Integers.*[1] The sequence of integers

$$u_0,\ u_1,\ u_2,\ \cdots \tag{1}$$

is uniquely defined in terms of the initial numbers $u_0, u_1, \cdots, u_{k-1}$ by the recurrence relation

$$u_{x+k} + \alpha_1 u_{x+k-1} + \alpha_2 u_{x+k-2} + \cdots + \alpha_k u_x = \alpha, \tag{2}$$

in which $\alpha, \alpha_1, \alpha_2, \cdots, \alpha_k$ are given integers. We assume that $\alpha_k \neq 0$. We use m to denote a given positive integer and p to denote a given prime; and often (as occasion arises) we think of p as a possible value of m.

Let

$$r_0,\ r_1,\ r_2,\ \cdots \tag{3}$$

be in order the least non-negative residues of the integers $u_0, u_1, u_2, \cdots$ with respect to the modulus m. It is easy to show that the sequence (3) always contains a set of k consecutive residues which is repeated (as an ordered set) infinitely often in the sequence. The infinite sequence beginning with this set in any position in the sequence is the same as that beginning with the same set in any other position. We may therefore say that the residues (3), with the possible exception of a finite number at the beginning, repeat themselves periodically in cycles. If μ is the number of residues in the smallest cycle which is thus repeated we shall call μ the characteristic number of (1) modulo m, and we shall write

$$\mu = \mu(m\ ;\ k\ ;\ \alpha\ ;\ \alpha_1, \alpha_2, \cdots, \alpha_k\ ;\ u_0, u_1, \cdots, u_{k-1}).$$

When a and m are relatively prime integers the relations

$$\mu(m; 1; a-1; -a; 0) = \mu(m; 2; 0; -a-1, a; 0, a-1) = \text{ the exponent of } a \text{ (modulo } m)$$

are easily verified in view of the sequence treated in § 1. Thus μ is in two ways a generalization of an important function in the classic theory. Furthermore, from the classic theory it follows that this particular μ is always a factor of $\phi(m)$. These special results are instances of propositions concerning the general function μ, propositions which may be attained without too great difficulty (see the memoir cited) when m and α_k are relatively prime.

The more important applications of the theory belong to the case in which equation (2) is homogeneous ($\alpha = 0$) and indeed to a case in which further conditions are also satisfied. The resulting sequences are then said to be restricted. By a restricted sequence (1) modulo m, where m and α_k are relatively prime, we shall mean a sequence

$$U_0,\ U_1,\ U_2,\ \cdots \tag{4}$$

such that U_n is a solution of the homogeneous equation

[1] For this section generally, see the Quarterly Journal of Mathematics, vol. 48 (1920), pp. 343–372.

$$(5) \qquad u_{x+k} + \alpha_1 u_{x+k-1} + \cdots + \alpha_k u_x = 0,$$

while the corresponding sequence of residues (3) modulo m,

$$(6) \qquad R_0,\ R_1,\ R_2,\ \cdots,$$

has a subset σ of k consecutive elements (which we call a chief subset) containing at least one element prime to m and being such that another and later subset σ_1 of k consecutive elements are in order congruent to the elements of the former set, each multiplied by one and the same integer τ prime to m. Then we call τ a multiplier modulo m of the sequence. If ρ is the exponent modulo m to which the multiplier τ belongs, then μ/ρ is an integer μ_τ which is called the restricted characteristic number modulo m of the restricted sequence (4) as to the chief subset σ and the multiplier τ. Some of the most useful applications of the theory are associated with the number μ_τ.

These ideas furnish the basis for a general theory of these recurrent sequences of integers. Since we are now to treat mainly certain special cases it is not necessary to outline the general theory. We may however give an idea of the latter by stating a few results for the very simple case when m is the prime power p^t and the polynomial $f(\rho)$,

$$f(\rho) = \rho^k + \alpha_1\rho^{k-1} + \cdots + \alpha_k,$$

is irreducible modulo p. When $k=1$ and $\alpha=0$ we suppose further that $f(\rho)$ is not identically equal to $\rho-1$. We suppose also that $p>k$. Then the characteristic number μ of the sequence modulo p^t is a factor of p^k-1 when $t=1$ and is a factor of $p^t(p^k-1)$ when $t>1$, α_k in each case being prime to p. As a special case we have the classic result that the exponent of a modulo p is a factor of $p-1$, a being prime to p. There are numbers a belonging modulo p to the exponent $p-1$. As a generalization of this classic result we have the theorem that for every value of k there are sequences (1) whose characteristic number modulo p is p^k-1. In a similar way it is possible to generalize other classic results concerning primitive roots modulo m.

There are also general theorems (see the Quarterly Journal of Mathematics, l. c.) asserting that a given number m is a prime when the characteristic number μ of a sequence modulo m has certain defined properties. For special cases of these results see the next section.

3. *The Functions*[1] D_n, S_n *and* F_n. The instances of the foregoing general theory which have been most fully treated are those which are associated with the functions

$$D_n \equiv D_n(\alpha,\beta) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad S_n \equiv S_n(\alpha,\beta) = \alpha^n + \beta^n,$$

both of which satisfy the equation

$$u_{n+2} - (\alpha+\beta)u_{n+1} + \alpha\beta u_n = 0,$$

[1] For this section generally, see the *Annals of Mathematics*, (2), vol. 15 (1913), pp. 30–70.

where $\alpha+\beta$ and $\alpha\beta$ are relatively prime integers. [The case when α and β are both roots of unity is excluded from consideration.] Associated with D_n and S_n is the function $F_n(\alpha, \beta)$, where

$$F_n(\alpha,\beta) = \beta^{\phi(n)}Q_n(\alpha/\beta)$$

and $Q_n(x)$ is the polynomial of degree $\phi(n)$ with leading coefficient unity whose roots are the primitive n^{th} roots of unity without repetition. When $n>1$ the value of $F_n(\alpha, \beta)$ is an integer; the value of F_1^2 is also an integer.

Then we have the obvious relation $D_{2n}=D_nS_n$ and the fundamental partial numerical factorization of D_n afforded by the formula

$$D_n(\alpha,\beta) = \prod_d{}' F_d(\alpha,\beta) ,$$

where d ranges over all the divisors of n except unity.

The properties of D_n, S_n and F_n have been developed in some detail. Perhaps the most fundamental properties of F_n are those represented in the following theorem:

I. Let ν be any positive integer and let p be any prime not dividing ν; then

(1) If $F_{\nu p^a}\equiv 0 \bmod p$, then $F_\nu^2\equiv 0 \bmod p$.

(2) If $F_\nu^2\equiv 0 \bmod p$, then each of the numbers $F_{\nu p}$, $F_{\nu p^2}$, $F_{\nu p^3}$, $\cdots$ is divisible by p, and none of them is divisible by p^2 except when $\nu=1$ in which case F_p may be divisible by p^2 and when $\nu=3$, $p=2$ in which case F_6 may be divisible by 2^2. Moreover, $F_k^2\not\equiv 0 \bmod p$ unless k is of the form νp^a.

(3) If $F_{\nu p^a}\not\equiv 0 \bmod p$, $a>0$, then $F_{\nu p^a}\equiv 1 \bmod p$ when $\nu>1$ or when $\nu=1$ and $p=2$; if $\nu=1$ and p is odd we have

$$F_{p^a} \equiv (\alpha-\beta)^{p-1} \equiv \pm 1 \bmod p .$$

Consider the sequence of integers F_1^2, F_2, F_3, $\cdots$. By a characteristic factor of F_n we mean a prime divisor of F_n which is not a factor of any number of the set F_1^2, F_2, F_3, $\cdots$, F_{n-1}. Similarly, a characteristic factor of $D_n[S_n]$ is a prime divisor of $D_n[S_n]$ which is not a factor of any $D_\nu [S_\nu]$ for which ν is a positive integer less than n. In this connection the principal theorems are the following:

II. A necessary and sufficient condition that a prime p which divides F_n shall be a characteristic factor of F_n is that p shall be prime to n.

III. A characteristic factor of $F_n[F_{2n}]$ is also a characteristic factor of $D_n[S_n]$.

IV. If α and β are real and if $n\neq 1, 2, 6$, then $F_n(\alpha, \beta)$ contains at least one characteristic factor in all cases except when

(1) α and β are suitably chosen irrational numbers and n is equal to an odd prime divisor of $(\alpha-\beta)^2$;

(2) $\quad n=3, \quad \alpha+\beta=\pm 1, \quad \alpha\beta=-2;$

(3) $\quad n=5, \quad \alpha+\beta=\pm 1, \quad \alpha\beta=-1;$

(4) $\quad n=12, \quad \alpha+\beta=\pm 1, \quad \alpha\beta=-1.$

V. If α and β are real and $n \neq 1, 2, 6$, then D_n contains at least one characteristic factor except when

$$n = 12, \quad \alpha + \beta = \pm 1, \quad \alpha\beta = -1.$$

VI. If α and β are real and $n \neq 1, 3$, then S_n contains at least one characteristic factor except when

$$n = 6, \quad \alpha + \beta = \pm 1, \quad \alpha\beta = -1.$$

VII. A characteristic factor of $F_n(\alpha, \beta)$ is of the form $kn \pm 1$. When a and b are relatively prime integers a characteristic factor of $F_n(a, b)$ is of the form $kn+1$.

From another point of view we have the following theorem:

VIII. If $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where the p's are distinct primes, and if m is prime to $\alpha\beta$ then $D_\lambda \equiv 0$ and $D_\phi \equiv 0$ mod m, where λ is the least common multiple of the integers

$$p_i^{\alpha_i - 1}[p_i - (\alpha, \beta)_{p_i}], \quad i = 1, 2, \cdots, k,$$

and where ϕ is their product, the symbol $(\alpha, \beta)_p$ for odd prime p having the value 0, 1, or -1 according as $(\alpha - \beta)^2$ is divisible by p, is a quadratic residue of p, or a quadratic non-residue of p, while $(\alpha, \beta)_2$ is 1, 0 or -1 according as (a) $\alpha\beta$ is even, (b) $\alpha\beta$ is odd and $\alpha+\beta$ is even, or (c) $\alpha\beta$ and $\alpha+\beta$ are both odd.

The function ϕ here introduced is a generalization of the Euler ϕ-function. Many of the generalizations of this function which have been employed may be associated similarly with the general theory indicated in § 2; and the latter general theory may be so developed as to bring a certain measure of unity into the known results concerning Euler's ϕ-function and its generalizations. In a similar way some unity may likewise be introduced into the generalizations of the theory of primitive roots modulo m.

By means of the functions treated in this section it may be shown that each of the sequences

$$p^k x - 1, \; 3 \cdot 2^k x - 1, \; 4x + 1, \; 4x - 1, \; 6x + 1, \; 6x - 1, \; x = 1, 2, 3, \cdots,$$

where p is any given odd prime and k is any given positive integer, contains an infinite number of primes.

In another range of ideas we have the following theorems:

IX. A necessary and sufficient condition that a given odd number p shall be a prime is that there shall exist relatively prime integers $\alpha+\beta$ and $\alpha\beta$ such that $F_{p-1}(\alpha, \beta)$ shall be divisible by p.

X. A necessary and sufficient condition that a given odd number p shall be a prime is that an integer a shall exist such that $F_{p-1}(a, 1)$ shall be divisible by p.

XI. A necessary and sufficient condition that an odd number p shall be

a prime is that there shall exist relatively prime integers $\alpha+\beta$ and $\alpha\beta$ such that $F_{p+1}(\alpha, \beta)$ shall be divisible by p.

XII. If $p=2^2+1$, $n>1$, and if r is any odd prime of which p is a quadratic non-residue, then a necessary and sufficient condition that p shall be a prime is that

$$r^{(p-1)/2}+1 \equiv 0 \mod p.$$

[For each p one may use 3 for r.]

XIII. A necessary and sufficient condition that $2^{k+1}\cdot q+1$, where q is an odd prime, shall be a prime is that an integer a shall exist such that

$$\frac{(a^{2^kq}+1)}{(a^{2^k}+1)} \equiv 0 \mod 2^{k+1}\cdot q+1.$$

It is by means of theorems of this sort that the discovery of large primes has been effected. Perhaps the whole of the known related theory may be unified by means of such theorems and their generalizations arising in connection with the general theory of recurrent sequences of integers.

We may suggest as worthy of attention two generalizations of the important function $D_n(\alpha, \beta)$, namely, the following:

$$G_n = \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_1^{k-2} & \alpha_2^{k-2} & \cdots & \alpha_k^{k-2} \\ \alpha_1^n & \alpha_2^n & \cdots & \alpha_k^n \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_k^2 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \cdots & \alpha_k^{k-1} \end{vmatrix}}, \quad H_n = \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1^n & \alpha_2^n & \cdots & \alpha_k^n \\ \alpha_1^{2n} & \alpha_2^{2n} & \cdots & \alpha_k^{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_1^{(k-1)n} & \alpha_2^{(k-1)n} & \cdots & \alpha_k^{(k-1)n} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_k^2 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \cdots & \alpha_k^{k-1} \end{vmatrix}},$$

where $\alpha_1, \alpha_2, \cdots, \alpha_k$ are distinct roots of the equation

$$x^k + c_1x^{k-1} + \cdots + c_k = 0,$$

in which $c_1, c_2, \cdots c_k$ are given integers. The investigation of the properties of these functions will lead to two generalizations of the theory associated with the function $D_n(\alpha, \beta)$, which is the special case for $k=2$ of both G_n and H_n. It seems likely that this investigation will lead to something of interest.

In G_n the numerator is formed by replacing the last row in the denominator by $\alpha_1^n, \alpha_2^n, \cdots, \alpha_k^n$. Other useful functions may also be formed similarly by putting these elements for other rows in the denominator.

The theory of recurrent sequences of integers, together with certain closely related matters, and particularly the special case treated in this section, may be shown to pervade, or at least to be implicit in important ways in, ten of the twenty chapters of volume I of Dickson's *History of the Theory of Numbers.* Around this theory may obviously be developed practically the whole of the theory of chapters XV, XVI, XVII, and VI; and it has important connections with chapters I, III, V, VII, and XIV, as one may readily see; that one may also bring into relation with it certain parts of chapter VIII is apparent from our §5 below.

It thus appears that the general theory of recurrent sequences of integers may be used as a unifying element for a significant portion of that part of the theory of numbers which has been treated in the volume mentioned. So far as I am aware, no systematic analysis of this theory has been made with reference to the problem of generalization and extension in such a way as to lead to further unification. What unification is brought about by the theory in its present state seems to have been accidental rather than purposed or foreseen; and yet what is present is significant. It appears that there is some ground to hope that further progress toward coordinating what are now diverse elements might be made here both by particular investigations undertaken with this in mind and by a systematic exposition from this point of view. One must not expect too much from such an investigation; but even the probable partial successes which are to be anticipated will justify the effort required.

4. *Propositions Discovered by Fermat.* On a previous occasion[1] I have insisted on the fact that many of the results announced by Fermat (See list in his *Oeuvres*, Vol. IV, pp. 231–237) may be associated closely with the properties of recurrent sequences of integers. In fact, a systematic examination of this matter brings out the following facts:—

(1) The results which may be derived readily from this theory by further methods known to have been employed by Fermat comprise about one-third of his principal discoveries in the theory of numbers as listed on pp. 231–237 of the fourth volume of his *Oeuvres*.

(2) In interest and value these results are fully up to the high level of Fermat's work in general.

(3) Approximately another third in the list in Fermat's *Oeuvres* may be selected of such sort that one can see from Fermat's writings a natural way by which he was led to each result in such portion of the total list and to its proof.

(4) Of the remaining third it can be determined from Fermat's work itself that some of the most remarkable of them were by him associated with the

[1] Quarterly Journal of Mathematics, vol. 48 (1920), pp. 363–372.

results which may readily be derived from the theory of recurrent sequences of integers; as, for instance, the fact that the theorem about polygonal numbers is a consequence of the fact that every prime of the form $4x+1$ is a sum of two squares.

Moreover, an analysis of the proofs of the general theorems about these recurrent sequences, with reference to the simplifications which would result for just the cases actually to be used in deriving Fermat's theorems, reveals the fact that the proofs in these cases are of extremely simple character and involve only those conceptions into which Fermat may naturally have been led from certain other matters to which he is known to have given much attention.

Furthermore, the theory of these recurrent sequences has many elements analogous to the theory of singly periodic functions; and this suggests inevitably the question whether there is a corresponding situation in the theory of numbers (not yet brought to explicit notice, to be sure) in which we should have elements corresponding to the theory of doubly periodic functions.

These considerations tend to suggest the conjecture that we may have in the theory of recurrent sequences of integers a reconstruction (and extension) of certain of the methods employed by Fermat in some of his most remarkable discoveries in the theory of numbers. While the evidence is of too uncertain a character to justify insistence upon the conjecture it is yet true that there is too much plausibility in the suggestion for it to be ignored in presence of the fact that Fermat's work is so important both in the history of the theory of numbers and with respect to its actual content in its present state.

In order to make clear the nature of the relations insisted upon, we present a few instances of theorems of Fermat which belong to the range of ideas now in consideration.

One of Fermat's important results is the following: If p is an odd prime number and a^t is the lowest power of a, such that a^t-1 is divisible by p, then t is a factor of $p-1$; if t is odd no number of the form a^t+1 is divisible by p; if t is the even number 2τ, then $a^\tau+1$ is divisible by p. All these propositions follow readily from the theory of recurrent sequences of integers.

If q is a prime divisor of 2^p-1, when p is a prime, then the restricted characteristic number modulo q of the sequence 2^n-1, $n=0, 1, \cdots$, as to the chief subset (0, 1) is a divisor of p, and hence equal to p. Therefore p is a factor of $q-1$, as asserted by Fermat.

It is obvious that the Fermat numbers $2^{2^x}+1$ are intimately associated with the theory of recurrent sequences.

As a final example, let us consider the properties of an odd prime factor p of the sum a^2+b^2 of two relatively prime squares. For D_n we now take $D_n=(a^n-b^n)/(a-b)$. Then we have

$$D_0 = 0, \quad D_1 = 1, \quad D_2 = a + b, \quad D_3 = a^2 + ab + b^2, \quad D_4 = (a + b)(a^2 + b^2).$$

Then we see that the restricted characteristic number modulo p of the sequence $D_0, D_1, D_2, \cdots$ as to the chief subset (0, 1) is 4; whence it follows from the

general theory that $p-1$ is divisible by 4, or that p is of the form $4x+1$. That is, every odd prime factor of the sum of two relatively prime squares is of the form $4x+1$. Thence we have readily other results stated by Fermat: No divisor of a sum of two relatively prime squares can be of the form $4x-1$; no number of the form $4x-1$ can be the sum of two squares, either integral or fractional; no number of the form $9x\pm 3$ can be the sum of two squares, either integral or fractional.

If we turn to the converse of the theorem that every odd prime factor p of the sum of two relatively prime squares is of the form $4x+1$ we may derive readily (by means of recurrent sequences) the conclusion that every prime of the form $4x+1$ is a factor of the sum of two relatively prime squares. Having reached this stage one observes that certain of these primes $4x+1$ may actually be represented as the sum of two squares, and naturally raises the question whether this is a common property of all of them. That this question is to be answered in the affirmative is then readily shown by the method of infinite descent, a method known to have been employed by Fermat in proving this theorem.

Such considerations as these lead to a realization of the fact that many of the apparently unrelated theorems stated by Fermat constitute so many fragments of a general theory of certain particular classes of recurrent sequences of integers.

5. *Galois Fields. Higher Congruences.* In order to associate the Galois field theory with the theory of recurrent sequences of integers, let us consider a primitive mark ω of the Galois field $GF[p^k]$ where p is a prime and k is a positive integer. Then integers $c_1, c_2, \cdots, c_k$ exist in the field such that

$$\omega^k = c_1\omega^{k-1} + c_2\omega^{k-2} + \cdots + c_{k-1}\omega + c_k. \tag{7}$$

Thence it follows that integers $u_n^{(1)}, u_n^{(2)}, \cdots, u_n^{(k)}$ exist such that

$$\omega^n = u_n^{(1)}\omega^{k-1} + u_n^{(2)}\omega^{k-2} + \cdots + u_n^{(k-1)}\omega + u_n^{(k)}. \tag{8}$$

Moreover, when $0 \leqq n < k$ we have $u_n^{(i)}$ equal to zero except when $n=k-i$ in which case the value is 1. Multiplying (8) by ω and employing (7) we have

$$\omega^{n+1} = u_n^{(1)}(c_1\omega^{k-1} + c_2\omega^{k-2} + \cdots + c_k) + u_n^{(2)}\omega^{k-1} + u_n^{(3)}\omega^{k-2} + \cdots + u_n^{(k)}\omega.$$

Comparing this equation with what is obtained from (8) on replacing n by $n+1$ we see that

$$\begin{aligned} u_{n+1}^{(1)} &= c_1 u_n^{(1)} + u_n^{(2)}, \\ u_{n+1}^{(2)} &= c_2 u_n^{(1)} + u_n^{(3)}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ u_{(n+1)}^{(k-1)} &= c_{k-1} u_n^{(1)} + u_n^{(k)}, \\ u_{n+1}^{(k)} &= c_k u_n^{(1)}. \end{aligned} \tag{9}$$

Eliminating from this system all the functions u except $u_n^{(1)}$ we find that $u_n^{(1)}$ is a solution of the following recurrence equation:

$$u_{n+k} = c_1 u_{n+k-1} + c_2 u_{n+k-2} + \cdots + c_n u_n. \tag{10}$$

Thence it is obvious that each of the functions $u_n^{(1)}, u_n^{(2)}, \cdots, u_n^{(k)}$ satisfies this recurrence relation. They are then uniquely determined by the help of their initial values as already given. (It is to be observed that the function-values may be reduced modulo p to numbers of the set $0, 1, 2, \cdots, p-1$.) It may be seen from (8) that $u_n = \omega^n$ is a solution of (10) in the $GF[p^k]$, whence it follows that the successive powers of a primitive mark in a $GF[p^k]$ satisfy a recurrence relation of order k with integral coefficients.

That p^k-1 is the characteristic number modulo p of the sequence $u_n^{(1)}$, $n=0, 1, 2, \cdots$, follows readily from the fact that the order of ω is p^k-1, proof being made by means of (9) and (8). Hence for every prime p and given k there exists a recurrence relation (10) with integral coefficients such that p^k-1 is the characteristic number modulo p of that solution of equation (10) for which $0, 0, \cdots, 0, 1$ are the initial constants. It may be shown that this is a property of prime numbers which belongs to no composite number.

Consider now the k solutions $u_n^{(1)}, u_n^{(2)}, \cdots, u_n^{(k)}$ of equation (10) where n ranges over the set $0, 1, 2, \cdots$. The corresponding sequences may be indicated as in the following array:

$$\begin{array}{lllll}
000\cdots 001 & u_k^{(1)}, & u_{k+1}^{(1)}, & u_{k+2}^{(1)}, & \cdots, \\
000\cdots 010 & u_k^{(2)}, & u_{k+1}^{(2)}, & u_{k+2}^{(2)}, & \cdots, \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
010\cdots 000 & u_k^{(k-1)}, & u_{k+1}^{(k-1)}, & u_{k+1}^{(k-1)}, & \cdots, \\
100\cdots 000 & u_k^{(k)}, & u_{k+1}^{(k)}, & u_{k+2}^{(k)}, & \cdots,
\end{array}$$

Now consider the p^k-1 ordered sets of k elements each, one set being formed from each of the first p^k-1 columns of the foregoing array by taking the elements in order in such a column beginning at the top and reducing each modulo p to a number of the set $0, 1, \cdots, p-1$. Then no two of these p^k-1 columns give rise to the same symbol, and no one of the symbols consists entirely of zeros. Hence the p^k-1 symbols may be denoted by $(\lambda_1, \lambda_2, \cdots, \lambda_k)$ where each of the λ's is a number of the set $0, 1, \cdots, p-1$ and in no symbol are all the λ's zero. Adjoin the symbol $(0, 0, \cdots, 0)$. The p^k symbols may then be used to represent the marks of the Galois field $GF[p^k]$ with the rules of addition and multiplication defined as follows: the element $(0, 0, \cdots, 0)$ enjoys both the additive and the multiplicative properties of 0; the sum of any two symbols is defined as the symbol obtained from the given symbols by adding corresponding elements and reducing modulo p to numbers of the set $0, 1, 2, \cdots, p-1$; the product of the elements formed from the m^{th} column and the n^{th} column is the element formed from the $(m+n-1)^{\text{th}}$ column. It is not difficult to show that the elements so defined, together with the de-

fined operations of addition and multiplication, constitute the $GF[p^k]$. Thus the Galois field theory is exhibited as belonging to the general theory of recurrent sequences of integers reduced modulo p where p is a prime number.

From this conclusion it may be seen that a significant part of the general theory of higher congruences may be intimately associated with the recurrent sequences here in consideration.

If the theory of the finite geometries is considered as a geometric phrasing of certain matters in the theory of numbers, then this division of number theory is intimately connected with recurrent sequences of integers. Thence one is led to a large part of the theory of Abelian groups, as is shown by the connection of that theory with the finite geometries.

We have now said enough to make it apparent that a considerable and a highly significant portion of that part of the theory of numbers analyzed in the first volume of Dickson's history of number theory is capable of a marked unification through a treatment of it in intimate connection with the theory of recurrent sequences. We have not had time to go into details. But the results presented indicate that here is an important work of exposition (and, to a smaller extent, of discovery) which should be carried out systematically. It would certainly bear fruit of interest.

6. *Connections with Finite Groups.* Perhaps I may be allowed a digression from the main theme of the paper for the purpose of presenting an empirical connection between some of the numbers here treated and certain properties of groups of finite order.

In 1905 (Göttingen Nachrichten) Dickson presented the following conjectured theorem, which he verified numerically in a wide range without however obtaining a proof of its validity:

I. If G is any group of order p^n-1, where p is a prime and n is an odd integer greater than two, then G contains a self-conjugate subgroup of order a power of a prime q, where q is a factor of p^n-1 but not of any p^m-1 where $0<m<n$.

Dickson found a need for this theorem in investigating the existence of certain finite algebras which generalize Galois fields. The problem of the existence of these algebras is intimately connected with that of the existence of doubly transitive groups of degree p^n and order $p^n(p^n-1)$. In connection with a study of the latter problem I have lately encountered a need for just such a theorem; and in analyzing the matter have come to raise several questions concerning theorems analogous to that conjectured by Dickson.

Dickson verified the foregoing theorem for the cases of 144 values of p^n-1. These I have checked and have continued the verification through 15 additional cases. Neither of us found any exception to the theorem.

The foregoing theorem is intimately connected with the following:

II. When α and β are real and the exceptional cases of theorem V of §3 are excluded (as far as may be necessary) and when n is odd and greater than 2, then a group G of order $D_n(\alpha, \beta)$ has a self-conjugate subgroup whose order is of the form $q^t (t>0)$ where q is a characteristic prime factor of $D_n(\alpha, \beta)$.

No method has been discovered which so much as promises to yield a proof of this theorem. A verification has been carried out for each of 230 cases for orders $D_n(\alpha, \beta)$; and no failure of the theorem has been discovered. For the purposes of this verification I have employed the known tables of factors of the numbers $D_n(\alpha, \beta)$ for particular values of α and β and indeed have constructed some additional tables for the problem in hand. Owing to the rapid increase of $D_n(\alpha, \beta)$ for increasing n (α and β being fixed) it is very laborious to obtain tables with a large number of entries.

The examination of these propositions has led to the consideration of others, the principal ones of which will be mentioned.

III. This is theorem I with the removal from the hypothesis of the condition that n shall be odd and with the further restriction that $n \neq 4, 6$.

This has been verified for 183 values of $p_i - 1$, twenty-four of them being cases for which n is even. The presence of the named exceptional cases indicates the possible presence of others.

IV. This is theorem II with the removal from the hypothesis of the condition that n shall be odd.

This theorem has been verified for 285 cases of values of $D_n(\alpha, \beta)$, fifty-five of the verifications being for even values of n.

A theorem implied by, but (apparently) not implying, theorem IV is the following:

V. When α and β are real and the exceptional cases of theorem IV of §3 are excluded, then a group G of order $F_n(\alpha, \beta)$, $n > 2$, contains a self-conjugate subgroup whose order is of the form $q^t (t > 0)$ where q is a prime factor of $F_n(\alpha, \beta)$ but is not a factor of n.

This theorem has been verified for each of 639 cases for the order $F_n(\alpha, \beta)$. It was found indeed that for each of these cases a Sylow subgroup serves for the invariant subgroup whose existence is asserted by the theorem.

In the case of propositions III and IV the conclusion may be weakened without destroying the value of the resulting theorems, if in their weaker form it should be possible to prove them. Writing $n = 2\nu$, we may thus weaken the conclusion by allowing q to be a characteristic factor of either $p^{2\nu} - 1$ or of $p^{\nu} - 1$ in the case of theorem III and of either $D_{2\nu}$ or D_{ν} in the case of theorem IV.

No one of the foregoing five theorems has been proved, and indeed no possible method of proof is apparent. No exception has been found to any one of them. So far as I have been able to find out, some characteristics of these propositions do not appear in any of the demonstrated theorems in group theory. If this opinion is well founded, then these propositions deserve especial attention; and that is my reason for calling them to your notice. If the propositions are true a demonstration of them would establish an interesting connection between group theory and the additive theory of numbers; for here the orders of the groups are determined by additive means while the conjectured properties of the groups have to do with fundamental matters pertaining to the structure of these groups.

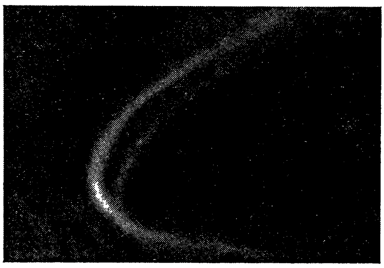
A LINE-CONIC CAMERA[1]

By L. R. FORD and G. L. LOCHER, Rice Institute

1. *Theory of the camera.* The camera to be discussed here is one in which the image of a straight line in space is a conic section. For a non-specialized line the image is a proper conic; for particular positions, however, the image may degenerate into a straight line.

The camera has no lens. It consists of a dark chamber with two walls in front. These walls are opaque except for a rectilinear slit in each through which light can pass. The light which passes through the slits falls upon a plane photographic plate lying within the chamber. The slits are skew lines.

The mathematical theory of the apparatus is simple. It is clear that a point before the camera will have a single image on the plate, for a single ray from the point meets the two skew slits. Consider a straight line l in space—the edge of a building, for example. The rays from the points of this line which reach the plate meet the two slits. They are thus the rays meeting three straight lines. It is well known that a straight line which moves so as to interest three skew lines generates a quadric surface. Hence if l is skew to both slits the rays issuing from the points of l and getting through to the plate lie on a quadric surface. And the section of the quadric surface of the rays by the plane of the plate is a conic (degenerate for particular positions of the plate). These remarks establish the principal property of the camera.



A LAMP FILAMENT

A ROW OF LIGHTS

Photographs of individual lines are shown in the first two figures. The first photograph, unlike all the others, was made with the object within the chamber and the plate outside. An incandescent lamp was put in the chamber and the image of one of its straight filaments was made. In this case a lens was used to reduce the size of the image. The secondary curve appearing in this figure is probably the reflection of the filament from the glass bulb.

2. *Details of construction of the camera.* The slits, which are made of steel knives fastened on wood discs, are each 4.5 cm. long and about .15 mm. wide. One of these is fixedly attached to the box of the camera, while the other can

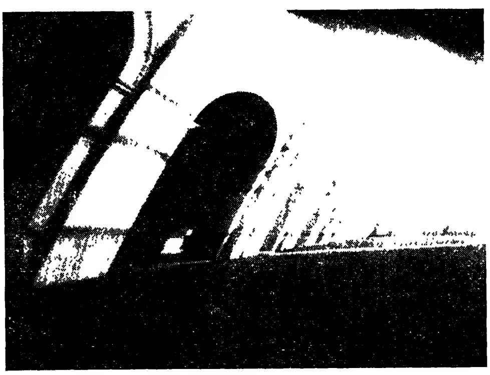
[1] Read before the Texas Section of the Mathematical Association of America, College Station. Texas, Jan. 28, 1928.

be set at any angle to the first, in a parallel plane one cm in front. Longer slits would have been advantageous, as they would have allowed a larger field of view.

The camera box is 12 by 18 by 15 cm. in size and is fitted with a removable back for loading and adjustment of the photographic plate. The plate-holder is a universal clamp which allows the plate to be held in any desired position, making any angle with either slit. The absence of lenses gives universal focus. For adjustment, a ground glass plate is used in the plate-holder and the image on it is viewed under a dark cloth.

It was found by trial that the angle between the slits giving the best images is 8° to 10°. Smaller angles give images approaching a degenerate case, while much larger angles give images like those of a pinhole camera. In our best photographs each slit was inclined at an angle of 30° to 40° to the plate. In bright sunlight, about one and one-half minutes exposure is required for lantern-slide plates.

3. *Degenerate images.* Let the slits be the skew lines m and n; and let the plane of the plate be π, not containing m or n. Let m and n meet π in the points M and N respectively. To avoid the special consideration of parallels we shall adopt the conventions of projective geometry about points at infinity.



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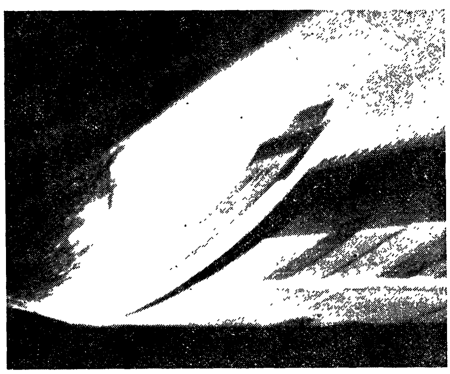

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Let l be a line whose image we are to consider, and let L be its intersection with π. We consider first those positions of l for which the image is not a proper conic.

If l meets both m and n the image is a point.

If l meets one and only one of the lines m and n the ray surface is a degenerate quadric. Let l and m meet in a point T, and let the plane σ containing l and m meet n in S. Then any ray joining a point of l to S gets through to the plate. So also does any ray from T to a point of n. The ray surface is, in theory, a pair of planes. Practically, however, the ray surface consists of the plane σ alone. For in the actual apparatus, a slit is a line segment and does not extend

to meet the actual illuminated point T on a line whose photograph is taken. The image of l is the intersection of σ and π—a straight line.

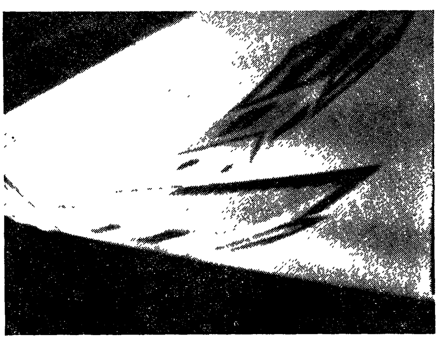
If l meets neither m nor n the ray surface is a proper quadric. The quadric is a hyperboloid of one sheet or a hyperbolic paraboloid; and l, m, n are generators of one system. Certain of the plane sections of the quadric are degenerate.

If the ray surface is a proper quadric the image is degenerate if and only if the plate is a tangent plane of the quadric.

If π is tangent to the quadric it contains the two generators through the point of tangency. The intersection is thus a pair of lines. Conversely if the image contains a line l' the complete intersection of π with the surface consists of l' and some other line l''. At the point of intersection of l' and l'', π is tangent to the quadric.

This condition can be put in a different form:

If a line does not meet either slit then its image is degenerate if and only if the line and the two slits meet the plate in collinear points.



Palmer Memorial Chapel, Houston

Administration Building, Rice Institute

If L, M, N, lie on a line l'' this line lies on the ray quadric, and is part of the intersection of π and the quadric. The remainder of the intersection is a line l'. Since the slits are not actually constructed up to their intersection with the plate the line l'' does not appear on the photograph. Conversely if the image contains a line l', the complete intersection of π and the quadric consists of l' and a second generator l''. These generators belong to different systems. The lines l, m, n do not meet and are therefore generators of one system. The generator l'' of the other system meets all of them; hence, L, M, N are collinear.

4. *Proper conic images.* Putting aside the degenerate cases we now state some theorems relative to the proper conics which appear as images of straight lines. We shall call a line non-specialized if it does not belong to one of the classes mentioned in the preceding treatment.

The image of a non-specialized straight line is a conic passing through the points where the slits meet the plane of the plate.

Through M a line can be drawn to meet the two skew lines n and l. This line is a ray of the ray surface determined by l; hence M lies on the image. Likewise N lies on the image.

The images of straight lines are thus conic sections which pass through two fixed points. From mechanical considerations these points do not lie in the photograph.

If either slit is parallel to the plate the conics extend to infinity and there are no elliptic images. If both slits are parallel to the plate all images have distinct common points at infinity. They are thus all hyperbolas with asymptotes parallel to the slits. They are similar hyperbolas.

The images of non-specialized lines lying in a plane are conics with three common points.

Let the lines lie in a plane τ; and let τ meet m and n in M' and N' respectively. Let the line $M'N'$ meet π in a point U. Then U is a point on each of the images. For, the line $M'N'$ meets both slits and meets any line l lying in τ; hence it lies on the ray quadric determined by l. Consequently U is on the image of l. All images pass thus through U, M, and N.

The point U lies on the intersection of the plane τ with the plane of the plate. It is distinct from M and N unless τ passes through one of these points.

The images of non-specialized concurrent lines are conics with three common points.

The image V of the point through which the lines pass lies on all the conics.

The images of non-specialized concurrent lines lying in a plane are conics with four common points.

The images pass through M and N and through the points U and V determined by the plane and the point on which the lines lie. For special situations some of these points may coincide.

In a photograph of a building, the images of the lines lying in one face have three common points. The images of the vertical lines, whether co-planar or not, have three common points. The images of vertical lines of one face, as in the accompanying photograph of the Physics Laboratory, have four common points

The rays which meet the two slits constitute a line congruence. There is one ray through a general point of space; that is, the congruence is of the first order There is one ray lying in a general plane in space; that is, the congruence is of the first class. The various theorems on congruences can be interpreted in terms of the properties of photographs taken with the camera.

ON A CYCLO-SYMMETRIC DIOPHANTINE EQUATION[1]

By H. A. SIMMONS, Northwestern University

1. *Introduction.* The purposes of this paper are: to exhibit a solution in positive integers of the cyclo-symmetric equation

$$\frac{1}{x_1x_2\cdots x_r}+\frac{1}{x_2x_3\cdots x_{r+1}}+\cdots+\frac{1}{x_nx_1\cdots x_{r-1}}=\frac{1}{a},\tag{1}$$

in which a, n, r are positive integers with $n\geqq r>1$ and each term of the left member involves r of the n variables $x_1, x_2, \cdots, x_n$; then, letting L stand for the left member of (1), to solve the equation

$$L=b/\{(m+1)b-1\},\tag{2}$$

when[2] $n>r>1$ and b, m are positive integers; to prove that there is a maximum number x that can appear in a solution in positive integers of (2); to point out properties of our solution of (2) which tend to convince one that it contains the maximum number x just mentioned when the variables x_i satisfy Definition 2 below; and finally to propose to the reader several problems which are suggested by the results of this paper and related literature[3]—problems which we have not been able to solve.

In our discussion we shall use the following definitions:

Definition 1. The term *solution* will mean *solution in positive integers.*

Definition 2. The order of magnitude of the integers x_i will be $x_1\leqq x_2\leqq\cdots\leqq x_n$.

Definition 3. In (2) suppose x_1 is first taken as small as possible; then with x_1 so selected, let x_2 be taken as small as possible; then with x_1, x_2 so selected, let x_3 be taken as small as possible; etc., until the first $(n-1)$ of the x's have been selected. We shall find that this method of selecting the x_i $(i=1, 2, \cdots, n-1)$ when $n>r$ gives to x_n in (2) an integral value, and we shall call the set $x_1, x_2, \cdots, x_n$ thus obtained the *principal solution*[4] of (2). Furthermore, any

[1] On May 14, 1927, the author presented to the Mathematical Association of America a paper on the case $r=2$, $a=1$ of equation (1) below.

[2] If $n=r$, (2) may not have a solution. For example, $1/(x_1x_2)+1/(x_2x_1)=3/5$ has no solution, as can be quickly shown by the method of trial and error.

[3] D. R. Curtiss, *On Kellogg's Diophantine problem*, this Monthly, vol. 29 (1922), pp. 380–387. He found the maximum value that x_n can assume in the equation $1/x_1+1/x_2+\cdots+1/x_n=1$ when $x_1, x_2, \cdots, x_{n-1}$ are positive integers and $x_1\leqq x_2\leqq\cdots\leqq x_n$, and showed how to extend his theory so as to obtain a similar result for the equation $1/x_1+1/x_2+\cdots+1/x_n=b/[(m+1)b-1]$ when b, m are positive integers and $n>r$. Tanzo Takenouchi, *On an indeterminate equation*, Proceedings of the Physico-Mathematical Society of Japan, vol. 3, No. 6, pp. 78–92. He obtained the results just mentioned by a method which is entirely different from that of Curtiss.

[4] This definition is somewhat like one which Takenouchi used in his paper referred to in footnote 3.

subset $x_1, x_2, \cdots, x_i$ $(r \leqq i < n)$ of the numbers which form the principal solution of (2) will be called a *principal set relative to* (2).

If (1) be used in place of (2) in Definition 3, it will still hold since (2) reduces to (1) when $b=1$ and $m=a$.

Definition 4. The symbol ${}_n\phi_r(x)$ will be defined by the equation

$${}_n\phi_r(x) = 1 + x_{n-r+1} + x_{n-r+1}x_{n-r+2} + \cdots + x_{n-r+1}x_{n-r+2}\cdots x_{n-1}.$$

For example, ${}_6\phi_4(x) = 1 + x_3 + x_3x_4 + x_3x_4x_5$.

2. *Inductive attack*. When $n=r$, (1) becomes

$$\frac{1}{x_1x_2\cdots x_r} + \frac{1}{x_2x_3\cdots x_rx_1} + \cdots + \frac{1}{x_rx_1\cdots x_{r-1}} = \frac{1}{a},$$

which obviously has the solution

$$x_i = 1 (i = 1, 2, \cdots, r-1), \quad x_r = ar, \tag{3}$$

a *principal solution* (c. f. Definition 3). Hereafter we shall take $n > r$.

When $r=2$, equation (1) reduces to

$$\frac{1}{x_1x_2} + \frac{1}{x_2x_3} + \cdots + \frac{1}{x_{n-1}x_n} + \frac{1}{x_nx_1} = \frac{1}{a}, \tag{4}$$

which, in the cases $n=3, 4, 5$ has (perhaps among others) the solutions below.

$n = 3$: $x_1 = 1$, $x_2 = a+1$, $x_3 = a(a+2)\cdot$

$n = 4$: $x_1 = 1$, $x_2 = x_3 = a+1$, $x_4 = a(a+1)(a+2)$.

$n = 5$: $x_1 = 1$, $x_2 = x_3 = a+1$, $x_4 = a(a+1)+1$,

$x_5 = a(a+1)^2[(a(a+1)+1)+1]$.

These solutions lead one to expect that the numbers $(x_1, x_2, \cdots, x_n) = (u_1, u_2, \cdots, u_n)$ defined by

$$\begin{aligned} &u_1 = 1, \ u_2 = a+1, \ u_i = au_1\cdots u_{i-2} + 1 (i = 3, 4, \cdots, n-1), \\ &u_n = au_1\cdots u_{n-2}(1 + u_{n-1}) \equiv au_1\cdots u_{n-2}\cdot{}_n\phi_2(u) \end{aligned} \tag{5}$$

are a solution of (4) for all integral values of $n > 2$.

Similarly when $r=3$, equation (1) reduces to

$$\frac{1}{x_1x_2x_3} + \frac{1}{x_2x_3x_4} + \cdots + \frac{1}{x_nx_1x_2} = \frac{1}{a}, \tag{6}$$

which, in the cases $n=4, 5, 6$ has (perhaps among others) the solutions below.

$n = 4$: $x_1 = x_2 = 1$, $x_3 = a+1$, $x_4 = a(a+3)$.

$n = 5$: $x_1 = x_2 = 1$, $x_3 = x_4 = a+1$, $x_5 = a[(a+1)^2 + (a+1) + 1]$.

$n = 6$: $x_1 = x_2 = 1$, $x_3 = x_4 = x_5 = a+1$, $x_6 = a(a+1)[(a+1)^2 + (a+1) + 1]$.

These solutions lead one to expect that the numbers $(x_1, x_2, \cdots, x_n) = (v_1, v_2, \cdots, v_n)$ defined by

$$\text{(7)} \quad \begin{aligned} &v_1 = v_2 = 1,\ v_3 = a+1,\ v_i = av_1 \cdots v_{i-3} + 1 (i = 4, 5, \cdots, n-1), \\ &v_n = av_1 \cdots v_{n-3}(1 + v_{n-2} + v_{n-2}v_{n-1}) \equiv av_1 \cdots v_{n-3} \cdot {}_n\phi_3(v) \end{aligned}$$

are a solution of (6) for all integral values of $n > 3$.

THEOREM 1. *The numbers* $(x_1, x_2, \cdots, x_n) = (w_1, w_2, \cdots, w_n)$ *defined by*

$$\text{(8)} \quad \begin{aligned} &\text{(8a)} \quad w_i = 1 (i = 1, 2, \cdots, r-1), \\ &\text{(8b)} \quad w_r = a + 1, \\ &\text{(8c)} \quad w_i = aw_1 \cdots w_{i-r} + 1 (i = r+1,\ r+2, \cdots, n-1), \\ &\text{(8d)} \quad w_n = aw_1 \cdots w_{n-r} \cdot {}_n\phi_r(w) \end{aligned}$$

are a solution of (1) *for all integral values of* $n > r > 1$.

This theorem is true. To prove it, take $n = k > r$ in (1); substitute the w_i $(i = 1, 2, \cdots, k)$ of (8) into (1); denote by C the sum of the first $(k-r)$ resulting terms; then, using 1 in place of the w_i of (8a) in the last r terms of (1), we have, by hypothesis,

$$\text{(9)} \quad C + \left[\frac{1}{w_{k-r+1}w_{k-r+2} \cdots w_{k-1}} + \frac{1}{w_{k-r+2}w_{k-r+3} \cdots w_{k-1}} + \cdots + \frac{1}{w_{k-1}} + 1\right]\frac{1}{w_k} = \frac{1}{a} \cdot$$

Substituting into (9) for w_k its value in terms of the $w_i (i = 1, 2, \cdots, k-1)$ as given in (8), reducing the expression in brackets of (9) to a common denominator, and then cancelling out of the numerator and denominator of the resulting quotient the common factor ${}_k\phi_r(w)$, which is the numerator of that quotient, we obtain

$$\text{(10)} \quad C + \frac{1}{aw_1 \cdots w_{k-1}} = \frac{1}{a} \cdot$$

Now taking $n = k+1$ in (1) and substituting the w_i $(i = 1, 2, \cdots, k+1)$ of (8) into (1), and using 1 in place of the w_i of (8a) in the last r terms of (1), we obtain the relation which we wish to prove,

$$\text{(11)} \quad C + \frac{1}{w_{k-r+1}w_{k-r+2} \cdots w_k} + \left[\frac{1}{w_{k-r+2}w_{k-r+3} \cdots w_k} + \frac{1}{w_{k-r+3}w_{k-r+4} \cdots w_k} + \cdots + \frac{1}{w_k} + 1\right]\frac{1}{w_{k+1}} = \frac{1}{a},$$

in which $w_k = aw_1 \cdots w_{k-r} + 1$ since $n = k+1$ [c. f. (8c)]. If (11) is true, one can see from (10) and (11), after collecting the expression in brackets of (11), that

$$(12)\qquad \frac{1}{aw_1\cdots w_{k-1}}=\frac{1}{w_{k-r+1}w_{k-r+2}\cdots w_k}+\frac{{}_{k+1}\phi_r(w)}{w_{k-r+2}w_{k-r+3}\cdots w_kw_{k+1}}\,.$$

Transposing the first term in the right member of (12) to the left member we obtain

$$(13)\qquad \frac{w_k-aw_1\cdots w_{k-r}}{aw_1\cdots w_k}=\frac{{}_{k+1}\phi_r(w)}{w_{k-r+2}w_{k-r+3}\cdots w_kw_{k+1}}\,.$$

The numerator in the left member of (13) equals 1 since w_k is defined by (8*c*). Hence the solution of (13) for w_{k+1} is

$$(14)\qquad w_{k+1}=aw_1\cdots w_{k-r+1}\cdot{}_{k+1}\phi_r(w)\,.$$

As (14) is the formula which (8*d*) gives for w_n when $n=k+1$, we conclude that (11) holds; consequently Theorem 1 is proved.

3. *Generalization of Theorem* 1. For equation (2), with $n>r$, we shall now exhibit a solution which reduces to solution (8) of equation (1) when $b=1$ and $m=a$. For brevity write $a=(m+1)b-1$. Then the expression

$$E\equiv(b/a)-1/\{[a/b]+1\},$$

$[a/b]$ standing for the greatest integer in a/b, equals a fraction with a unit numerator. In the case $b=1$, $m=a$, studied above, $E=1/a-1/(a+1)=1/a(a+1)$. To generalize the work of §§ 1, 2, for $n>r$, one needs only to prove two statements. (*i*) if $n=r+1$, the numbers $(x_1, x_2, \cdots, x_{r+1})=(w_1, w_2, \cdots, w_{r+1})$ defined by $w_1=1$ $(i=1, 2, \cdots, r-1)$, $w_r=[a/b]+1=m+1$, $w_{r+1}=a(m+r)$ are a solution of (2); (*ii*) if the numbers $(x_1, x_2, \cdots, x_n=(w_1,)$ $w_2, \cdots, w_n)$ defined by

$$(15)\quad\begin{aligned}&(15a)\quad w_i=1(i=1,2,\cdots,r-1),\\&(15b)\quad w_r=m+1,\\&(15c)\quad w_i=aw_1\cdots w_{i-r}+1(i=r+1,r+2,\cdots,n-1),\\&(15d)\quad w_n=aw_1\cdots w_{n-r}\cdot{}_n\phi_r(w)\end{aligned}$$

are a solution of (2) when $n=k$, so are they when $n=k+1$. The proof of (*i*) can be made by mere substitution; that of (*ii*), by the method of §2. Thus one obtains

THEOREM 2. *If in equation* (2), *b and m are positive integers, the numbers* (15) *are a solution of* (2) *for all integral values of* $n>r>1$.

4. *Number of solutions finite.* As (2) reduces to (1) when $b=1$, $m=a$, we shall prove that the number of solutions of (1) is finite if we show that this is true of (2) for every pair of the positive integers $a=(m+1)b-1$ and b. In (2), $x_1\leqq(na/b)^{1/r}$ because $(b/na)\leqq(1/x_1x_2\cdots x_r)\leqq(1/x_1{}^r)$, (c.f. Definition 2). Hence there are only a finite number, n_1, of choices of x_1. If one of

these values of x_1 be substituted into (2), an equation in $x_2, x_3, \ldots, x_n$ will be obtained in which there are only a finite number of choices of x_2. Therefore in all of the n_1 equations obtained from (2) by substituting into it one-by-one the n_1 choices of x_1, there are only a finite number, n_2, of choices of x_2; similarly, by considering n_1n_2 equations in $x_3, x_4, \cdots, x_n$, one finds that there are only a finite number of choices of x_3; etc. for $x_4, x_5, \cdots, x_n$. This proof, the first paragraph of § 2, and footnote 2, altogether, give

THEOREM 3. *The number of solutions of* (2) *is finite. If* $n=r$, (2) *has a principal solution when* $b=1$, *and does not always have a solution when* $b>1$.

From Theorem 3 the following corollary is obvious.

COROLLARY 1. *There exists a maximum number* x *that can appear in a solution of* (2).

5. *Properties of solution* (15). For solution (15), properties (I) and (II), which we state and prove below, are of special interest because (I) is an analog of a property which Takenouchi[1] used much in finding the maximum number x that can appear in a solution of the equation $1/x_1+1/x_2+\cdots+1/x_n=b/[(m+1)b-1]$, while (II) is an analog of the initial inequality in a chain of inequalities which Curtiss[1] used in finding the same x.

(I). *The numbers* (15) *are the principal solution of* (2) *when* $n>r$.

(II). *In* (15), $x_n\equiv w_n$ *has attained its functional upper bound.*

Proof of (I). When $n=r+1$, the numbers (15),

$$w_i = 1(i = 1,2, \cdots, r-1), \quad w_r = m+1, \quad w_{r+1} = a(m+r),$$

are obviously a principal solution of (2). Suppose the numbers (15) are a principal solution of (2) when $n=k$, k being any integer $>r>1$. Then, by hypothesis, $w_1, w_2, \cdots, w_{k-1}$ are a principal set relative to (2). We wish to prove that when $n=k+1$ the numbers (15) are the principal solution of (2). It will be sufficient to show that the numbers $w_1, w_2, \cdots w_k$ defined by (15a), (15b), (15c) are a principal set relative to (2). Obviously $w_1, w_2, \cdots, w_{k-1}$ are a principal set relative to (2) when $n=k+1$, since they are such a set when $n=k$. What we need to prove is that if $w_1, w_2, \cdots, w_{k-1}, x_k$ are a principal set relative to (2) when $n=k+1$, then $x_k=w_k$. Considering only such numbers x_i $(i=1, 2, \cdots, n)$ as satisfy Definition 2 and equation (2), we define the symbol $f_p(x)$ (>0) by the equation

$$(16) \quad \frac{1}{f_p(x)} = \frac{b}{a} - \frac{1}{x_1x_2\cdots x_r} - \cdots - \frac{1}{x_{p-r+1}x_{p-r+2}\cdots x_p}, \quad r \leqq p \leqq n-1.$$

Consequently

[1] Cf. footnote 3.

$$x_p \geqq \frac{f_{p-1}(x)}{x_{p-r+1}x_{p-r+2}\cdots x_{p-1}} + 1.$$

Since this inequality holds for every set of the $x_i (i=1, 2, \cdots, p)$ which satisfy Definition 2 and belong to a solution of equation (2), we have

$$(17) \qquad x_p \geqq \frac{f_{p-1}(w)}{w_{p-r+1}w_{p-r+2}\cdots w_{p-1}} + 1,$$

where the w's are defined by (15). It will follow now that

$$(18) \qquad x_p \geqq aw_1 \cdots w_{p-r} + 1 (r \leqq p \leqq n-1)$$

if it can be shown that $f_{p-1}(w) = aw_1 \cdots w_{p-1}$. This fact we prove in

Lemma 1. $f_q(w) = aw_1 \cdots w_q$ if $r \leqq q \leqq n-1$.

Proof. When $q=r$, we have, by (16),

$$\frac{1}{f_r(w)} = \frac{b}{a} - \frac{1}{w_1w_2\cdots w_r} = \frac{b}{a} - \frac{1}{m+1} = \frac{b}{(m+1)b-1} - \frac{1}{m+1} = \frac{1}{a(m+1)},$$

so that $f_r(w) = a(m+1) = aw_1 \cdots w_r$, and Lemma 1 is true. Suppose it is true when $q=s$, where s is an integer such that $r \leqq s \leqq n-2$; then, by hypothesis, $f_s(w) = aw_1 \cdots w_s$. To prove that $f_{s+1}(w) = aw_1 \cdots w_{s+1}$, observe that, by (16),

$$\frac{1}{f_{s+1}(w)} = \frac{1}{f_s(w)} - \frac{1}{w_{s-r+2}w_{s-r+3}\cdots w_{s+1}} = \frac{w_{s-r+2}w_{s-r+3}\cdots w_{s+1} - f_s(w)}{w_{s-r+2}w_{s-r+3}\cdots w_{s+1}f_s(w)}.$$

After replacing $f_s(w)$ in the last fraction by its value, $aw_1 \cdots w_s$, and cancelling from the numerator and denominator the factor $w_{s-r+2}\, w_{s-r+3} \cdots w_s$, we obtain

$$(19) \qquad \frac{1}{f_{s+1}(w)} = \frac{w_{s+1} - aw_1\cdots w_{s-r+1}}{aw_1\cdots w_s w_{s+1}}.$$

Since $w_{s+1} = aw_1 \cdots w_{s-r+1} + 1$, by hypothesis [c. f.(15)], the solution of (19) for $f_{s+1}(w)$ is $f_{s+1}(w) = aw_1 \cdots w_{s+1}$, and Lemma 1 is true.

Application of Lemma 1 in (17) now shows that (18) is true. If we take $p = n-1 = (k+1)-1 = k$ in (18) we obtain $x_k \geqq aw_1 \cdots w_{k-r}+1$. Comparison of this relation with the value given for w_k in (15c) shows that if $w_1, w_2, \cdots, w_{k-1}, x_k$ is a principal set relative to (2), then $x_k = w_k$. Hence (15) is the principal solution of (2) when $n = k+1$, as was to be proved.

Thus far we have assumed $r>1$. We may regard

$$(20) \qquad \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = \frac{b}{(m+1)b-1}$$

as the case $r=1$ of (2). If we do so, the principal solution (15) should reduce

to Takenouchi's principal solution of (20), namely,[1] $(x_1, x_2, \cdots, x_n) = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, where

$$\alpha_1 = m + 1,\ \alpha_i = a\alpha_1 \cdots \alpha_{i-1} + 1 \equiv \alpha_{i-1}(\alpha_{i-1} - 1) + 1 (i = 2, 3, \cdots, n - 1),$$
$$\alpha_n = a\alpha_1 \cdots \alpha_{n-1} \equiv \alpha_{n-1}(\alpha_{n-1} - 1).$$

To see that this is the case, we need only to omit the w_i of (15a), whose subscripts $\geqq 1$ and $\leqq 0$, and replace ${}_n\phi_r(w) \equiv {}_n\phi_1(w)$ in (15d) by 1, since there is no subscript $\geqq n-r+1=n$ and $\leqq n-1$.

Proof of (II). Instead of using the expression given for w_n in (15d), we now find it convenient to employ an equivalent homogeneous expression involving all $w_i (i=1, 2, \cdots, n-1)$, namely,

$$(21)\quad \begin{aligned} w_n = a w_1 \cdots w_{n-r}(&w_1 w_2 \cdots w_{r-1} + w_2 w_3 \cdots w_{r-1} w_{n-r+1} \\ &+ w_3 w_4 \cdots w_{r-1} w_{n-r+1} w_{n-r+2} + \cdots + w_{r-1} w_{n-r+1} w_{n-r+2} \cdots w_{n-2} \\ &+ w_{n-r+1} w_{n-r+2} \cdots w_{n-1}). \end{aligned}$$

Denote by $\theta(w)$ the part of (21) which is in parentheses. To prove (II), we only need to show that

$$(22)\qquad x_n \leqq a x_1 \cdots x_{n-r} \cdot \theta(x)$$

when the x_i $(i=1, 2, \cdots, n)$ satisfy Definition 2 and belong to a solution of equation (2). From (16) and (2), one can easily verify that $1/f_{n-1}(x)$ equals the sum of the terms of (2) which involve x_n. After collecting these terms, then, we have

$$\frac{1}{f_{n-1}(x)} = \frac{\theta(x)}{x_1 x_2 \cdots x_{r-1} \cdot x_{n-r+1} x_{n-r+2} \cdots x_{n-1}} \cdot \frac{1}{x_n},$$

so that

$$(23)\qquad x_n = \frac{f_{n-1}(x) \cdot \theta(x)}{x_1 x_2 \cdots x_{r-1} \cdot x_{n-r+1} x_{n-r+2} \cdots x_{n-1}}.$$

From (16), one sees that $f_{n-1}(x)$ equals a fraction whose numerator is $a x_1 \cdots x_{n-1}$ and whose denominator is a positive integer. Hence $f_{n-1}(x) \leqq a x_1 \cdots x_{n-1}$. Using this inequality in (23), we obtain

$$(24)\qquad x_n \leqq a x_r \cdots x_{n-r} \cdot \theta(x),$$

in which x_r is not one of the numbers $x_{n-r+1}, x_{n-r+2}, \cdots, x_{n-1}$; that is, $r \leqq n-r$. Furthermore it is to be understood that if $r = n-r$, x_r occurs exactly once in the coefficient of $\theta(x)$ in (24). To see that (24) implies (22) now, we only need to observe that $x_1 x_2 \cdots x_{r-1} \geqq 1$. Hence (II) is true.

6. *Unsolved problems.* Properties (I), (II), and Corollary 1, together with the results mentioned in footnote 3 lead one to expect an affirmative answer

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[1] See p. 81 of Takenouchi's article referred to in footnote 3.

to the first question below. For several particular pairs of values of n and r, we have taken $b=m=1$ in (2) and obtained an affirmative answer. All evidence which we have collected also favors an affirmative answer to the last two problems. We invite the reader to answer the following questions.

Problem 1. If $n>r$, is the w_n of (15d) the maximum number that can appear in a solution of (2)?

Problem 2. If $n>r$, is $w_1w_2 \cdots w_n$ the maximum product of any set of n numbers which constitute a solution of (2)?

Problem 3. If $n>r$, is $w_1+w_2+\cdots+w_n$ the maximum sum of any set of n numbers which constitute a solution of (2)?

SYNTHETIC MUSICAL SCALES

By J. MURRAY BARBOUR, Wells College

1. *Busoni's problem.* In his little book, *A New Esthetic of Music,*[1] Ferrucio Busoni describes a method of forming musical scales by raising or lowering various tones of the scale of C major. By this method he has obtained 113 scales, the majority of which differ from the ordinary major and minor scales in having their intervals differently arranged. Apparently his results have not been questioned since his book was published, for two rather recent works[2] accept them as authoritative.

A serious objection to Busoni's scheme is that, in accordance with the usual method of notation and with the conception of a seven-tone scale on successive alphabetical degrees, his octave contains twenty-one different tones instead of the twelve that belong to our system of enharmonic temperament on the piano. This would seem to be an unnecessary complication—and restriction—in a proposal that is otherwise so novel.

The question also arises as to whether it is proper to include all the cyclic permutations of any given scale or whether a certain arrangement of intervals should be counted only once, irrespective of the point at which the series begins. From the modern tonal view-point, all of the medieval church modes are variants of the scale of C major. Should one look at these new scales according to the medieval or the modern standard?

2. *The harp, as basis.* A good way to avoid both the difficulties mentioned above is to restate the problem in terms of the harp. The octave of the harp contains the twenty-one tones, of which only seven can be used at any one time. On the harp it is literally impossible to form a scale containing the tones $A\flat$, A, $A\sharp$; on the piano there is no practical reason why these tones should not

[1] Ferrucio Busoni, *Entwurf einer neuen Aesthetik der Tonkunst,* (1907), translated by Dr. Th. Baker, (1911), pp. 29–30.

[2] George Dyson, *The New Music,* 2nd edition (1926), pp. 94–5–7. Albert A. Stanley, *Greek Themes in Modern Musical Settings,* (1924), introduction, p. xiii.

occur in a scale empirically formed, instead of the enharmonically equivalent $G\sharp$, A, $B\flat$ that would be written in one of Busoni's scales.

The difficulty of cyclic permutations is also eliminated by confining our attention to the harp. One simply asks in how many ways a harp can be tuned. The harp has seven pedals, each controlling all of the strings of a particular letter. The "natural" key of the harp is $C\flat$ major. The C pedal will raise all of the $C\flat$ strings either a semitone or a tone to C or $C\sharp$. The other pedals operate similarly. Thus each string may produce one of three sounds, and the total number of ways the harp may be tuned is 3^7 or 2187.

Although this answer is correct, the number given above includes many tunings that, aurally considered, are not scales but chords. Often a composer desires the glissando of the harp to sound like a seventh chord, arpeggiated with extreme rapidity. Rimsky-Korsakow, for example, in his orchestral suite "Scheherazade" uses the following tunings: $C\flat, D, E\sharp, F, G\sharp, A\flat, B$; $C, D\sharp, E\flat, F\sharp, G\sharp, A\flat, B\sharp$; $C, D\sharp, E\flat, F, G\sharp, A\flat, B\sharp$; $C, D\flat, E\flat, F\sharp, G, A, B\flat$. Of these, the first two are chords of the diminished seventh; the third is a chord of the "added sixth"; and only the fourth, which, however, is typical of many in the composition, might properly be called a scale.

In accordance with the customary definition of the term "scale" we must rule out such a case of tonal duplication as $G\sharp - A\flat$ and the rarer ones of tonal overlapping, $B\sharp - C\flat$ and $E\sharp - F\flat$. These restrictions are sufficiently obvious and reasonable.

The ordinary major and minor scales contain minor seconds, $(C - D\flat)$, major seconds, $(C - D)$, and augmented seconds, $(C - D\sharp)$. One of Busoni's examples contains the doubly augmented second, $F\flat - G\sharp$. In this paper, therefore, the interval of four semitones is included with the more familiar types.

3. *Method and results.* The method used here to compute the number of scale-tunings is exemplified in Table 1. In it is found first the number of two-tone scales containing "inflections" ($\sharp$, $\natural$, and $\flat$) of C and B—either 1 or 0 for each of the 9 combinations, as determined by the given restrictions. For example, $C\sharp - B\sharp$, $C - B$, $C\flat - B\flat$ are possible, but $C - B\sharp$, $C\flat - B\sharp$, and $C\flat - B$ are not.

Then inflections of A are added to form three-tone scales. These are cumulative. For example, $A\sharp$ may occur in a scale with $B\sharp$ or B, and the numbers opposite $A\sharp$ are the sums of those opposite $B\sharp$ and B in each of the three respective columns.

Generally speaking, if in any column the numbers opposite the three inflections of a letter are a, b, and c, those opposite the inflections of the letter below, if it is a semitone lower, are a, $a+b$, and $a+b+c$. If it is a tone lower, the numbers are $a+b$, $a+b+c$, and $a+b+c$. The sole principle involved is the avoidance of tonal duplication and overlapping. The totals are the sums of the last three numbers in each column. The sum of the three columns, 363, is the result which we are seeking; i.e., the number of ways the harp may be tuned to form "scales."

If we start with any letter other than C, the totals of the three columns will differ, but their sum will remain constant. Therefore the number of scales that may be formed on all the twenty-one tones in the octave will be 7×363 or 2541. One may wonder if there is much enharmonic duplication of entire scales by this insistence on the separate identity of tones a "diminished second" apart. Of the 363, however, there are only four such pairs of scales, all of them well-known. They are $D\flat - C\sharp$ maj., $G\flat - F\sharp$ maj., $C\flat - B$ maj., and $D\flat - C\sharp$ min., melodic form.

Table 1

	$C\sharp$	C	$C\flat$
B	1	0	0
	1	1	0
	1	1	1
A	2	1	0
	3	2	1
	3	2	1
G	5	3	1
	8	5	2
	8	5	2
F	13	8	3
	21	13	5
	21	13	5
E	13	8	3
	34	21	8
	55	34	13
D	47	29	11
	102	63	24
	0	63	24
Totals	149	155	59

Table 2

Number of scale-tunings for each tone

$B\sharp$	57	$F\flat$
$E\sharp$	59	$G\flat$
$A\sharp$	81	$G\flat$
$D\sharp$	87	$D\flat$
$G\sharp$	105	$A\flat$
$C\sharp$	149	$E\flat$
$F\sharp$	153	$B\flat$
B	153	F
E	155	C
A	177	G
D	189	D

By a method similar to that illustrated in Table 1, the number of scales that do not contain the doubly augmented second can be found for any tone. For C it is 134. Thus Busoni's figure of 113 is somewhat less than the correct number of scales without the uncommon interval and much less than the number of scales (155) formed in accordance with his declared method. Evidently he followed no scientific plan, but was content with writing down the entire 113,—a task indeed. Oddly enough, this number is not far from one third the sum of the three inflections of any letter: $363 \div 3 = 121$.

4. *Applications to number theory.* If the numbers representing possible scales for the different degrees are put in order, Table 2 results. It is interesting to observe that each number in it can be represented by the equation $s = 11a + 13b$.

This table, to a musician, is the most remarkable and interesting feature of the entire discussion. Each tone is a perfect fifth distant from the tone next

above or below it in the table. Considered as the tonic or key-note of a major scale, B (at the top of the left-hand column) would have a key signature of 12 sharps. For each tone below there would be one less sharp in the signature. D has two sharps, C none, and each tone above C in the right-hand column has a signature of one additional flat. The whole is often termed by musical theorists "the circle of fifths" and it is then written in the form of a circle by using the enharmonic coincidence of $G\flat$ and $F\sharp$ and omitting the tones beyond them.

Table 3

$B\sharp$	3	8	21	39	96	57
$E\sharp$	3	8	15	37	96	59
$A\sharp$	3	8	15	37	96	81
$D\sharp$	3	6	15	39	102	87
$G\sharp$	3	6	15	39	72	105
$C\sharp$	3	8	21	55	102	149
$F\sharp$	3	8	21	39	96	153
B	3	8	21	39	96	153
E	3	8	15	37	96	155
A	3	8	15	37	96	177
D	3	6	15	39	102	189
G	3	6	15	39	72	177
C	2	5	13	34	63	155
F	2	5	13	24	59	153
$B\flat$	2	5	13	24	59	153
$E\flat$	2	5	9	22	57	149
$A\flat$	2	5	9	22	57	105
$D\flat$	2	3	7	18	47	87
$G\flat$	2	3	7	18	33	81
$C\flat$	1	2	5	13	24	59
$F\flat$	1	2	5	9	22	57

Table 4

Combinations of intervals	Number of permutations
1122222	$\frac{7!}{5!2!} = 21$
1112223	$\frac{7!}{3!3!} = 140$
1111233	$\frac{7!}{4!2!} = 105$
1111224	$\frac{7!}{4!2!} = 105$
1111134	$\frac{7!}{5!} = 42$
Total	413

Table 3 is formed by adding by threes the numbers in the columns of Table 1, for all the twenty-one tones, arranged as in Table 2. It will be seen that S_1 (the first term of a series) is 3 from $B\sharp$ to C, 2 from C to $C\flat$, and 1 from $C\flat$ to the end of the table. In general, following a semitone (as F to E), $S_n = 3(S_{n-1} - S_{n-2})$; following a whole tone (as G to F), $S_n = 3S_{n-1} - S_{n-2}$. Also, for the first 9 tones in the table, $S_6 = 2S_5 - 3S_4$, after a semitone, and $S_6 = 2S_5 - S_4$, after a tone. We must assume S_0 to be 1 in every case.

The series $S_n = 3(S_{n-1} - S_{n-2})$ is not of great interest, mathematically, except for its fluctuations and changes of sign. If the first two terms are a and b, the series is:

$$a,\ b,\ 3b-3a,\ 6b-9a,\ 9b-18a,\ 9b-27a,\ -27a,\ -27b,\ \cdots.$$

The entire series will consist of repetitions of the 6-term portion given above, multiplied each time by the constant factor (-27). The expression $(-27)^{n/6}$ is a sort of symbol of the series.

The terms of the series $S_n = 3S_{n-1} - S_{n-2}$ are a, b, $3b-a$, $8b-3a$, $21b-8a$, $\cdots$. Since a and b occur in every term, it is only necessary to derive

a formula for the coefficient series 1, 3, 8, 21, $\cdots$. These terms may be written as follows: $1=1$; $3=3$; $8=3^2-1$; $21=3^3-2\cdot 3$; $55=3^4-3\cdot 3^2+1$; $144=3^5-4\cdot 3^3+3\cdot 3$; etc. By induction

$$S_n = 3^{n-1} - (n-2)3^{n-3} + \tfrac{1}{2}(n-3)(n-4)3^{n-5} - \tfrac{1}{6}(n-4)(n-5)(n-6)3^{n-7} + \cdots.$$

The ath term of the above series is

$$\frac{(-1)^{a-1}(n-a)!\,3^{n-2a+1}}{(a-1)!\,(n-2a+1)!}$$

If n is even the last term is $(-1)^{(n-2)/2}(n/2)\cdot 3$. If n is odd the last term is $(-1)^{(n-1)/2}$.

This series is of the greatest importance in any generalization of the problem. We have seen that in a single column it applies whenever there is a whole tone. But when the totals of three columns are added it is necessary to have at least 3 successive whole tones before the law of the series becomes operative.

5. *Permutation of intervals.* Similar to Busoni's problem is that of finding in an octave of 12 semitones the possible combinations of the given intervals with all their permutations. This is a simple algebraic problem and its solution is clearly shown in Table 4, where the numbers in the column headed "combinations of intervals" refer to the size of the intervals: 1, minor second; 2, major second; 3, augmented second; 4, doubly augmented second.

Since, under the conditions of the Busoni problem, no more than 4 semitones may occur in succession, the 14 permutations in the fifth row in which all 5 semitones come together should be subtracted from the total in order to give the number of different scale-forms actually occurring in the scale-tunings.

With this change, it is an interesting fact that the number of permutations representing scales without an augmented second (21), those with a singly augmented second (266), and those with a doubly augmented second (399) are exactly the same as the cyclic permutations of the scales containing $B\sharp$ or $F\flat$, using these intervals. The reason therefor has not been found.

Since the figures in Table 4 include cyclic permutations, dividing by 7 will give numbers to be compared with the 363 scale-tunings. For example, there are 15 major-scale-tunings (natural, 7 sharps, and 7 flats); but of the 3 permutations of major and minor seconds in the first row of Table 4, only one (2212221) is called a major scale. At the other extreme is the 4th combination of this table, with the intervals arranged as given. Only one such scale can be so constructed; viz., $C\sharp$, D, $E\flat$, $F\flat$, $G\flat$, $A\flat$, $B\sharp$.

This shows very clearly the limitations of the Busoni method. On the piano this last-mentioned permutation of intervals might begin on any white or black key. Of course its notation would involve the use either of one letter twice and the omission of another, or else of double sharps or flats.

6. *Scales not heptatonic.* In musical composition the whole-tone scale of 6 tones is common, pentatonic scales (as on the black keys of the piano) are used in the folk-songs of certain countries, and the minor scale might very properly

be said to contain 9 tones (in A minor, $-A$, B, C, D, E, F, $F\sharp$, G, $G\sharp$). Therefore it is not making the question purely academic to pursue the line of inquiry shown in Table 4 in respect to scales containing less or more than 7 tones. The method is identical; the results appear in Table 5.

When we leave the heptatonic scale we leave also the troublesome question of notation in alphabetical sequence. However, in the 2nd column of Table 5 the four-semitone interval is still the largest one used. Since this restriction is not in accordance with the freedom of this phase of the inquiry, the size of the intervals in the 3rd column is unrestricted, making the problem the simpler one of finding the number of permutations of 11 things, taken $(n-1)$ at a time.

Table 5

Number of tones in scale	Permutations including doubly augmented second	Permutations including all intervals
2	0	11
3	1	55
4	31	165
5	155	330
6	336	462
7	413	462
8	322	330
9	165	165
10	55	55
11	11	11

One may well ask how profitable to the composer is the knowledge that he is free to select any one of thousands of hitherto unknown scales, as the foundation for his creative work. Unfortunately, the whole problem is of greater theoretical interest than of practical worth. At present, there is the most astounding license in composition; there seems to be an intuitive attempt to obtain euphony (if not harmony in the classical sense) by the combination of the most diverse tonalities; tonality itself as applied to melodies is almost a thing of the past. To compose on the basis of any artificially created scale would be to fasten on again the shackles that were slipping in Wagner's day and that were thrown off entirely in the early years of this century. The composers of this generation seem to have attained the ultimate freedom possible under the system of duodecuple division of the octave.

A NOTE ON FOUCAULT'S PENDULUM

By JAMES PIERPONT, Yale University

1. If one consults the standard treatises on dynamics relative to Foucault's pendulum, one finds that they one and all (as far as I know) make use of the force of Coriolis. In studying how a Foucault pendulum behaves in elliptic space I found myself obliged to go back to Hamilton's principle or the corresponding equations of Lagrange

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = \frac{\partial W}{\partial q}\,. \tag{1}$$

I wish in the following to treat Foucault's pendulum in Euclidean space from this standpoint. This method has two advantages. First it avoids introducing the fictitious force of Coriolis which proves a stumbling block to some students. Secondly the integration of the equation giving the rotation of the plane in which the pendulum is swinging is immediate, whereas in the other method it is rather artificial.

2. We take two sets of rectangular axes. First, a set x, y, z fixed in space and having as origin O, the center of the earth, regarded as rotating about the z-axis with constant angular velocity k. Secondly, a set ξ, η, ζ whose origin O' is the point of suspension of the pendulum; the negative ζ axis passes thru O, the positive ξ, η axes point south and east respectively. Let the coördinates of O' relative to the x, y, z axes be

$$a = r\cos\phi\cos\theta, \qquad b = r\cos\phi\sin\theta, \qquad c = r\sin\phi.$$

The relation between the two systems of coördinates is given by the table

(2)

	ξ	η	ζ
$x-a$	$\sin\phi\cos\theta$	$-\sin\theta$	$\cos\phi\cos\theta$
$y-b$	$\sin\phi\sin\theta$	$\cos\theta$	$\cos\phi\sin\theta$
$z-c$	$-\cos\phi$	0	$\sin\phi$

Let the ξ, η, ζ coördinates of the bob of the pendulum be

$$\xi = l\sin\psi\cos\omega, \qquad \eta = l\sin\psi\sin\omega, \qquad \zeta = -l\cos\psi. \tag{3}$$

Then ψ measures the deviation from the vertical, while ω is the azimuth of the plane of vibration.

The velocity of the bob, regarded as a particle, is given by $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$. The table (2) gives x, y, z in terms of ξ, η, ζ and these are given in terms of ψ,ω by (3). Setting $k = \dot{\theta}$ we have

$$\begin{aligned} v^2 = {} & k^2\{l^2\sin^2\psi\sin^2\omega + (r\cos\phi + l\sin\psi\sin\phi\cos\omega - l\cos\phi\cos\psi)^2\} \\ & + l^2\dot{\psi}^2 + l^2\sin^2\psi\cdot\dot{\omega}^2 + 2k\dot{\psi}\{lr\cos\phi\cos\psi\sin\omega - l^2\cos\phi\sin\omega\} \\ & + 2k\dot{\omega}\{lr\cos\phi\sin\psi\cos\omega + l^2\sin\phi\sin^2\psi - l^2\cos\phi\sin\psi\cos\psi\cos\omega\}. \end{aligned} \tag{4}$$

The kinetic energy is $T=\frac{1}{2}mv.^2$

The equation (1) for $q=\omega$, is, since there is no force along ω,

$$(5)\quad l^2\frac{d}{dt}(\sin^2\psi\dot{\omega}) + kl^2\sin\phi\frac{d}{dt}(\sin^2\psi) = k^2l^2\cos\omega\sin\omega\sin^2\psi$$

$$- k^2(r\cos\phi + l\sin\phi\cos\omega\sin\psi - l\cos\phi)(l\sin\phi\sin\omega\sin\psi).$$

In Foucault's experiment in the Pantheon $l=6, 7 . 10^3$cm. As $r=6, 4 . 10^8$cm, and $k=\dot{\theta}=7{,}3.10^{-5}$, we have

$$k^2r = 3{,}4, \qquad k^2r/2l = 2{,}5\cdot 10^{-4}.$$

If we suppose the vibrations of the pendulum to be small, we may take $\sin\psi=\psi$. Thus neglecting small quantities of higher order, (5) becomes

$$(6)\qquad \dot{\psi}(\dot{\omega} + k\sin\phi) + \tfrac{1}{2}\psi\ddot{\omega} = 0,$$

an equation satisfied by

$$(7)\qquad \dot{\omega} = - k\sin\phi$$

as $\dot{\omega}$ is now a constant. The solution (7) is that found by the usual method.

We may carry the solution of (5) a little farther. At the end of each swing of the pendulum $\dot{\psi}=0$, and is very small near such positions. When $\dot{\psi}=0$, (5) becomes, neglecting small quantities of higher order,

$$(8)\qquad \ddot{\omega} = - (k^2r/l\psi)\cdot\sin\phi\cos\phi\sin\omega.$$

If we suppose that the bob has a swing of 1 meter on either side, the maximum numerical value of the right side of (8) is 0, 034. Thus (8) indicates a small positive or negative speeding up of the plane of oscillation of the pendulum. As however ψ changes its sign each time the pendulum passes the vertical, these changes in the velocity $\dot{\omega}$ destroy each other and lead us back to the average value given in (7).

THE INAUGURATION OF THE INSTITUTE HENRI POINCARÉ IN PARIS

In November, 1928 was formally inaugurated in Paris a new Institute. It was both the official opening of a new building and the beginning of new courses of lectures, all to be a part of the Faculty of Sciences of the University of Paris.

The building is now ready but the internal arrangement and furnishing will not be ready before some time. It was however considered a good thing to hold the ceremony in the building in order to attract public attention to the opening of the lectures and to the foundation of the Institute.

It was desired to express the gratitude of the University of Paris towards those who had provided the necessary means. The history of this Institute is brief. It had been noted by the International Education Board that several opportunities had led them to give very large sums of money to different Universities in Europe and that gifts to French ones had been on a much smaller scale. Noting the importance of the French mathematical school, it was thought that helping mathematics in France was perhaps one of the best ways of helping science all over the world.

The decision was taken after consultations in which Professor Trowbridge, as representing at that time in Paris the International Education Board, and Professor Birkhoff, as a great mathematician, took decisive parts.

It was decided to ask Professor Émile Borel to draw a plan. The plan, which was approved creates under the name of "Institut Henri Poincaré" a center widely opened to teaching and researches in mathematical physics and calculus of probabilities.

The new teaching positions have been given to three men. The courses on "Physical Theories" will be delivered by Professor Léon Brillouin and M. Louis de Broglie (to be distinguished from physicists of the same names, both members of the "Académie des Sciences"). Professor Léon Brillouin has made himself known by his deep researches on the theory of quanta and its applications; and he was called last year to expound them in several universities of the United States and Canada. Dr. Louis de Broglie is the creator of those wave mechanics which, yesterday born, play a leading part in mathematical physics and was the source of many works renovating their aspects.

Those who are interested in theoretical physics will find in Paris that although this is a very important addition, there were already (existing) courses on this subject among which were those of Professor Brillouin and Professor Langevin at the Collège de France and of Professor Eugène Bloch and Professor Villat at the Sorbonne.

As to calculus of probability, it had already its great exponent at the Sorbonne in Professor Émile Borel. His researches on this subject and his personal activities have done much to revive in France the interest in this science which owes so much to French scientists such as Pascal, Fermat, Laplace, Poisson, Bienaymé, Cauchy, Cournot, Bertrand, and Henri Poincaré.

To Professor Borel's course will now be added a new course by Maurice Fréchet, formerly professor at the University of Strasbourg. His theory of abstract spaces and functions has already made him known in America where he was called to expound it at the University of Chicago in the 1924 summer quarter. But he has, of late, devoted much attention to the theory of probability on which he published (in collaboration with Professor Halbwachs) "Le calcul des probabilités à la portée de tous."

Let us also recall that the applications of probabilities to social sciences are taught in the already existing "Institut de Statistique" of the University of Paris.

But the action of the Institute Henri Poincaré will not be confined to the new courses. It aims at being international in scope. The attendance at these courses is very cosmopolite indeed. But the Institut will also have an international staff of lecturers. In addition to the standing courses, single lectures or brief series of lectures will be given by distinguished scientists. Professors Vito Volterra of Rome and de Donder of Bruxelles have already promised their coöperation; other engagements will soon be announced.

Finally, as the ever increasing numbers of lecturers and students at the Sorbonne called for new measures, it was decided to seize upon the opportunity and erect a new building where not only the new courses but all the advanced courses in mathematics will be given and where the mathematical library will be moved. The International Education Board is to contribute one hundred thousand dollars to these expenses; Baron Edmond de Rotschild has contributed twenty five thousand dollars and the French Ministry for Education, three hundred thousand francs.

It is to be hoped that among those students and scholars who would like to complete their scientific education or to go on with their researches in Europe, some will remember that, thanks chiefly to American generosity, a great scientific international center for mathematical physics and calculus of probability has been created in Paris.

RECENT PUBLICATIONS

Edited by Roger A. Johnson, Hunter College, New York, N. Y., to whom books and communications should be sent.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Algebra for Secondary Schools, Based on the Worded Problem. By Stephen Emery and Eva E. Jeffs. D. Van Nostrand Company, 1928. 626 pages. $1.85.

This book certainly carries out the promise of its sub-title and is unique in the number of worded problems. The first 208 pages contain 3019 problems, 2676 of which are grouped under such headings as work, motion, frames and borders, percentage, geometrical figures, etc. They are in the main well selected and well graded and in this respect this book will be a treasure trove to teachers of mathematics, who are always looking for extra problems. Every teacher will want a copy for personal use.

The development and explanations have been prepared with meticulous care and a mature student should be able to obtain a good knowledge of the subject from this book without the aid of a teacher. The aim here is to teach the student to "use the book and free much of the class room period for the teachers

inspiration and guidance in other important ways." Only experience will demonstrate whether this wholly desirable aim can be realized.

There is ample material here to meet the requirements and recommendations of the College Entrance Board and the Regents of the University of New York. However, the book departs drastically from the recommendation of the National Committee that the function concept be made the keynote of mathematics. Variation receives slight treatment, though the introductory sentence in the preface, "Education is life; life is work and activity," seems full of promise. Logarithms and numerical trigonometry are not made integral parts of the book but are detached in the appendix.

There are several good chapters on graphs with some new and interesting data which has been carefully compiled.

There has apparently been no attempt made to carry out the recommendation of the National Committee that "drill in algebraic manipulation should be limited, for here we find pages on the square root of polynomials; examples in radicals involving no less then seven radical signs; nests of parentheses; the theory of exponents carried to an extreme; and the Euclidean method of finding the highest common factor. The pupil and even the inexperienced teacher will likely be bewildered by these hundreds of pages of finely printed matter. Indeed the weakness of this book is the wealth of material.

Caroline Coman

Précis d'Analyse Mathématique. By E. Lainé. Vol. I. and II., Librairie Vuibert, Paris, 1927. 231+351 pages.

This work is not a treatise going deeply into any one phase of analysis, but it gives briefly and clearly a treatment of fundamental topics with which every mathematician should be familiar. It was written for the use of candidates for the "certificat de calcul différentiel et intégral." It could perhaps be used to advantage as a basis for minimum requirements in the field of calculus for our candidates for the doctorate, with the understanding that a candidate must go further into some of the main divisions of the work. The volumes are warmly recommended to American mathematicians for consideration for this purpose. It is to be regretted that we cannot require of our candidates for a master's degree the mastery of all of the topics presented; perhaps the first volume, however, could be taken as a basis for a set of minimum requirements in the field of calculus for such candidates.

There are five main divisions of the work, as follows: Book I, Theory of functions of real variables (112 pages); Book II, Theory of analytic functions (86 pages); Book III, Theory of differential equations (62 pages); Book IV, Differential geometry (146 pages); Book V, Partial differential equations (90 pages). In addition there is a preliminary complement of algebra and analytic geometry (28 pages).

The author assumes that the reader has a knowledge of the elements of algebra, analytic geometry, and calculus. In the preliminary pages he includes

a concise treatment of elimination, algebraic plane curves, algebraic surfaces, and homography and duality. The treatment is rather brief, but clear. The essential ideas are adequately presented for simple cases, then the generalizations are stated without proofs, references to standard treatises being given for details.

The reviewer admires particularly the author's presentation of the theory of functions of real variables. The purpose is to give the reader a good working knowledge of the fundamentals of the subject as quickly as possible. The matter of selection of material is a difficult one. It is easy to bewilder the student with niceties of arguments about limit points, uniform continuity, and the like. By a wise discretion in the choice of theorems to be proved and those to be stated without proof, and by an adherence to things that are essential, the author opens up a large field to the reader in but a little more than a hundred pages. Among the topics included are Taylor's series, Fourier's series, infinite products, multiple integrals, implicit functions, functional determinants, improper and line integrals, elliptic and Eulerian integrals, free and restricted extremes of functions of several variables, calculus of variations, Green's and Stokes's theorems, and change of variables for multiple integrals. Of course these topics cannot be treated fully in the space alloted to them, but it is interesting to see how much the author has succeeded in giving with a high standard of mathematical rigor.

The book on the theory of analytic functions is also highly commendable. Assuming that the reader is familiar with the elements of the theory of complex quantities, the author quickly derives the well known Cauchy-Riemann differential equations. Writing $z=x+iy$ he shows that $u=z^m$, m being a positive integer, is analytic at every point of the finite plane. After a short discussion of infinite series he introduces the exponential and the circular functions by means of their series. He turns to multiplevalued functions, irrational functions, branch-points, and the elementary inverse functions. Then follows a chapter on integrals and the development of a function in series of Taylor and of Laurent, following traditional lines. The next chapter, on integration by the method of residues, is noteworthy for the large number of illustrative examples which it contains. The final chapter deals with integrals as functions of their upper limits, and with the inversion of integrals, ending with a very brief introduction to elliptic and hyperelliptic integrals. We find as appendices articles on analytic prolongation and analytic functions of several variables, which are very important for the subsequent treatment of differential equations.

The third book, which comes at the beginning of the second volume, is devoted to a concise treatment of ordinary differential equations, making free use of the theory of analytic functions. Preparatory to proving existence theorems for such equations, there is a discussion of dominating functions and double series, and a proof of an existence theorem for implicit functions.

The existence theorems are proved only for the simplest types of cases, but the general theorems are stated. This commendable procedure enables the reader to get most easily the kernel of the method of proof and the general result. Likewise in discussing singular solutions of the equation $F(x, y, y')=0$. The author gives a fairly complete study of the singular solutions of the equation

$$y'^2 - 2P(x,y)y' + Q(x,y) = 0,$$

and then states the facts for more general cases. In the third chapter of this book particular methods of integrating differential equations are given, with discussions of linear equations and the equations of Riccati and of Laplace. The fourth and final chapter is on the theorem of Fuchs, the details being given only for the second order equation.

The longest of the five books is that on differential geometry. It is the one which is perhaps most likely to be adversely criticized, the reason being that the author has adopted the vector notation and uses the algebra of vectors. The reviewer, however, feels that the simplifications in exposition thus obtained amply justify the procedure. The author is able in about 140 pages to cover a wide range of topics, and to do it neatly and clearly. The field treated includes an introduction to vector analysis, the theory of space curves, ruled surfaces, developable surfaces, contact, envelopes, surfaces in curvilinear coordinates, the theorem of Meusnier, asymptotic lines, conjugate nets, lines of curvature, geodesic lines, evolute of a surface, the formula of Ossian Bonnet, line congruences, isometric surfaces, conformal representation, contact transformations, equations of Monge, line complexes, and continuous groups of transformations.

The last of the five books, which is on partial differential equations, is written by G. Bouligand. The presentation, while noticeably different in style from that of Lainé, maintains a high standard of excellence. The writer is concerned with points of view and an appreciation of the significance of results rather more than with a rigorous development of the subject. His treatment of partial differential equations as limiting cases of difference equations is very interesting. For example, the Laplace equation

$$(\partial^2 u/\partial x^2) + (\partial^2 u/\partial y^2) = 0$$

is considered as the limit as h approaches zero of the equation

$$4f(x,y) = f(x+h,y) + f(x-h,y) + f(x,y+h) + f(x,y-h).$$

In the latter equation the value of the function $f(x, y)$ at the point P (x, y) is the arithmetic mean of its values at four points surrounding P. It readily follows by purely algebraic processes that $f(x, y)$ cannot have a maximum or a minimum at an interior lattice point of a suitably defined region, and that there exists a unique solution of the equation which takes on given boundary values. From this proposition Dirichlet's principle is inferred, the author being careful to point out that the reasoning is incomplete.

Partial differential equations of the first order are discussed at some length. The presentation is in terms of vectors and the notions introduced in differential geometry; geometric intuitions are constantly called into play in developing the theory. A final chapter is devoted to an introduction to the theory of the Monge-Ampère equation.

At the end of each of the five books we find a list of exercises, the total number of them being 166. They are for the most part taken from the French examination papers. The reviewer has taken time to solve only a half dozen of them. Perhaps he was unfortunate in his sample, for one exercise contained a technical term not explained in the book and another contained an important misprint. The exercises were noticeably more difficult than would be found in most American texts. I think that an American professor, if he were to use these volumes as the basis for a course of lectures, would feel the necessity of supplying more and simpler exercises.

E. J. MOULTON

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3365. *Proposed by R. E. Gaines, University of Richmond.*

An open pan with a square bottom (a frustum of a pyramid) having a given total surface is to be made so as to have a maximum content. Find its dimensions.

3366. *Proposed by Otto Dunkel, Washington University.*

Given the two equations

$$x^{r+2} - bx^{r+1} \pm l(x - a) = 0, \qquad 0 < a < b, \qquad l > 0,$$

where r is any positive number, let r_1 and r_2 be the smallest and largest positive roots of the second equation (with the lower sign). Prove that all of the roots of the two equations except r_1 and r_2 lie within the circular ring with radii r_1 and r_2 about the origin as center.

Let $OA=a$, $OB=b$, $OR_1=r_1$, $OR_2=r_2$, and let R', R_2' be the harmonic conjugates of R_1, R_2 with respect to A and B. Show that neither equation has positive roots on the segments R_1R', R_2R_2' except the two roots of the second equation r_1, r_2.

3367. *Proposed by Harry Langman, Averne, L. I., N. Y.*

Given any triangle. On each side construct an equilateral triangle externally. The centers of these triangles determine another equilateral triangle A. Similarly an equilateral triangle B is determined by constructing the equilateral triangle internally. Show that the difference between the areas of the triangles A and B is equal to the area of the given triangle.

3368. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

About a given quadrilateral to circumscribe a rhombus similar to a given rhombus.

3369. *Proposed by J. Rosenbaum, Milford, Conn.*

Given two equilateral triangles one within the other, to construct a third equilateral triangle which shall be inscribed in the outer and circumscribed about the inner.

3370. *Proposed by Paul Wernicke, Washington, D. C.*

Write down an orthogonal transformation from rectangular Cartesian coordinates X, Y, Z to x, y, z having the same origin such that the z-axis becomes the line $X=Y=Z$ and that the y-axis lies in the plane through the Y and z axes.

3371. *Proposed by Harry Langman, New York City.*

Let ABCD be any simple quadrilateral (convex or cross) inscribed in the circle whose center is O. Let AB and DC meet in F, BC and AD in E. Let M be the midpoint of the third diagonal, EF, and MU and MV tangents at U and V. Let EU and FV meet in P; EV and FU meet in Q. Take the point G on EF so that $\angle DGF=\angle DAF$, and let AC cut OG in the point R. Let the secants GA, GB, GC, GD, cut the circle again in the points A', B', C', D', respectively. Take $OA=r$. Then prove the following:

(a) G is the Clifford point of the quadrilateral $ABCD$—i.e., the common intersection of the circles about the four possible triangles formed by the sides of the quadrilateral.

(b) The square of EF is equal to the sum of the squares of the tangents to O from E and F (Casey: *Sequel to Euclid*).

(c) The circle on EF as diameter cuts the circle O orthogonally (Casey).

(d) $AC'//BD'//A'C//B'D//EF\perp OG$, proving the theorem that the

Clifford point of an inscribed quadrilateral is the foot of the perpendicular from the center to the third diagonal.

(e) P and Q are the intersections of OG and the circle O.

(f) BD and UV pass through R, proving the theorem that the perpendicular from the center on the third diagonal of an inscribed quadrilateral passes through the intersection of the other two diagonals.

(g) If a quadrilateral be circumscribed about a circle at the vertices of an inscribed quadrilateral, the two pairs of diagonals intersect in a common point.

(h) Any obtuse-angled triangle may be the self-conjugate triangle of an inscribed quadrilateral. If the triangle be given, the center and radius of the circle are determined; but, when one side of the triangle has been chosen as the third diagonal of an inscribed quadrilateral, the quadrilateral is not thereby determined, there being one degree of freedom.

UNSOLVED PROBLEMS

Solutions are requested for the following unsolved problems proposed in 1927. The number of each problem is printed in italics, with the page number following.

1927

3233, 45; *3236*, 97; *3239*, 98; *3243*, 98; *3251*, 216; *3255*, 217; *3271*, 335; *3278*, 381; *3279*, 381; *3281*, 438; *3283*, 438; *3285*, 438; *3287*, 438; *3289*, 491; *3290*, 491; *3294*, 492; *3298*, 537.

SOLUTIONS

2928 [1921, 467].

Show that if through the end P (opposite from the origin) of the loop of (a) the folium of Descartes, $x^3+y^3=3axy$, (b) the strophoid or logocyclic curve, $x(x^2+y^2)+a(x^2-y^2)=0$, a straight line be drawn meeting the curve again in Q and R, then QR always subtends a right angle at the origin (compare *1916*, 90–92; also Basset, *Treatise on Cubic and Quartic Curves*, Cambridge, 1901, p. 82). Are these results particular cases of a general result for a certain class of cubic curves?

Solution by R. M. Mathews, West Virginia University, and Otto Dunkel, Washington University

A solution of this problem is contained in the article *Strophoidal curves and cubics* printed in this Monthly, Vol. 35 (1928), pp. 544–547.

2960 [1922, 129]. *Proposed by E. P. Lane, University of Chieago.*

When do two cones circumscribing a sphere intersect in two ellipses, and when are the planes of the ellipses perpendicular?

Solution by Rufus Crane, Ohio Wesleyan University

Let the center of the sphere be O; let the two cones have vertices V_1 and V_2; and let a plane through V_1, V_2, and O cut the two cones and the sphere in the

lines V_1A, V_1B, and V_2C, V_2D and the circle $ACBD$. Let the vertical angle of the cone at V_1 be 2α, at V_2 be 2β. Two cones circumscribing a sphere intersect along a degenerate space quartic consisting of two conics. If the vertex of each cone is external to the other cone, one of these two conics is an ellipse, while the other is either a hyperbola, parabola, or ellipse, according as the angle V_1OV_2 is less than, equal to, or greater than $\alpha+\beta$. (See MacCord's *Elements of Descriptive Geometry*, pp. 135, 136, for special cases.)

The lines V_1A, V_1B, V_2C, V_2D form a quadrilateral circumscribed to the circle (O), two of the diagonals of which are the orthogonal projection of the curve of intersection of the two cones. The diagonal line triangle of this quadrilateral is also the diagonal point triangle of the inscribed quadrangle $ACBD$, and is self polar with respect to the circle (O). Now, in order that a triangle, self polar with respect to a real, non degenerate circle, shall have a right angle at one of its vertices, one of its other vertices must lie at infinity. Hence, either $CB//AD$ or $AC//BD$. In either case, $AB=CD$. Hence, the required condition that the planes shall be perpendicular is that the bases of the cones shall be equal, i.e., that the cones shall be congruent. This condition is easily seen to be necessary and sufficient.

Note by the Editors: With the notation above let AOA', BOB' be two diameters of the circle (O), and $A'V'$, $B'V_1'$, the two tangents at A', B', meeting in V_1' and cutting V_1B, V_1A in M, L, respectively. The sides of the rhombus $V_1LV'M$ produced divide the part of the plane exterior to (O) into twelve regions which may be grouped as follows: I. The region of the interior angle at V_1 bounded by the arc AB, the region of exterior angle at V_1 vertical to the first angle, and the two similar regions at V_1'. II. The regions of the exterior angles at L, M. III. The regions of the interior angles at these two points. IV. The four infinite strips with bases V_1M, MV', $V_1'L$, LV_1. If V_2 lies within the regions of I or II, the intersections of the two cones are two ellipses; if within III, two hyperbolas; if within IV, an ellipse and an hyperbola.

If V_2 lies on the straight line $V_1'M$ or $V_1'L$ produced both ways, one intersection is in general a parabola. It will suffice to consider the line $\infty MA'V_1'\infty$. At V_1' the two intersections are an ellipse (circle), ellipse (infinite circle); at ∞, an ellipse, a degenerate parabola. At all other points one intersection is a parabola, while within the segments ∞M, $A'V_1'$, $V_1'\infty$, the other intersection is an ellipse; at M, a degenerate parabola; within MA', an hyperbola.

If V_2 lies on the element of the cone V_1M or V_1L, one intersection is a degenerate parabola. It will suffice to consider $\infty V_1BM\infty$. If V_2 lies at ∞ or within the segments ∞V_1, V_1B, $M\infty$, the other intersection is an ellipse; if within BM, an hyperbola; at M, a parabola. The cases in which V_2 lies at A, B, A', B' or at any point on (O) may be disregarded since the cone reduces then to a plane.

this case there are three distinct real solutions of the problem. If the expression is less than zero, the cubic has three distinct real roots, and there are three distinct pairs of real straight lines. In this case there are four distinct real solutions of the problem. If finally the expression is greater than zero, the cubic has two imaginary roots and there are two pairs of imaginary straight lines. In this case there are only two distinct real solutions of the problem.

Only in special cases may these solutions be constructed by ruler and compass; for the cubic in (4) has no linear factor with rational coefficients in a and b if their values are unrestricted. From the above discussion of (4) we see that, if $a-b=0$, one root is $\lambda=-1$; if $a+b=0$, one root is $\lambda=1$. In these simple special cases the solutions may be constructed by ruler and compass. Other special cases may be easily obtained by placing suitable restrictions upon a and b. After having obtained a root λ, the values of t and u may be obtained by use of the resulting linear equation with one of the equations in (1). This method cannot, however, be recommended for numerical computation. It is much easier to use Horner's method for the determination of, say, t from the equation obtained from (1)

$$t^4 - 2at^3 + (a^2 + b^2 - 1)t^2 + 2at - a^2 = 0.$$

If we take $a=2$, $b=1$, we obtain a case which cannot be constructed with ruler and compass. Here $a^2+b^2>1$ and there are only two real solutions given by the two real values of $t=.77473$, $-.94697$.

3206 [1926, 385]. *Proposed by D. H. Lehmer, University of California.*

Prove the following theorems and show how they may be used in finding the factors of R;

Theorem 1: Let R be a non-square integer of the form $8n+k$ and let $2^p(2m+1)$ be any even denominator of a complete quotient occurring in the expansion of $R^{1/2}$ in a continued fraction, then, if $k=1$, $p\geqq 3$; if $k=4$ or 0, $p\geqq 2$; if $k=5$, $p=2$; if $k=2$, 3, 6, or 7, $p=1$.

Theorem 2: If R contains a square factor, k^2, then every multiple of k appearing as a denominator of a complete quotient must also contain a factor k^2.

Solution by the Proposer.

This problem is the same as 3194 [3182; 1926, 278] by the same proposer. A solution by the proposer was printed in Vol. 34 (1927), p. 381.

3301 [1927, 538]. *Proposed by J. B. Reynolds, Lehigh University.*

From the corners of a square sheet of tin are cut out quadrilaterals so that when the sides are turned up the pan formed will have a maximum volume. Find the top and bottom dimensions of the pan.

Solution by Emma M. Gibson, Central High School, Springfield, Mo.

Let y be a side of the lower base and x a side of the upper base of a square pan which has been made from a square sheet of tin of side a. Then the altitude h of the pan is given by $4h^2=(a-y)^2-(x-y)^2$, and its contents V is given by

$$6V = [(a-y)^2-(x-y)^2]^{1/2}[x^2+y^2+xy].$$

The partial derivatives V_x, V_y are given by

$$12hV_x = (2x+y)[(a-y)^2-(x-y)^2]-(x-y)[x^2+y^2+xy],$$

$$12hV_y = (2y+x)[(a-y)^2-(x-y)^2]-(a-x)[x^2+y^2+xy].$$

We are to determine the values of x and y which make $V_x=V_y=0$, and in doing this we may discard the values $x=y=0$ and those values of x and y for which h is zero. we then find

$$(x-y)/(a-x) = (2x+y)/(2y+x),$$

and from a transformation of this equation we have

$$(a-y)/(x-y) = 3(x+y)/(2x+y).$$

Substituting the value of $(a-y)$ obtained from this equation in $V_x=0$, making certain reductions, and then setting $x/y=r$, we find

$$r^3+2r^2-3r-3=0.$$

This equation has only one positive root $r=1.46$; the negative roots must be discarded. Hence $x=1.46y$, $y=.54a$, $x=.78a$; and these values give the pan of maximum content.

Also solved by R. E. Gaines, J. Q. McNatt and A. Pelletier. The results obtained in a different way by R. E. Gaines are $h=.19672a$, $x=.78230a$, $y=.53564a$, $V=.08642a^3$.

Note by Otto Dunkel, Washington University

It remains to prove that the volume obtained is really the greatest of all possible volumes. In the solution above we may restrict x and y to a region R of the xy-plane defined by $0\leqq x\leqq a$, $0\leqq y\leqq (a+x)/2$. On two of the boundaries of R, $x=a$ and $2y=a+x$, we have $V=0$. On the other two, we have $6V=x^2(a^2-x^2)^{1/2}$ and $6V=y^2(a^2-2ay)^{1/2}$. The corresponding maxima are $3^{1/2}a^3/27$ and $5^{1/2}\,2a^3/375$, the first of which is the greater. Thus V on the boundary of R is not as great as at a point in its interior where $V=.08642a^3$. Since V is a continuous function of x and y within and on the boundary, it must attain an absolute maximum value at one or more points within or on the boundary of R. As shown above the point or points must be *within* R. Moreover V has definite partial derivatives at every point *within* R, and we know that at such points the partial derivatives must be zero when there exists a maximum at any given one. Hence there exists a point or points within R where V attains

an absolute maximum value and at such points the partial derivatives are zero. But as found in the above solution there is only one point within R where the two partial derivatives are zero, and hence it must be at this point that the absolute maximum exists. This completes the proof.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor Arnold Emch, who is on leave of absence from the University of Illinois, has been invited by Professor Fueter, editor-in-chief of the new Swiss mathematical journal, "Commentarii Mathematici Helvetici," to send a contribution to that journal. In September Professor Emch read a paper before the International Mathematical Congress at Bologna, Italy, on "Finite groups and their geometric representations," and on January 8, he read a paper (by invitation) on "Cremona transformations and algebraic curves" before the mathematical colloquium of the University of Zurich and the Eidgenössische Technische Hochschule.

Professor R. E. Moritz, head of the department of mathematics of the University of Washington, is on leave of absence for the winter and spring quarters of 1929. He is making a tour of the world, proceeding from Seattle to Japan, China, Dutch Indies, Straits Settlements, India, Suez Canal, Southern Europe, France, and back to New York and Seattle in time for the fall quarter. Professor A. F. Carpenter is acting as department head during Professor Moritz's absence.

Dr. C. M. Cramlet will return to the University of Washington for the academic year 1929–30 after a two years' stay at Princeton, where he has held a National Research Fellowship in mathematics. Miss Hermance Mullemeister has been promoted to an assistant professorship of mathematics at the University of Washington.

Mr. A. O. Hickson, of Brown University, and Mr. E. R. C. Miler, of Rice Institute, have been elected assistant professors of mathematics at Duke University, effective September, 1929.

Mr. J. G. Chaney and Mr. Dan Hall have been made acting instructors of mathematics at the Agricultural and Mechanical College of Texas.

The following courses in mathematics are announced for the summer of 1929:

University of Colorado, first term, June 18 to July 21; second term, July 23 to August 24. In addition to the usual elementary work in algebra, trigonometry, analytic geometry, and calculus, the following courses will be offered. First term—By Professor Light: Teachers' course in mathematics; History of mathe-

matics; Partial differential equations. By Professor Kempner: Algebraic analysis; Projective geometry; Differential equations. Second term—By Professor Light: Statistics; Teachers' course (repeated); Partial differential equations (continued). By Professor Kempner: Projective geometry (continued); Differential equations (continued). By Professor Kendall: Theory of equations.

Columbia University, July 8 to August 16. In addition to courses in trigonometry, solid geometry, college algebra, analytic geometry, and calculus, and a series of courses for teachers of secondary mathematics, the following advanced courses are offered. By Professor G. D. Birkhoff: Mathematical elements of art; Introduction to relativity. By Professor W. B. Fite: Differential equations. By Professor J. F. Ritt: Theory of numbers. By Professor B. O. Koopman: Functions of a real variable.

Cornell University, July 6 to August 16. In addition to the usual elementary work, the following advanced courses will be offered. By Professor Virgil Snyder: Teachers' course; Projective geometry. By Professor F. R. Sharpe: Advanced analytic geometry. By Professor W. A. Hurwitz: Advanced calculus. By Professor W. B. Carver: Theory of numbers. By Professor C. F. Craig: Elementary differential equations. Reading and research will be directed by Professors J. I. Hutchinson, Virgil Snyder, F. R. Sharpe, W. A. Hurwitz, W. B. Carver, D. C. Gillespie, C. F. Craig, and C. F. Roos.

University of Illinois, July 17 to August 10. In addition to the usual courses in college algebra, trigonometry, analytic geomtry, and calculus, the following advanced courses are offered. By Professor G. A. Miller: The theory of numbers. By Professor R. D. Carmichael: Partial differential equations. By Assistant Professor E. B. Lytle: Teachers' course; Theory of equations and determinants. By Assistant Professor H. Levy: Geometric transformations. By Dr. V. A. Hoersch: Advanced calculus. By Dr. F. C. Ogg: Projective geometry.

University of Indiana, June 13 to August 9. In addition to the usual courses in college algebra, trigonometry, analytic geometry, and calculus, the following advanced courses are offered. By Professor S. C. Davisson: Theory of functions of a complex variable; Differential equations; Theory of equations. By Professor D. A. Rothrock: Partial differential equations; Advanced calculus. By Assistant Professor H. E. Wolfe: College geometry; Analytic mechanics. By Associate Professor C. B. Hennel: General mathematics; Analytic geometry.

University of Iowa, first term, June 8 to July 19. In addition to courses in college algebra, trigonometry, analytic geometry, and calculus, the following subjects are offered. By Dr. M. A. Nordgaard: Subject matter and teaching of mathematics. By Dr. Conkwright: Ordinary differential equations; Theory of numbers. By Professor Wylie: Celestial mechanics; Mathematics of finance; Descriptive astronomy. By Professor Ward: Modern geometry. By Professor Chittenden: Advanced calculus; Orthogonal functions. By Professor Rietz;

Actuarial theory and practice; Statistics. Second term, July 22 to August 23. By Mr. McCoy: Matrices and determinants. By Dr. Nordgaard: The history of mathematics. By Professor Ward: Modern geometry; Differential equations. By Professor Reilly: Algebra for high school teachers; Linear difference equations.

Johns Hopkins University, July 1 to August 9. In addition to elementary course, the following advanced course will be given. By Dr. John Williamson: Theory of functions of a complex variable.

University of Kansas, First term, June 12 to July 20. In addition to elementary courses in college algebra, trigonometry, analytic geometry, differential calculus, and integral calculus, the following courses are offered. By Professor C. H. Ashton: Advanced calculus; Seminar. By Professor U. G. Mitchell: Projective geometry, I; Teachers' course in mathematics; Seminar. Second term July 22 to August 17. By Professor Mitchell: Projective geometry, II; History of mathematics; Seminar.

University of Michigan, June 24 to August 16. In addition to courses in algebra, trigonometry, analytic geometry, elementary calculus, statistics, and the theory of interest and insurance, the following advanced courses are offered. By Professor J. W. Bradshaw: Higher algebra; Projective geometry. By Professor P. Field: Vector analysis; Applied mathematics, engineering problems. By Professor W. B. Ford: Advanced calculus; Infinite series with special reference to Fourier series. By Professor T. H. Hildebrandt: Theory of functions of a real variable; Partial differential equations. By Professor L. C. Karpinski: Teaching of geometry; History of mathematics. By Professor T. R. Running: Empirical formulas. By Professor H. C. Carver: Advanced mathematical theory of statistics. By Professor L. A. Hopkins: Analytic mechanics; Celestial mechanics. By Professor N. H. Anning; Differential equations. By Professor C. J. Coe: Integral equations. By Professor J. A. Nyswander: Theory of probability; Finite differences. By Professor R. L. Wilder: Foundations of mathematics By Mr. N. C. Fisk; Graphical methods. By Mr. D. K. Kazarinoff: Aerodynamics.

University of Minnesota, first term, June 18 to July 27. In addition to the usual elementary work, the following courses will be offered. By Professor Dunham Jackson: History of ancient and modern mathematics. By Assistant Professor Elizabeth Carlson: Differential equations. By Professorial Lecturer James V. Uspensky: Theory of numbers. By Professor Jackson: Fourier; Legendre, and Bessel series. By Professors Raymond Brink, Dunham Jackson, and Professorial Lecturer J. V. Uspensky: Reading in advanced mathematics. Second term, July 29 to August 31. By Professorial Lecturer J. V. Uspensky: Recent developments in the mathematical theory of probability. By Associate Professor A. L. Underhill and Professorial Lecturer J. V. Uspensky: Reading in advanced mathematics.

Ohio State University, June 18 to August 30. In addition to the usual courses in college algebra, analytic geometry, and calculus, the following courses are offered. By Professor S. E. Rasor: The teaching of mathematics; Theory of functions of a complex variable; Advanced calculus. By Professor A. D. Michal: Continuous groups; Tensor analysis. By Professor Grace Bareis: Projective geometry.

University of Pennsylvania, July 1 to August 10. In addition to elementary courses, the following advanced courses are offered. By Professor H. H. Mitchell: Galois theory of equations. By Professor J. R. Kline: Functions of a complex variable. By Professor F. D. Murnaghan, of Johns Hopkins University: Inversive geometry; Linear differential equations. By Professor J. M. Thomas: Integral invariants.

Stanford University, June 20 to August 31. In addition to the usual courses in calculus and differential equations, the following advanced courses will be given. By Professor W. A. Manning (Stanford): Group theory; Theory of functions. By Assistant Professor G. T. Whyburn (Texas): Point-set theory.

University of Wisconsin, July 1 to August 9. By Professor R. W. Babcock: Vector analysis. By Professor H. W. March: Differential equations; Definite integrals. By Professor E. B. Skinner: Differential geometry; Finite groups; Infinite series. Special nine weeks session for graduates, July 1 to August 30. By Professor M. H. Ingraham: Higher algebra; Theory of approximations. By Professor Warren Weaver: Theory of relativity; Advanced electrodynamics; Complex variable theory. (Only one of the two last named courses will be given, the choice depending upon the demand. Prospective students should communicate with the chairman of the department indicating their preference.) By Professor J. H. Van Vleck of the department of physics: Introduction to atomic theory and line spectra; Dielectric and magnetic media; Quantum mechanics and chemistry. (Only one of the two last named courses will be given, the choice depending upon the demand.)

University of Wyoming, first term, June 17 to July 24. In addition to courses in algebra, trigonometry, analytic geometry, and the elementary calculus, the following advanced courses will be offered. By Professor O. H. Rechard: Advanced integral calculus; Differential equations. Second term, July 15 to August 30. By Professor O. H. Rechard: Differential equations; Solid analytic geometry.

The Chauvenet Prize

In the year 1925, the ASSOCIATION established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a liberal cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

The retiring President of the Association, Professor W. B. Ford, has given an additional endowment for this prize whereby it will hereafter be awarded every three years. The next award, however, will be in December, 1929, for the period 1925-1928 inclusive.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

The Carus Mathematical Monographs

The CARUS MONOGRAPHS are already fulfilling their mission as intended by the generous donor, MRS. MARY HEGELER CARUS and her son, DR. EDWARD H. CARUS.

Somewhat more than one-half the members of the Association have taken advantage of the distribution at cost of the first three Monographs already published. Those who neglected to do so at the start may still have the privilege by applying to the Secretary. Each member is entitled to one copy of each Monograph at this special price.

It would be a great tribute to the donor and an honor to the Association if a large majority of the members would subscribe for the complete series.

It is believed that the Association is rendering a great service to mathematics by this enterprise, and a liberal support from the membership constitutes an appropriate vote of confidence in the undertaking.

The publication of the fourth Monograph has been delayed on account of unavoidable circumstances. It is now ready for the printer and will be announced at an early date. Still other Monographs are in preparation.

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EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Thirteenth Summer Meeting of the Association, Boulder, Colorado, August, 1929.

Fourteenth Annual Meeting, Des Moines, Iowa, December 31, 1929, January 1, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled.

ILLINOIS, Carthage, Ill., May 3-4.

INDIANA, Culver Military Academy, May 3-4.

IOWA, Fairfield, Iowa, April 26-27.

KANSAS, Topeka, Kansas, February 2.

KENTUCKY.

LOUISIANA-MISSISSIPPI.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, May 4.

MICHIGAN, Ann Arbor, Mich., March 16.

MINNESOTA, St. Paul, May 8.

MISSOURI, Kansas City, Mo., November.

NEBRASKA.

OHIO, Columbus, Ohio, April 4.

PHILADELPHIA, University of Pennsylvania, November 30.

ROCKY MOUNTAIN, Greeley, Colo., April 12-13.

SOUTHEASTERN, Macon, Ga., April 29.

SOUTHERN CALIFORNIA, University of Redlands, March 9.

TEXAS, Houston, Texas, Jan. 26.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teacher's agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to department or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

THE FUNDAMENTAL MATHEMATICAL REQUIREMENTS OF BIOLOGY[1]

By J. ARTHUR HARRIS, Department of Botany, University of Minnesota

I. *Introductory*

In coming before the Mathematical Association of America to discuss the fundamental mathematical requirements of biology, I must do so without any pretense of placing myself in the ranks of a group of scholars which I both envy and admire—the trained mathematicians.

Since some explanation of my ignorance of things purely mathematical is in order, I may say that from about ten years of age I was determined to devote my life to biology. At that time the natural sciences were divided into two great groups, the *exact* and the *descriptive*. There were many to tell me that mathematics was essential to the development of broad scholarship, and in preparation for work in astronomy, in physics and perhaps in chemistry, but none to say—and few to imagine—that mathematics might be essential in preparation for work in biology.

Having honestly disclaimed any place among a group of mathematicians, I shall remember the Latin proverb, *Ne sutor ultra crepidam*, and, cobbler that I am, shall stick to my last, speaking only as a biologist.

The writings of Karl Pearson fired the imaginations of a few of us at a period when naturalists were first gaining full consciousness of the baffling complexity of biological phenomena. Work in taxonomy had shown that

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[1] Prepared by invitation for presentation before the Mathematical Association of America at Nashville, December, 1927.

exploration had assembled only a fraction of the world's species as conceived by Linnaeus, and that the Linnaean species are themselves complex groups. Investigation in morphology, embryology, histology and even in cytology was beginning to require the undivided efforts of specialists. The pioneer writings of Warming and Schimper were diverting the attention of some students of classification, morphology and physiology to the problem of the influence of environmental factors on the distribution of organisms over the earth's surface. Research in physiology was beginning to feel the influence of physics and chemistry, but with the first result of focusing attention on the problem of vitalism rather than of emphasizing the possibility of biology's becoming an exact science.

The whilom leaders in biology were, therefore, faced on the one hand by the assertion that mathematics is an exact and final science, and on the other hand by the fact that biological phenomena are complex and extremely variable because of the influence of innumerable uncontrollable factors. They were unable to harmonize the rigid finality claimed for mathematics with the flux which they saw everywhere about them in the objective reality of biological phenomena. In consequence they asserted "Biology can never be an exact science." It was my special good fortune to have to work under opposition, which made me more determined in the position I have held for these many years—that the great task of the biologist of this generation is to place biology alongside physics and chemistry in the ranks of the exact sciences.

The establishment of biology in a place among the exact sciences may be realized only by progress along two lines.

First, precise measurement must, in so far as is practicable and desirable, replace qualitative description in biological research.

Second, mathematical methods of description and analysis of the quantitative data of observation and experimentation must be more extensively introduced into the biological sciences.

The idea that biology requires the service of mathematics is by no means new. It has been enunciated many times in the past by mathematicians themselves. But their assertions, unsupported by actual demonstrations of what mathematics can contribute to biology, instead of furthering the introduction of mathematics into the biological field, may have retarded its application by creating an attitude of antagonism on the part of biologists.

Biologists are not altogether to blame for their historical attitude toward mathematics, for the older mathematical literature had relatively little that biologists could apply to the solution of their problems. Today conditions are much improved. Through the work of Karl Pearson, and of others whom he has inspired, there is now at hand a wealth of mathematical theory which will be fruitful of biological results when it is applied to the interpretation of biological data. So much, indeed, has been accomplished in the way of development of mathematical theory especially adapted to biological research that, in my opinion, the most urgent present need in biology itself is not the

amplification of mathematical theory applicable to biological problems—important as this unquestionably is. The pressing need is for biologists themselves to apply more extensively, intelligently, and critically, theory which has already been made available to them.

History seems to demonstrate that mathematics can not be introduced into any science by mandate. It must take its place in the physical, biological and social sciences through the normal evolutionary development of these sciences themselves.

Physical and biological sciences have passed through, or are now in, periods of observation and description, of classification, of comparison, and of experimentation. Apparently all the physical and biological sciences move in their development toward a final period, not yet surpassed in any of the sciences, of mathematical description and analysis, and of the formulation of mathematical laws. It is in part because of an earlier adoption of mathematical notation and methods of reasoning that in their development physics and chemistry shot ahead of biology. It is because of the difficulty of obtaining adequate quantitative data and because of the unwillingness of many of the workers to employ quantitative methods of analysis that sociology lags far behind biology.

The evolution of physics and of chemistry to the stage of mathematical sciences has been much more rapid than has that of biology. In exoneration of the biologist, however, we must remember that the amount of descriptive work which he has had to do is enormously greater than that of the chemist or physicist, and that the problems of structure and function with which he has to deal are vastly more complex and difficult.

While most biologists were long hostile to the introduction of mathematical methods into biological research, it was inevitable that in time they would find it necessary to avail themselves of the powerful tools of mathematical theory.[1]

As I have indicated in an earlier address, mathematics has, during the past quarter of a century, been entering biology on two broad fronts.

First, biology in our generation has become an experimental science. This, in itself, represents an evolutionary advance over its purely descriptive and comparative stages. Biology entered the experimental period when physics and chemistry were already far advanced as experimental sciences. It was, therefore, only natural that the experimentation of biologists should consist very largely in the introduction into biology of the methods of the more highly developed sciences. Controlled experimentation is essentially quantitative in both methods and results. Biology has, of necessity, taken over with the experimental methods of physics and chemistry the mathematical methods of description and analysis which alone are capable of dealing with quantitatively

[1] The progress in this field is a stimulating chapter in the history of science, which cannot be given here, even in outline.

measured variables and which, in consequence, early became an integral part of these two highly developed physical sciences.

Second, the work of Quetelet, Galton and Pearson, and of others who have been associated with Pearson in the movement which has often been designated as the Biometric School, has opened up to mathematical treatment a vast group of problems. These pertain to the description and analysis of phenomena which are highly variable and which can not, in many cases, be experimentally controlled.

Our time is brief, and since no special plea is required for the mathematics which form a part of physics and chemistry as independent sciences, or of physics and chemistry as they are a part of biology, I shall limit my discussion to the needs of biology for mathematics as it may be applied in the second group of problems. In relation to the first line of advance, I will merely say in passing that, in my opinion, we shall soon reach a stage in biology at which the type of mathematics which has served physics and chemistry will be inadequate, and at which experimental results obtained under controlled conditions will themselves have to be subject to the statistical method of treatment.

II. *The Specific Mathematical Requirements of Biology*

Because the biologist deals with phenomena which are all but infinitely complex, his demands upon the mathematician may ultimately be many in number and varied in kind. The specific needs which can now be recognized must be presented in barest outline. They may be grouped under four main heads.

1. The Need for Quantitative Description

Those groups of plant and animal organisms which are sufficiently different that they can be assigned to the categories which the biologist calls species may be numbered by hundreds of thousands. Each of these groups has its own unknown evolutionary history and its own range of tolerable environmental conditions. Experimental work indicates that the great majority of these so-called species are not homogeneous entities but heterogeneous groups of sub-species, minor species, varieties, or strains—whatever one cares to call them—each of which shows some constancy of character from generation to generation. The individuals of a species may be subject during the course of their development to (and may therefore have their characteristics modified by) endless permutations of those extrinsic factors which constitute the environment. Thus the first need of biology was observation and description of the actual living things which now inhabit, and have in the past inhabited, the planet. Centuries of effort have been given to qualitative work in this field.

As biological research progresses, description becomes more, not less, essential, but the description must be to an increasing extent in terms of mathematical constants and equations. Both biologists and mathematicians have sometimes failed to realize that many of the statistical constants are, in and of themselves, simply descriptive.

While such quantitative description has in the past been largely devoted to the physical and physiological characteristics of individuals at one and the same (generally the mature) stage of development, it is evident that the course of the development of the individual and the distribution of organisms through time and space must also be described in terms of mathematical expressions. Limiting our attention for the moment to the quantitative description of masses of individuals, we may review briefly some of the accomplishments which have been made and suggest some of the needs which must still be met.

a. The Description of Type and of Deviation from Type

Suppose N variates, $x_1, x_2, x_3, \cdots x_n$, all recognized by the trained biologist as belonging to the same category, to have been measured. If all individual values of x were identical, it would be unnecessary to determine even a simple mathematical expression to represent them. The value required would be obvious by inspection. Since the values of x differ among themselves, so that the characteristics of no one of them can be accepted as a wholly valid basis of generalization, it is necessary to obtain some description of the series as a whole based on a large number of individuals.

While such values as the most frequent or modal class, the maximum and the minimum class, were widely used in earlier semiquantitative biological work, they are inadequate since they neglect a material (and often a large) fraction of the available measurements. $\bar{x}=\Sigma(x)/N$ furnishes a description of the magnitude of x based upon all of the individuals considered. If the values of x have been accurately measured, $\bar{x}$ is, within the limitations evident in the formula, a true description of the magnitudes of the individuals actually measured.

The description of x in terms of $\bar{x}$ is inadequate, since by its very nature the average obliterates certain of the most salient features of the series of variates. For biological comprehensiveness the variability of x must also be expressed. The biologist has learned to do this in terms of the "standard deviation" or "root mean square deviation."

$$\sigma_x = (\Sigma[(x - \bar{x})^2]/N)^{1/2}$$

Such quantitative descriptions of the characteristics of series of variates as $\bar{x}$, $\bar{y}$, etc., are directly comparable for samples within the same category. The values for different categories may often be rendered comparable by expressing $\bar{x}$, $\bar{y}$, etc., as ratios to some properly chosen standard values, $\overline{X}$, $\overline{Y}$. σ_x, σ_y, etc. may be directly compared, or compared after reference to their mean values, $V_x=\sigma_x/\bar{x}$, $V_y=\sigma_y/\bar{y}$, etc., in order to correct for differences in the average magnitudes of the variables.

While such measures of type and variability are among the simplest of the mathematical descriptions of the characteristics of groups of organisms, there are as yet largely undeveloped possibilities in biological research through the proper use of these constants. The problems are primarily biological.

Research for the most part requires only the use by the biologist of formulae which are already available.

It will be clear that while only the biologist can make the work of the mathematician fruitful of valuable results through the application of his constants, only the mathematician can justify the use of the root of the average squared deviation of the variates from their mean as the most practicable and generally usable measure of the amount of variability. In many cases the root mean square deviation is altogether inadequate as a full description of the biological facts. In some instances I believe it may be quite erroneously applied. Certainly the latter is true of the coefficient of variation, $V_x = \sigma_x / \bar{x}$.

Furthermore there are still undeveloped possibilities in the theory underlying these descriptive constants. One such field is the combination of the ordinary biometric theory with the theory of measurements. In biological work it is ordinarily assumed that the individual measurements represent the true values of the variates, and that the trustworthiness of the constants derived from N such values is determined solely by the magnitude of the errors of random sampling. This assumption is valid in only a very approximate way. As work of greater precision is undertaken in biology it will be necessary to take errors of measurement into account in the interpretation of results.

Finally, there are always dangers associated with methods which are so simple that they can be used in a wholly routine way. Even at the present stage of development of theory and application, it is important to keep clear of dogmatism. On the one hand, the mathematician should not be too positive in his assertion that such constants as means, standard deviations, and coefficients of variation are the best descriptions of the actual observations, because he does not know all of the special needs of biological work. On the other hand, the biologist should not be too ready to apply blindly what the mathematician may recommend. He should know something of the reasons for the recommendation given, and should assure himself that the reasons are wholly cogent for the specific case with which he has to deal.

b. The Description of Frequency Distributions

Such constants as $\bar{x}$, σ_x, V_x convey vastly more and more valuable and dependable information concerning a distribution in which frequencies are plotted as ordinates against the values of x as abscissae than any other kind of description (*e. g.*, the empirical mode and the observed minimum, maximum and range). These fundamental constants are, however, inadequate as a full portrayal of the distribution of the magnitudes of x. The frequencies may be distributed symmetrically about the mean, may be wholly skew, or may take any other conceivable form. Some description of the frequency distribution as a whole is therefore essential.

The biologist can represent empirical frequencies graphically, and can compare two or more distributions of the values of different variables or of the same variable observed under different conditions. Before much progress

can be made, however, the empirical distributions must be described or graduated by some kind of frequency curve equations. Quite obviously the mathematician and not the biologist must provide suitable equations.

Early in the history of the description of the frequency distributions of biological measurements it was assumed that in general such variates are distributed in accordance with the Gauss-Laplace or normal curve. We now know that this holds rigidly in only a fraction of the cases. The form of the frequency distribution of x, as well as its average magnitude and standard deviation, may be characteristic of the particular variable under consideration, and may be of great significance in the analysis of biological phenomena.

Two courses are open to the mathematician in attempting to describe such a distribution by an equation. *First*, he may try to obtain an equation which will give exactly the observed frequencies of x. In general this will be exceedingly laborious, and in most cases practically impossible. *Second*, he may seek to represent the results as well as they can be represented by an equation involving moment coefficients of a lower order than those required for the more accurate description of the actual facts of observation.

The proper solution of this problem is of more interest to the biologist than to the pure mathematician. The mathematician may readily provide symbolic expressions of any degree of complexity. The practicability of applying these to biological data, and the extent to which they will be used in actual biological research, will be determined very largely by the arithmetical routine required in the numerical evaluation of the equations.

Furthermore, there is a limit beyond which precision of mathematical description of biological phenomena ceases to be of biological value. The results of biological observation and experimentation are irregular because of the inevitable errors of measurement and because of the errors of random sampling from a population made up of individuals which are highly variable because they have been subjected during their development to the influence of the permutations of an unknown but generally large number of innate or extrinsic factors, which themselves may be of highly varying potency.

If the mathematician followed the first course indicated above to the limit, he would merely reproduce by his equations the very irregularities which the biologist would like to see eliminated. He would leave in the archives of biology a statement which equals in complexity and irregularity that of the original biological data upon which it was based. If, however, he adopts the second alternative, he does not reproduce, but graduates, the frequencies. The biologist requires the simplest description which is adequate to represent his observations reasonably well, that is, well enough for the purposes of biological science as it exists today. It must be clearly recognized, however, that at any time more refined, and in consequence more complex, mathematical formulae may be required and may be justified by the progress in exactness of biological research.

Large contributions of method have been made by mathematicians. As

mere descriptions of individual frequency distributions, their equations may be adequate. The work of description in such terms has not, however, been sufficiently extensively and systematically pursued by biologists to have made the work very fruitful of purely biological results. It is doubtful whether this end will be realized until biologists as a class have much more thorough general mathematical training than is possible at present.

Furthermore, biological phenomena as they appear on the surface may not be interpreted by description, however accurate. A theoretical curve which gives frequencies in terms of areas corresponding to given intervals on the scale of abscissae may be erroneous because the units of the latter scale, while numerically comparable, have not the same biological significance. There is here a practically untouched field awaiting the attention of the biologist with an adequate mathematical background.

Another group of important problems in the mathematical description of biological data must be passed with but a word of comment. Frequency curves, which describe the form of the distribution of magnitudes of biological variables at any given stage of development or under any given set of conditions, fail to take account (except in the terms of end results) of the fact that organisms acquire their magnitudes at the given stage through the developmental processes which the biologist summarizes by the term *ontogeny*.

Comparative and experimental embryology have vastly increased our knowledge of ontogeny, but development is a quantitative process and to be fully understood must be dealt with, in some of its phases at least, in terms of mathematical descriptions.

A flood of papers has in recent years been devoted to the mathematical description of growth—both of the individual and of populations of individuals. Much of this work is superficial in the extreme, but sufficient results of value have been obtained to indicate the importance of the development of the field from both the mathematical and the biological sides.

c. The Quantitative Description of Inter-relationship

The biologist cannot be satisfied with the exact description of single biological variables. He cannot limit his attention to x alone but must also consider the relation between x and any number of other variables, $a, b, c, \ldots, \alpha, \beta, \gamma, \ldots$. He knows that in nature the magnitudes of x and of these other variables are not independent but that they are interdependent because x may be causally related to one or more of the other variables, or because x and the other variables may be subject in a similar manner but in varying degrees to the influence of the same constellation of factors which determine their magnitudes.

In conventional terms, the biologist is interested in the problem of correlation. He requires to know, in the quantitative terms of a universally comparable scale, the closeness of interrelationship between any pair of variables.

The correlation coefficient—the brilliant conception of Francis Galton's

fertile mind—has long been familiar in biology. It was soon adopted in psychology and economics, and it is now finding its way into all of the sciences, including a field already so highly developed mathematically as astronomy.

Pearson early expressed Galton's conception of correlation in terms of

$$r_{xy} = \frac{\Sigma[(x-\bar{x})(y-\bar{y})]/N}{\sigma_x \sigma_y}.$$

Since the numerator of the right hand side involves the product moment of x and y about their respective means as origins, both x and y must be quantitatively measurable. When x and y are wholly independent and N is large the product tends to vanish, and r_{xy} to approach 0. Since the deviations of the variates from their respective means are expressed in terms of $(x-\bar{x})/\sigma_x$, $(y-\bar{y})/\sigma_y$, $r_{xy}=1$ when the relationship between the two variables is linear and when there is no variability in the arrays of one variable associated with the several classes of the others. If the deviations of both x and y from their respective means are preponderantly of like sign when both magnitude and frequency are considered, the value of r_{xy} will be positive. If deviations of pairs of x and y from their respective mean values are preponderantly of unlike sign, the value of r_{xy} will be negative. Interrelationships between two variables both measured on a quantitative scale and linear in their relation to each other are ideal for the biologist. Fortunately the experience of the past forty years has shown that the majority of relationships between morphological variables may be approximately expressed by straight line equations.

In the extension of the conception of correlation beyond the first simple case, three main difficulties have had to be faced. Possibly there are others which have not yet been so clearly distinguished.

First, the relationship between x and y may be non-linear. In this case r_{xy} does not furnish a wholly valid description of the relationship between them.

Second, some characteristics of organisms cannot as yet be measured on a strictly quantitative scale, but must be expressed as qualitative categories. Thus the coat color of cattle or the hair or eye color of human beings may be more readily described than measured.

This introduces all the difficulties associated with the real or imaginary discontinuity of variation of many biological variables. Whether or not really discontinuous, a large percentage of biological variables must be expressed on a discontinuous scale. This introduces theoretical questions as to the applicability to discontinuous phenomena of mathematical theory based on the assumption of continuous variation. Even if these theoretical difficulties be ignored, or considered of insignificant importance in dealing with biological phenomena, there still remain in some cases large practical obstacles which may be more readily considered later.

Third, the number of classes which can be distinguished in the case of measurable characters, and particularly the number of categories which may be recognized in the case of non-measurable characters or attributes, may be

either large or small. Thus individuals may fall for purposes of registration statistics into as few as two alternative groups—for example, survived and died, or vaccinated and not vaccinated, or married and unmarried. When the number of classes of one or both of the two variables is either too small or too large, difficulties may be encountered in the measurement of the interrelationship between them. In some cases it is possible to increase the number of categories by logical subdivision, when too small, or to decrease the number by logical combination, when too great. In other instances the data available to the mathematical biologist are not such as to make this possible. Even in the most careful biological research it is sometimes practically necessary to recognize two alternative categories only, or to retain a number of categories so great as to render the interpretation of contingency coefficients uncertain.

Limiting our attention for the moment to the third difficulty of biological origin, with a promise to return to the first and second classes later, we may note that in biological research, when conducted in a broad and comparative way, we have for either of our two variables, x and y, all possible gradations between measures on a uniformly divided quantitative scale, numerous multiple categories and two alternative categories. In the determination of measures of interrelationship these may be encountered in every possible permutation.

It is for this reason that the theory of correlation has had to be extended to include the description in quantitative terms of interrelations between values of x and y measured or appraised in multiple categories only, and in various combinations of quantitative and non-quantitative categories. Thus we have the following possible combination of our two variables.

First Variable x (or y)	Second Variable y (or x)
Quantitatively measured: classes unlimited in number	Quantitatively measurable; classes unlimited in number
"	Quantitatively measurable; classes limited in number
"	Classed in multiple categories; classes many
"	Classed in multiple categories; categories few or two only
Classed in multiple categories: categories many	Classed in multiple categories; categories many
"	Classed in multiple categories; categories few or two only
Classed in multiple categories; categories few or two only	Classed in multiple categories; categories few or two only

All of these interrelationships have been considered in a series of masterly papers by Pearson and others who have followed him. Formulae have been deduced which have in common the expression of the interrelationships in terms of a scale which is universally comparable in that it has a range of 0 to 1. Ideally, the point 0 on this scale represents an entire absence of interrelation-

ship between the two variables, while ideally $+1$ or -1 indicates such closeness of interrelationship that knowing the value of one variable, x, we also know the value of the other variable, y, of the same pair.

This description of the interrelationship between two variables on a quantitative, universally comparable, and mentally comprehensible scale has been one of the most fertile conceptions in biology. A volume would be required to list the applications which have already been made.

Illustration of the importance of biological accomplishments already realized by the application of available correlation and contingency theory is less pertinent to our present purposes than the indication of some of the difficulties which still remain to be overcome. Of fundamental importance in relation to the problem of biological methodology is the fact that the description of the relationships between biological variables as written in the quantitative terms of the various interrelationship coefficients are not always wholly consistent.

Consistency of the end results of mathematical reasoning has long been recognized as one of the touchstones by which the validity of the premises and the deductions from them are tried. Inconsistency in quantitative descriptions of interrelationships may be due either to the inadequacy of the theory as far as it is at present developed and tested, or it may be due to the fact that the two quantitative descriptions really portray different features of the biological interrelationships.

Some of the difficulties are such that they will long remain insurmountable, but both biologists and mathematicians should keep before them the fact that the needs for the adequate description of the relationship between the variables measured or appraised for N individuals or associated pairs of individuals has been only partly met.

Returning now to the first and second difficulties we may note the following.

While the measurement of interrelationship between variables in the terms of a universally comparable scale has been one of the most fruitful conceptions in modern science, measures on a universally comparable scale must fail to express the relationship between two variables in terms which are the most useful for purposes of many biological researches. It is for this reason that measures of interrelationship in terms of correlation must also be expressed in terms of regression equations. These equations may be of various orders. Most fortunately for biological research, experience has shown that the linear equation

$$(y - \bar{y}) = r_{xy}\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

is realized to a very close degree of approximation for an enormous number of pairs of variables. When regression is not linear, the theory of the correlation ratio involving ${}_x\eta_y$, ${}_y\eta_x$ must be employed to lead to equations of degrees higher than the first.

In the use of the correlation ratio as a method of approach to non-linear regression equations, a number of difficulties are incurred. While some progress has been made toward overcoming these, much still remains to be done in the development of mathematical theory and in testing it through its application in actual biological research.

In the case of characters or attributes which are not quantitatively measurable, we must note that it is not yet possible to deal with many of these in terms of regression at all. In some cases, indeed, all that is possible is to express the deviation of the system from independent probability in terms of the correlation scale. Much must be done by both mathematicians and biologists before we shall be able to deal satisfactorily with all the problems of the interrelationship between variables of this kind regularly encountered in biological research.

In the foregoing discussion, we have dealt with the problems of the interrelationship between pairs of variables, x and y. In certain cases we have, instead of individual pairs, a number of values of x or y or of x and y which are associated in groups or classes, which groups or classes may be delimited one from the other by reason of their position in time or space, or by some other valid condition. Such groups of x, or of any other variable, may be designated as a class.

Now all of the n values of x which constitute one of the m classes which together make up the $\Sigma(n) = N$ individuals of the sample may be assumed to be subject to the influence of a constellation of intrinsic or extrinsic factors in some measure peculiar to that class and competent to determine or to modify their values. In short, the biologist frequently has to deal with variables the magnitudes of which may be associated with or differentiated by time or space, or by both time and space, or by some other differentiating factor which is not covered by these terms as they are ordinarily employed.

Because of these conditions, there will be a tendency for the individuals of the several classes to be differentiated from class to class. This differentiation may be measured in terms of the intra-class correlation coefficient. Such intra-class correlation coefficients may be computed for values of x or of y or of any other variable. Not infrequently values of both x and y, or of more variables, are measured on individuals of the same class, or on individuals of associated classes. In such cases it is possible to deal with direct or cross intra-class correlations, or with direct or cross inter-class correlations. In some instances it may be desirable to recognize sub-classes, in which case fractional intra-class or inter-class correlation coefficients may be determined.

Large biological accomplishments await the further development and application of intra-class and inter-class correlation theory.

Two cases in point may be noted. *First*, in many instances the influence of the factors associated with time and space on the values of a variable must be measured in terms of the correlations between the magnitudes of the associated variates themselves, because the constellation of influences associated

with time or space is so complex that the worker has no means of determining directly the correlation between these environmental factors and the magnitudes of the variable. *Second*, if the biologist is to determine the true values of the variables with which he has to deal, he must eliminate the differentiating influence of time or space or of both time and space on the empirical measurements which he has made.

The foregoing discussion has been limited to the measurement of the interrelationship between two variables. Cases in which more than two variables must be considered at one time will be more conveniently treated later.

2. The Need for Criteria of the Adequacy of Mathematical Description

Having, as a biologist, urged that the equations to be employed in the description of organic phenomena should be deduced by processes approved by the trained mathematician, I must, as a biologist, insist that such mathematical descriptions or equations should be satisfactory to the biologist.

If mathematical descriptions of biological phenomena are to be written, we require rigorous mathematical tests of the closeness of agreement between the empirical and the theoretical distributions. We need to know, in short, whether the equation adopted describes the actual biological facts so closely that differences between the empirical values and the theoretical values as given by the equation may be no larger than those which might reasonably be supposed to have arisen from errors of measurement or from errors of random sampling.

Thus the biologist requires not merely a system of mathematical equations which will describe his observational or experimental data, but he must be assured that the agreement between observational data and mathematical description is as good as can be expected from the random sampling of a population of the given variables, or at least that it is good enough for the purposes of his investigation. If the results do not show this concordance, then he may properly be dissatisfied with the given equation as a basis of generalization concerning his measurements, and he must seek some other equation to represent his empirical results.

It will be clear that in addition to providing the equations for mathematical description of series of biological data, the mathematician has a two-fold task. *First*, he must provide criteria of the closeness of agreement of his mathematical description with the emprical data of observation or experimentation. *Second*, he must provide a system of probable errors which determine the trustworthiness of his expressions as bases of generalization. Naturally the processes underlying these two undertakings are interrelated.

Limiting our attention for the moment to the first of these two requirements, we may merely note the importance of Pearson's χ^2 and P criterion of goodness of fit, more recently extended to the testing of the sameness of origin of two independent samples, and of Pearson's and Blakeman's work on criteria of linearity of regression. These tests of the adequacy of mathematical description

have been so fruitful that it is earnestly to be hoped various and much needed extensions will be made along different lines.

3. The Need for Criteria of Bases of Generalization

The purpose of biological work is to derive from the series of observations which can be made and recorded as descriptions of relatively small samples of biological variables, systems of generalizations concerning the biological phenomena of the universe.

As descriptions of the N observations or sets of observations made on any given biological variable, the physical constants (such as means, standard deviations, coefficients of variation, coefficients of correlation, regression equations, etc.) are absolutely valid except for (a) the limitations of the errors of measurement (which in some cases vanish on summation) and for (b) the limitations due to the assumptions made in the selection of the mathematical formulae employed.

Such validity is not, however, all that is required to fulfill the purpose of biology as indicated above. The precise determination of the characteristics of the individuals of a specific group, and the adequate description of the group in the terms of biometric constants, may have large personal interest or economic importance but the procedure lacks scientific value unless it can be made the basis for a generalization concerning the universe or population of individuals from which the finite sample investigated was drawn. Since, because of practical limitations, the whole of the population cannot be investigated, the characteristics of its individuals must be predicted from the sample. The mathematician must provide the biologist with the means of estimating how trustworthy his statistical constants are as a basis for generalization concerning the population, or the universe.

The validity as bases for generalization of the mathematical constants which describe series of biological measurements must depend upon four conditions:

(1) The method of drawing the sample must introduce no differentiation between the sample and the population other than that reasonably attributable to the errors of random sampling.

(2) The measurements must be made with a degree of precision sufficient for the purposes of the generalization to be drawn.

(3) The formulae in terms of which the final constants for the samples are expressed must introduce no source of error.

(4) The number of individuals measured must be adequately large for the generalization to be formulated.

Absolute assurance that the fraction which has been subjected to observation or measurement is really typical of the universe as a whole, as required by condition (1), is unattainable except by the process of determining the characteristics of all the individuals which constitute the universe or the population at large. Because of the impracticability of measuring all of the individuals of the population, the mathematician has quite properly laid down the principle

that the sample observed must be drawn at random. This is relatively easy when the population sampled is made up of colored balls or of dice contained in the mathematicians bag of infinite capacity.

Living organisms are not, however, found under these ideal conditions. They are not merely distributed through time and space, but their distribution is very definitely related to the differences in the conditions presented by time and space. A sample taken at one place or time may as a matter of fact be highly differentiated from other samples which might be taken. Only biological experience, with a consideration of all of the possible physical and biological factors involved, can make reasonably certain the adequacy of the sampling of the population in the case of any unique sample.

When more than a single sample is taken, the constants obtained from the several samples may be compared among themselves. In this case the mathematician can furnish criteria of the randomness with which the samples were taken. In my opinion there should be further provision of theory for dealing with the problem of the significance of the results of repeated sampling of what is presumably the same population.

In some cases biologists have followed the mathematician so blindly as to presume that a sample distributed at uniform intervals throughout the time or space occupied by a fraction of the population is less suitable than a sample drawn in a wholly haphazard manner. As a matter of fact, it often turns out that samples taken in a systematic manner are more representative of the population, and in consequence more valuable as a basis of generalization, than samples taken in what is supposed to be a random manner. This is true in part because (a) it assures the wider distribution of the individuals through time or space, and in consequence makes more probable the obtaining of a truly representative sample, and in part because (b) the individuals of the universe may be subject to the influence of differentiating factors associated with space or with time or with both space and time. Thus when the problem in hand is the comparison of two or more series of measurements (as is frequently or generally true in practical work in quantitative biology) it is possible by selecting pairs of individuals, x and y, which are distributed in a systematic manner through time or space, or through both, to obtain a smaller value of the probable error of the difference between them than would otherwise be possible. This is true because any positive correlation which may exist between x and y will tend to reduce the probable error of the difference between them.

The problem of accuracy of measurement (2) need not receive consideration here further than to note that it must be solved by the cooperation of biologist and mathematician. In most of the statistical theory as developed up to the present time, the assumption has been made that the measurements which are available for mathematical analysis represent the true values of the variables. In many cases this is far from true. The measurements themselves are in some degree erroneous. It will be necessary, therefore, in the future to combine the theory of measurement and the theory of variation to a greater extent than has heretofore been done in biological investigation.

Quite obviously only the mathematician can furnish the biologist assurance that condition (3) has been adequately met.

The mathematician must not, however, assume that this is a simple task. In biology constants to be of the greatest value must be comparable from investigation to investigation. The materials considered, the conditions under which the measurements were made or the units in which the measurements are recorded may vary greatly from case to case.

In consequence there may be a tendency on the part of the worker to use different methods of statistical analysis for the different series. This may, indeed, be necessary.

Consider by way of illustration the problem of the measurement of the relationship between the variables a and b when each may be classed in alternative categories only, a_1, a_2, b_1, b_2. Obviously the four-fold table is

	b_1	b_2	Total
a_1	$a_1\,b_1$	$a_1\,b_2$	$a_1b_1 + a_1b_2$
a_2	$a_2\,b_1$	$a_2\,b_2$	$a_2b_1 + a_2b_2$
Total	$a_1b_1 + a_2b_1$	$a_1b_2 + a_2b_2$	N

Now many methods of measuring the relationship between a and b under such conditions have been suggested, but certain of these methods may give very inconsistent results when applied under varying conditions. Clearly the mathematical theory underlying some of the formulae must be inadequate. The illustration is by no means unique. A vast amount of work still remains for mathematicians and biologists in the selection of the formulae most suitable for application in special cases.

Finally, with reference to (4), only the mathematician can decide how large a number of individuals must be measured in order to arrive at a generalization of any given degree of accuracy, and he can give the information required in any specific case only when he knows the experience of the biologist with respect to the variability of the character, or of the statistical constants for the character, in question.

Taking the simplest of the constants, such as $\bar{x}$ and σ_x, by way of first illustration of the difficulties of the problem, we may note that while $\bar{x}$ and σ_x are true descriptions, within limitations already indicated, of the sample of N individuals studied, they may be in some measure erroneous as descriptions of the population from which the values of x were actually drawn. Thus they have no probable error (except that due to random or non-systematic errors of measurement which do not vanish in the summation of the second and other even powers of the technical errors) as a description of the magnitude or the variability of the sample on which they are based, but are subject to the limitations of a probable error as a generalization concerning the population from which the finite (and

often small) sample was drawn. It is for this reason that we write $\bar{x} \pm E_{\bar{x}}$, $\sigma_x + E\sigma_x$.

While the theory of the probable errors of means and standard deviations is relatively easy as compared with that of the probable errors of constants involving higher moments, product moments, or products of independent probabilities, it is in itself by no means simple in either theory or practice. Thus, for example, the conventional formula, $E\sigma = .6754\sigma_x/(2N)^{1/2}$, usually but not invariably given in the text books, is valid only if the magnitude of the variable be distributed with normal frequency. To determine a value of $E\sigma$ which shall be correct for any distribution (and here neglecting the errors of generalization which may be due to either the size of N or the magnitudes of the standard deviation), the first four moment coefficients must be computed. When the probable errors are calculated by the two methods, there may be a difference of as much as 100 percent between them.

It must not be forgotten that in biological work the interpretation of the probable error involves the fundamental problem of the variability of the individuals of the sample and of the infinite population. If s_x be the standard deviation in the infinite population and σ_x the standard deviation of the N individuals of the sample, it is by no means certain that $\sigma_x = s_x$.

Since the standard deviation of the individuals of the infinite population s_x is unknown, the standard deviation of the individuals of the sample must be substituted as an approximate value. Both $\bar{x}$ and σ_x may be materially in error, expecially when N is small and s_x is large.

Furthermore, in many studies in practical biological or sociological work we deal with cases in which the population is not infinitely large. When N is a substantial part of the universe, the value of a given constant as a basis of generalization is somewhat different from that which it is when N is an infinitesimal part of the general population of which it is a random sample.

These points might be developed almost indefinitely. Quite obviously all statistical constants, including the many coefficients of correlation and contingency, have their probable errors. The deduction of the most suitable formulae for the probable errors is a task requiring mathematical ability and training of the highest order. An enormous contribution has been made to the field by Karl Pearson and his pupils, but much still remains to be done on the problem, which is one of the most important in the whole field of quantitative science.

4. The Need for the Powers of Symbolic Analysis

In the preceding sections it has been made sufficiently clear that the mathematician must, by processes of symbolic reasoning, provide the formulae which are needful for the mathematical description of biological phenomena for the testing of the adequacy of such description, and for the estimation of the value of such descriptions as bases of generalization from experience of a sample to the population from which the sample was drawn.

I now desire to emphasize the fruitfulness of the use of such reasoning in

the field of biology itself. Our problem is to consider whether in many instances mathematical analysis through processes of symbolic reasoning cannot supplement or in some cases largely replace other analytical methods in biology.

From the biological side the problem must be largely one of the adoption of the tested processes of the mathematician, with the substitution of constants based on the data of observation for the formal symbols of the mathematician. If it can be shown that this substitution of biological constants, and the subsequent treatment of these constants as though they were the formal symbols of the mathematician, is legitimate, the biologists power of reasoning about his observational data could be greatly extended.

Probably there are those who would categorically deny the propriety of such procedure. The final biological test of its legitimacy would seem to be two-fold: *First*, the consistency of the end results with those obtained by other methods of research; *Second*, the capacity of the methods for the prediction of the unknown.

Let me illustrate by a concrete case.

In the foregoing discussion of correlation two variables only were considered. In such cases inspection of the correlation or contingency surface may furnish graphic evidence of the existence of a relationship between the two variables. The relationship is mentally intelligible because it can be detected visually.

As a matter of fact such relationships between two sets of N paired variables represent only the simplest cases with which the biologists has to deal. Instead of considering merely x and y, he must deal with x, y and z, or with w, x, y and z, or with a far larger number of variables, all of which may be interrelated.

The complications of such a system of interrelationships are such that they cannot be readily represented graphically and they cannot be grasped by the ordinary mind. The complication is not merely that of a chess board on which moves may be made in two dimensions of space. It is the complication of a game in which the men may be moved in three or in many different dimensions of space.

Since such problems are so involved that the interrelationships cannot be visualized, they must be attacked by symbolic reasoning. Only the mathematician, of high ability and rigorous training, can write the equation which will give the probable consequences of any shift of the pieces.

For problems of this kind symbolic reasoning has led to the theory of multiple correlation and partial correlation. This in turn, has led to multiple regression or prediction equations.

For example, suppose we require to predict the basal metabolism—that is, the energy requirement per unit of time of the body when in a state of complete repose—of human individuals from various physical measurements. It is known that basal metabolism is related to sex, age, weight and stature. Individuals may be divided according to sex and equations determined for each sex separately. But even with this simplification, the problem is a relatively complex

one. We may measure in terms of the correlation coefficient the relation between age and basal metabolism, the relation between weight and basal metabolism, the relation between stature and basal metabolism. Thus all three variables, age, stature and weight must enter into our prediction equation. But these variables (age, stature and weight) are correlated among themselves, and these interrelations must be taken into account in determining equations which will give the most probable basal metabolism of an individual of known sex, age, stature and weight.

Only four variables are considered (though many others are doubtless involved) but each of these four variables may occur in many different magnitudes in the N individuals whose basal metabolism has been determined. Thus the problem becomes so complicated that no human mind can grasp or express the quantitative relations involved without the aid of symbolic reasoning.

That such multiple prediction equations will give reasonably good estimates of the values of unknown variables is proved by the results of a number of investigations in which they have been actually applied. Thus in the case of human basal metabolism it has been possible to predict, with a high degree of accuracy, the daily heat production of subjects which were unknown as far as the development of the equations was concerned.

Such examples should carry to both the biologist and the mathematician the conviction that within reason, and as established by experience, we may replace biological constants by mathematical symbols, and that these symbols, after treatment by legitimate mathematical processes, may again be replaced by their biological equivalents in our final results.

III. *Summary*

In an address which has for its purpose the outlining of the fundamental mathematical requirements of biology, it would be both undesirable and impracticable to suggest specific mathematical problems of importance for the biologist which await solution by the mathematician. I have, indeed, indicated that the greatest need is for biologists to apply more extensively the masterly series of results of Pearson and of those whom he has inspired. An enormous service mathematics has unquestionably rendered, but it has been through a few workers only. For the most part mathematicians have not felt it particularly worth their while to aid biologists in the solution of their problems.

In part, the past indifference of mathematicians has been due to the fact that biology was itself insufficiently advanced to permit the application of mathematical theory. I am inclined to think that it was in part due to the fact that the older mathematics had relatively little which could be demonstrated to be of value to the biologists and that mathematicians as a class had not yet become aware of the new fields of mathematical theory which had been developed with the specific purpose of meeting the needs of biology. Now that Quelelet, Galton and Pearson have broken down the barriers, mathematicians may find it worth their while to extend the field and to revise and refine the work already done under the stress of practical necessity.

The mathematician must not think that this problem of providing "mathematical tools" for the biologist is a simple one. In some respects it presents greater difficulties than those encountered in the field of pure mathematics.

The fact that much of the pertinent mathematical theory which is already available to biologists is still little used need not deter workers from adding to the literature. This condition of apathy on the part of biologists will not long prevail. The generation of leaders who opposed progress a few years ago will soon be without influence, and in biology we may look forward to growing appreciation and widening application of mathematical theory.

May I urge, therefore, that trained mathematicians maintain a more friendly attitude toward those biologists who are seeking through refinement of measurement and of mathematical description and analyses to place biology alongside physics and chemistry in the ranks of the exact sciences?

ON CERTAIN SEQUENCES OF CONICS AND ASSOCIATED SEQUENCES OF NUMBERS

By L. S. JOHNSTON, University of Detroit

The writer gives herein some properties of conics which he encountered while investigating the loci of the intersections of the bisectors of the angles of a triangle under certain conditions.

Theorem: *If two vertices of a triangle be taken as the foci of a conic traced by the third vertex, then the in-center and the three ex-centers trace two lines tangent to the initial conic at the vertices on the focal axis and two conics of the same species as the initial conic, concentric with the initial conic, and passing through its foci.*

Consider two points F_1 and F_2 fixed, and a variable point P tracing a conic C of eccentricity e_0, with F_1 and F_2 as foci. The bisectors of the angles at P are the tangent and the normal to C at P. If one of the foci be taken as the origin, C may be written in the form

$$\rho = k/(1 - e_0 \cos \theta)$$

or in the Cartesian forms

$$x^2(1 - e_0^2) - 2ke_0x + y^2 = k^2, \tag{1}$$

or

$$\left(x - \frac{e_0k}{1 - e_0^2}\right)^2\left(\frac{k}{1 - e_0^2}\right)^{-2} + y^2(1 - e_0^2)^{-1}\left(\frac{k}{1 - e_0^2}\right)^{-2} = 1 \qquad (e_0 < 1), \tag{2}$$

or

$$\left(x + \frac{e_0k}{e_0^2 - 1}\right)^2\left(\frac{k}{e_0^2 - 1}\right)^{-2} - y^2(e_0^2 - 1)^{-1}\left(\frac{k}{e_0^2 - 1}\right)^{-2} = 1 \qquad (e_0 > 1). \tag{2'}$$

ellipse, $EI^{r-1}C$ can be constructed by reflecting I^rC in the line $y-x=0$ and then magnifying by the factor $(1-e_r^2)^{-1/2}$.

The theory developed for the central conics holds for the parabola if we make the customary modifications due to the fact that one focus of the parabola is at infinity. One property common to the central conics and the parabola has already been mentioned; other modifications can easily be made. It is interesting to note that the expression for the abscissa of the limiting position of the vertices of $\{I^nC\}$ is exceedingly simple in this case. If the equation of C be given in the form $y^2=4px$, it is easy to show that the equation of IC is $y^2=p(x-p)$, which is a parabola through the focus of C with focal length one fourth that of the initial conic. Hence the limit point of the vertices of $\{I^nC\}$ is the point whose abscissa is

$$p(1 + 1/4 + 1/4^2 + \ldots) = 4p/3.$$

A NOTE ON NEWTON'S DIAGRAM FOR APPROXIMATING PLANE ALGEBRAIC CURVES AND SURFACES

By ALAN D. CAMPBELL, Syracuse University

The following approach to the subject of Newton's Diagram for approximations to plane algebraic curves and to algebraic surfaces (at the origin and at infinity) has been found useful by the author. First let us consider plane algebraic curves. If an n-ic passes through the origin we solve its equation simultaneously with an equation of the form $y=x^\alpha$, then we determine α (say $\alpha=\alpha'$) so that the resulting equation in x shall have as high a power of x as possible (say x^m) coming out as a factor. Then we see geometrically that the curve $y=x^\alpha$ is a good approximation to the n-ic near the origin, since the two curves have in common there m' coincident points (where $m'=m$ if m is an integer, otherwise m' is the greatest integer less than m). If there are two or more terms in the n-ic that give us terms in x^m when the n-ic is solved simultaneously with $y=x^{\alpha'}$, and we equate to zero these terms in the n-ic, we shall have perhaps an even better approximation to the n-ic than is the curve $y=x^{\alpha'}$.

Let us consider the quintic

$$(1) \qquad x^2y^3 - 2xy^4 + y^5 + x^2y = x^4 + y^4.$$

Substituting $y=x^\alpha$ in (1) we get

$$x^{3\alpha+2} - 2x^{4\alpha+1} + x^{5\alpha} + x^{\alpha+2} = x^4 + x^{4\alpha}.$$

Evidently we cannot determine α so as to have a common factor x^m in (1) with $m>4$. The origin is a triplet point on (1), with two coincident tangents given by $x^2=0$ and a third tangent $y=0$. Hence we should have two approximations to (1) at the origin. We try successively $\alpha+2=4$, $5\alpha=4$, $4\alpha+1=4$, $3\alpha+2=4$, $4\alpha=4$. The first and the fourth of these choices of α give us the approximations

we desire, since corresponding to them we have pairs of terms from the n-ic for our approximations. These two approximations are respectively $x^2y=x^4$ and $x^2y=y^4$.

To consider approximations at infinity we imitate the above procedure at the origin. For example (1) in the homogeneous form is

$$(1') \qquad x^2y^3 - 2xy^4 + y^5 + x^2yz^2 = x^4z + y^4z.$$

We see that (1′) passes through the points (1, 0, 0) and (1, 1, 0). To study (1′) at (1, 0, 0) we divide the equation through by x^5, then we put (1′) in the non-homogeneous form

$$(1'') \qquad y^3 - 2y^4 + y^5 + yz^2 = z + y^4z,$$

where $y=y/x$ and $z=z/x$. We note that (1, 0, 0) is not a multiple point on the curve. Now $z=y^\alpha$ when solved with (1″) gives

$$y^3 - 2y^4 + y^5 + y^{2\alpha+1} = y^\alpha + y^{\alpha+4}.$$

We see that $\alpha=3$ gives us the best approximation. From (1″) we then obtain the approximation $y^3=z$, to which there corresponds from (1′) $y^3x^2=x^4z$. To study the curve at the point (1, 1, 0) we can change this point to (1, 0, 0) by a transformation of coordinates. If the curve (1′) passed thru (0, 1, 0) the equation we should use to replace (1″) would contain the variables x and z but no y.

Now we continue somewhat as in Hilton *Plane Algebraic Curves* §3, p. 38. Let us plot the exponents β and γ of each term $x^\beta y^\gamma$ in an n-ic as coordinates referred to a set of u- and v-axes (rectangular or oblique) obtaining what is called Newton's diagram. We suppose the u-axis to be horizontal. The equation $y=x^\alpha$ has the corresponding points (0, 1) and $(\alpha, 0)$. A line l thru (β, γ) parallel to the line l' joining (0, 1) and $(\alpha, 0)$ has the equation $(u-\beta)+\alpha(v-\gamma)=0$ and cuts the u-axis at $(\beta+\gamma\alpha, 0)$. Suppose the term $x^{\beta'}y^{\gamma'}$ corresponds to a point (β', γ') on l, then $\beta'+\gamma'\alpha=\beta+\gamma\alpha$. But putting $y=x^\alpha$ in $x^\beta y^\gamma$ gives us $x^{\beta+\gamma\alpha}$ and in $x^{\beta'}y^{\gamma'}$ gives $x^{\beta'+\gamma'\alpha}$. Hence any term $x^{\beta'}y'^{\gamma}$ in the n-ic that corresponds to a point (β', γ') on l yields the same power of x as $x^\beta y^\gamma$ if we put $y=x^\alpha$, namely the power given by the u-intercept on l. Also any term $x^{\beta''}y^{\gamma''}$ where (β'', γ'') is to the left of and below l yields a power of x smaller than $\beta+\gamma\alpha$ if we put $y=x^\alpha$; whereas if (β''', γ''') is to the right of and above l the power of x yielded by $x^{\beta'''}y^{\gamma'''}$ is greater than $\beta+\gamma\alpha$. From this argument we see that if any line l joining two or more points of a Newton's diagram of an n-ic is such that all the other points of the diagram are to the right of and above l; then the terms of the n-ic corresponding to points on l, when equated to zero, give an approximation to the n-ic at the origin. Thus for (1) we have a diagram with the points (2, 3), (1, 4), (0, 5), (2, 1), (4, 0), and (0, 4). The two lines joining (0, 4) to (2, 1) and (2, 1) to (4, 0) give us the approximations at the origin that we obtained previously. Similarly we can draw a Newton's diagram for approximations at infinity. For example for (1″) we have a diagram (taking u for the exponents of the powers of y and v for powers of z) with the points

(3, 0), (4, 0), (5, 0), (1, 2), (0, 1), (4, 1). The line joining (0, 1) to (3, 0) gives us the approximation $y^3x^2 = x^4z$ that we obtained above.

Next we combine all these diagrams for an n-ic (for approximations at infinity as well as at the origin) into just one diagram. We take a triangle with sides u, v, w, where w cuts u at the point $(n, 0)$ and v at $(0, n)$ and w has n equal divisions marked on it. The term $x^\beta y^\gamma$ in homogeneous coordinates is $x^\beta y^\gamma z^{n-\beta-\gamma}$. We see that $n-\beta-\gamma$ is the distance from w to (β, γ) measured parallel to u or parallel to v. To each term $x^\beta y^\gamma z^{n-\beta-\gamma}$ there now corresponds in our diagram a point whose coordinates can be written $(\beta, \gamma, n-\beta-\gamma)$ where the distance $n-\beta-\gamma$ is measured parallel to u or to v. For approximations to the n-ic at the origin we use the u- and v-axes alone, with u representing powers of x and v powers of y and with the positive directions for u and v measured from the point of intersection of u and v. For approximations at (1, 0, 0) we use as axes the u- and w-lines alone, with u representing powers of z and w powers of y, with the distance w measured parallel to the w-axis, and with the positive directions for u and w measured from the point of intersection of the u- and w-axes. Similarly for approximations at (0, 1, 0) we use w for powers of x and v for powers of z. From this discussion we have the rule that if we draw a Newton's diagram (as above) to approximate an n-ic at the origin we can use the same diagram for approximations at infinity, because any line m joining two or more points of the diagram that leaves the rest of the diagram to the left and above or below gives us an approximation at (1, 0, 0), whereas any such line n that leaves the rest of the diagram to the right and below gives us an approximation at (0, 1, 0). For example the cubic

$$x^2y + y^2z - 2xyz - xz^2 = 0 \tag{2}$$

has a diagram with the points (2, 1, 0), (0, 2, 1), (1, 1, 1) and (1, 0, 2). The line l joining (0, 2, 1) to (1, 0, 2) gives the approximation $y^2 - xz^2 = 0$ at the origin; the line m joining (1, 0, 2) to (2, 1, 0) gives the approximation $-xz^2 + x^2y = 0$ at (1, 0, 0); the line n joining (0, 2, 1) to (2, 1, 0) gives the approximation $y^2z + x^2y = 0$ at (0, 1, 0).

We can generalize the above discussion to the approximation of the nth degree algebraic surfaces at the origin and at infinity. (In fact the method can be generalized to any number of dimensions.) To do this we solve $x = z^\alpha$, $y = z^\beta$ simultaneously with the nth degree equation of the surface and determine α and β so as to get the highest possible power of z (say z^m) coming out as a factor. Geometrically we are finding two cylinders $x = z^\alpha$, $y = z^\beta$ such that their curve of intersection C cuts the given surface in the greatest possible number of coincident points at the origin. Now if we find all the terms $x^\lambda y^\mu z^\nu$ in the equation of the surface that yield the same power of z (namely $m = \lambda\alpha + \mu\beta + \nu$) when we put $x = z^\alpha$ and $y = z^\beta$, then the locus of these terms (when equated to zero) gives us an approximation to the surface at the origin, since this locus contains one or more curves that like C cut the given surface in the greatest possible number of coincident points at the origin.

We draw a Newton's diagram for the equation of the surface, representing each term $x^u y^v z^w$ by a point (u, v, w) referred to rectangular (or oblique) u-, v-, and w-axes. For ease of discussion we take these axes as rectangular. We take the plane ω through $(0, 0, 1)$, $(1/\alpha, 0, 0)$, $(0, 1/\beta, 0)$, namely $\alpha u+\beta v+w=1$. Through any point (λ, μ, ν) we take a plane ω' parallel to ω, namely $\alpha(u-\lambda)+\beta(v-\mu)+(w-\nu)=0$. Any point (u', v', w') on ω' gives a term $x^{u'}y^{v'}z^{w'}$ such that when we substitute $x=z^\alpha$, $y=z$ we obtain $z^{\alpha u'+\beta v'+w'}$. But from the equation of ω' we have $\alpha u'+\beta v'+w'=\alpha u+\beta v+w$, hence any points of the diagram that lie on ω' correspond to terms in the equation of the surface that give rise to the same power of z (namely $m=\alpha\lambda+\beta\mu+\nu$). The distance from the origin to the plane ω' is

$$d \equiv (\alpha\lambda + \beta\mu + \nu)/(\alpha^2 + \beta^2 + 1)^{1/2}.$$

Suppose we have a point (λ', μ', ν') to the right and above ω' or to the left and below ω'; then any plane ω'' through (λ', μ', ν') parallel to ω' has as its distance from the origin

$$d' \equiv (\alpha\lambda' + \beta\mu' + \nu')/(\alpha^2 + \beta^2 + 1)^{1/2}.$$

Since $d'>d$ or $d'<d$ respectively we have $\alpha\lambda'+\beta\mu'+\nu'$ greater than or less than $\alpha\lambda+\beta\mu+\nu$ (i.e. when we put $x=z^\alpha$ and $y=z^\beta$ in $x^{\lambda'}y^{\mu'}z^{\nu'}$ we obtain a power of z greater than or less than $m=\alpha\lambda+\beta\mu+\nu$). From the above discussion we see that to get an approximation to the given surface at the origin we must find a plane ω' through three or more points of the diagram that leaves all the rest of the diagram to the right and above ω'; then the terms in the equation of the surface that correspond to points on ω' give (when equated to zero) the desired approximation.

If the u-, v-, w-axes of a Newton's diagram are not mutually perpendicular lines we can transform them into a set of rectangular axes having the same origin and apply the above discussion. Then if we transform our coordinates back so as to refer to the original u-, v-, w-axes the above results are still valid; because by the transformations of coordinates that were employed points on a plane go into points on a plane, parallel planes go into parallel planes, and points to the right or left of any plane ω' go into points to the right or left respectively of the plane ω_1' into which ω' goes. Hence our diagram and its theory is valid also for oblique u-, v-, w-axes as well as for rectangular.

Just as for the n-ic, so for an nth degree algebraic surface we can use similar diagrams to approximate this surface at infinity at the points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, and $(0, 0, 1, 0)$. Finally we wish to combine all such diagrams into a single diagram (as we did for the n-ic). We take the plane π through $(n, 0, 0)$ $(0, n, 0)$, $(0, 0, n)$ on a diagram (for approximation at the origin) that is referred to rectangular u-, v-, w-axes, namely $u+v+w=n$. Suppose π cuts the coordinate planes of the diagram in the line segments l_{uv}, l_{uw}, l_{vw}. We divide each such line segment into n equal lengths. The perpendicular distance from π to any point (λ, μ, ν) is $\rho\equiv(-\lambda-\mu-\nu+n)/\sqrt{3}$, whereas the distance from π to (λ, μ, ν) that is measured parallel to any one of the three axes is $\rho\sqrt{3}$

$\equiv n-\lambda-\mu-\nu$. Now we can use a tetrahedron with the u-, v-, w-axes and the plane π; to each point in the diagram we can assign homogeneous coordinates $(\lambda, \mu, \nu, n-\lambda-\mu-\nu)$. It is now evident that we can use π and the u-axis for a diagram to approximate the surface at (1, 0, 0, 0), π and the v-axis for (0, 1, 0, 0), π and the w-axis for (0, 0, 1, 0). In the first case we use l_{uv}, l_{uw} and the old u-axis as a set of oblique axes; in the second case l_{uv}, l_{uw} and the old v-axis; in the third case l_{uw}, l_{vw} and the old w-axis. Now we see that, having drawn a Newton's diagram with rectangular u-, v-, w-axes (oriented like the ordinary x-, y-, z-axes, respectively) to approximate an algebraic surface near the origin, we can use this diagram in the following way to approximate the surface at infinity. Any plane through three or more points of the diagram that leaves the rest of the diagram on the side toward the origin and above or below (as viewed from the u-axis by someone standing upright on the uv-plane) gives an approximation at (1, 0, 0, 0); any plane similarly placed with respect to the v-axis gives an approximation at (0, 1, 0, 0); and any plane similarly placed when viewed from the w-axis (by an observer standing on the vw-plane) gives an approximation at (0, 0, 1, 0). As an example we have the cubic surface

$$xyz + x^2y - xy^2 + x + y - 3z - 2xy + x^2z - xz^2 = 0.$$

Its diagram has the points (1, 1. 1, 0), (2, 1, 0, 0), (1, 2, 0, 0), (1, 0, 0, 2), (0, 1, 0, 2), (0, 0, 1, 2), (1, 1, 0, 1), (2, 0, 1, 0), and (1, 0, 2, 0). This gives us the respective approximations $x+y-3z=0$, $x^2y+x^2z-xz^2=0$, $y-3z-xy^2=0$, and $3z+xz^2+xy^2=0$.

LINEAR FRACTIONAL TRANSFORMATIONS ON QUARTIC EQUATIONS

By RAYMOND GARVER, University of California at Los Angeles

The purpose of this paper is to consider a number of linear fractional transformations on quartic equations from the standpoint of simple algebra. Most of the results obtained are not new, but the proofs ordinarily given make use either of invariant theory or of certain ideas of algebraic geometry, or of both.[1] We do not require a knowledge of either in this paper. Transformations on cubic equations have already been considered from the point of view of this article by Serret,[2] Niewenglowski[3] and Simonart.[4] That work, however, is much simpler, and the only interesting result seems to be that the general cubic equation can be reduced to a binomial equation by means of a linear fractional transformation. An incorrect statement, or conjecture, of Simonart to the

[1] See, for example, L. E. Dickson, *Modern Algebraic Theories*, p. 20; R. M. Winger, *Introduction to Projective Geometrie*, p. 209; E. B. Elliott, *Algebra of Quantics*, 1st ed., p. 275 ff.

[2] *Cours d'Algèbre Supérieure*, 5th ed., vol. 1, p. 140.

[3] *Cours d'Algèbre*, 5th ed., vol. 2, p. 309.

[4] Sur une transformation homographique d'une équation, Mathesis, vol. 36 (1922), pp. 212–218.

effect that three intermediate terms of an equation of degree greater than three can be removed by means of such a transformation led to the present paper.

We shall consider the general quartic equation in the convenient reduced form

$$x^4 + a_2x^2 + a_3x + a_4 = 0. \tag{1}$$

To this we apply the transformation

$$y = (c_1x + c_2)/(c_3x + c_4) \qquad (c_1c_4 - c_2c_3 \neq 0). \tag{2}$$

Our purpose is to so choose the parameters c_i that the transformed equation in y will be simpler than the original equation in x, in that certain of its coefficients will be zero. We thus require expressions for the coefficients of the transformed equation.

These coefficients are, of course, equal except possibly for sign to $\sigma_1 = \Sigma y_1$, $\sigma_2 = \Sigma y_1y_2$, $\sigma_3 = \Sigma y_1y_2y_3$ and $\sigma_4 = y_1y_2y_3y_4$, respectively. The second coefficient will then be zero provided we can make $\sigma_1 = 0$, the third if we can make $\sigma_2 = 0$, etc. To obtain explicit expressions for these equations of condition, we make use of the following relations:

$$y_i = (c_1x_i + c_2)/(c_3x_i + c_4), \quad \sum x_1 = 0, \quad \sum x_1x_2 = a_2, \quad \sum x_1x_2x_3 = -a_3,$$
$$x_1x_2x_3x_4 = a_4.$$

The equations $\sigma_i = 0 (i = 1, 2, 3, 4)$ reduce to the following:

$$4c_1c_3^3a_4 - 3c_1c_3^2c_4a_3 - c_2c_3^3a_3 + 2c_1c_3c_4^2a_2 + 2c_2c_3^2c_4a_2 + 4c_2c_4^3 = 0, \tag{3}$$

$$\begin{aligned} 6c_1^2c_3^2a_4 - 3c_1^2c_3c_4a_3 - 3c_1c_2c_3^2a_3 \\ + c_1^2c_4^2a_2 + c_2^2c_3^2a_2 + 4c_1c_2c_3c_4a_2 + 6c_2^2c_4^2 = 0, \end{aligned} \tag{4}$$

$$4c_1^3c_3a_4 - 3c_3c_1^2c_2a_3 - c_4c_1^3a_3 + 2c_3c_1c_2^2a_2 + 2c_4c_1^2c_2a_2 + 4c_4c_2^3 = 0, \tag{5}$$

$$c_2^4 + a_2c_2^2c_1^2 - a_3c_2c_1^3 + a_4c_1^4 = 0. \tag{6}$$

If (3) is satisfied, the second term in the transformed equation will vanish; if (4) is satisfied, the third term will vanish; if (5) is satisfied, the fourth term will vanish, and if (6) is satisfied the new constant term will vanish. We shall now attempt to choose the c_i so that two or more of these equations are satisfied simultaneously.

A natural first step is to attempt to satisfy (3) and (4) and thus arrive at the form $y^4 + b_3y + b_4 = 0$. However, this seems to require the solution of a fourth degree equation. The same reduction by means of a Tschirnhaus transformation, requiring merely the solution of a quadratic, is easily carried out. However, with respect to the quartic (or higher degree) equation, the Tschirnhaus transformation is much more general than the linear fractional.

It is possible, on the other hand, to remove the second and fourth coefficients without solving any equation of degree higher than the third. Before indicating the process it is useful to see that transformation (2) can be simplified. First,

in this and our later work we may take $c_1=1$, since it is not difficult to see that none of our results could be obtained with $c_1=0$. Second, we may also take $c_3=1$. For, in general, if the transformation $y=\phi(x)$ leads to an equation in y with certain coefficients missing, then $y=k\phi(x)$ will lead to a transformed equation with the same coefficients missing, where k is any constant. (The actual values of the non-vanishing coefficients will of course be different.) Thus if a certain group of coefficients can be removed by any transformation of type (2), they can be removed by a transformation in which $c_1=c_3=1$. Equations (3)–(6) then take the simplified form

$$(3') \qquad 4a_4 - 3c_4a_3 - c_2a_3 + 2c_4^2a_2 + 2c_2c_4a_2 + 4c_2c_4^3 = 0,$$

$$(4') \qquad 6a_4 - 3c_4a_3 - 3c_2a_3 + c_4^2a_2 + c_2^2a_2 + 4c_2c_4a_2 + 6c_2^2c_4^2 = 0,$$

$$(5') \qquad 4a_4 - 3c_2a_3 - c_4a_3 + 2c_2^2a_2 + 2c_2c_4a_2 + 4c_2^3c_4 = 0,$$

$$(6') \qquad c_2^4 + a_2c_2^2 - a_3c_2 + a_4 = 0.$$

If we subtract (5′) from (3′), and divide out $2(c_4-c_2)$, which is not equal to zero, we obtain

$$(7) \qquad a_2(c_2 + c_4) + 2c_2c_4(c_2 + c_4) - a_3 = 0.$$

And if we multiply (3′) by c_2 and (5′) by c_4, and subtract, we have

$$(8) \qquad 4a_4 - a_3(c_2 + c_4) - 4c_2^2c_4^2 = 0.$$

Combining these equations leads to a pair of cubic equations, one in c_2c_4, the other in (c_2+c_4):

$$(9) \qquad 8(c_2c_4)^3 + 4a_2(c_2c_4)^2 - 8a_4c_2c_4 + a_3^2 - 4a_2a_4 = 0,$$

$$(10) \qquad a_3(c_2 + c_4)^3 + (a_2^2 - 4a_4)(c_2 + c_4)^2 - 2a_2a_3(c_2 + c_4) + a_3^2 = 0.$$

Now both of these cubics are closely related to a well-known resolvent cubic of (1):

$$(11) \qquad r^3 - a_2r^2 - 4a_4r + 4a_2a_4 - a_3^2 = 0,$$

whose roots are[1]

$$r_1 = x_1x_2 + x_3x_4, \quad r_2 = x_1x_3 + x_2x_4, \quad r_3 = x_1x_4 + x_2x_3.$$

First, the roots of (9) are simply those of (11) multiplied by $-1/2$. And it can be easily verified, though it is less easy to derive, that the roots of (10)[2] are

$$a_3/(a_2 - r_1), \quad a_3/(a_2 - r_2), \quad a_3/(a_2 - r_3).$$

Consequently if we choose any root r of (11), c_2 and c_4 may be found by solving

[1] See, for example, L. E. Dickson, *Elementary Theory of Equations*, (1914), p. 39.

[2] I first obtained essentially this equation in a problem involving the Tschirnhaus transformation; see the Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 73–74.

$$(12)\qquad c_2c_4 = -r/2, \quad c_2 + c_4 = a_3/(a_2 - r).$$

This will require the solution of an additional quadratic. We thus state

THEOREM I. *The general quartic equation can be reduced to the trinomial form* $y^4+b_2y^2+b_4=0$ *by means of a linear fractional transformation, the determination of whose coefficients does not involve the solution of any equation of higher than the third degree.*

Since b_2 and b_4 could be computed explicitly without a great deal of difficulty, since the transformed equation is readily solvable and since x can be expressed in terms of y by means of (2), we have a possible method of solving quartic equations. And in some cases the solution could be performed conveniently. Thus when (11) has a rational root the process will involve only square roots. Speaking for the moment in terms of Galois groups, it is not difficult to verify that (11) will have at least one rational root if equation (1) has rational coefficients, is irreducible in the field of rational numbers, and has as its Galois group for that field one of the three possible imprimitive groups. Certain reducible quartics will also lead to an equation (11) with a rational root.

We may at once state, on the basis of our preceeding work,

THEOREM II. *The general quartic equation cannot be reduced to the binomial form* $y^4+b_4=0$ *by means of a linear fractional transformation.*

This follows at once since the left-hand side of (4′) does not become identically zero when c_2 and c_4 are chosen, as above, to satisfy (3′) and (5′). I have proved[1] the same theorem by a somewhat different method, still working, however, from equations (3), (4), and (5).

But there are certain quartics which can be reduced to binomial form; these are given by

THEOREM III. *Any quartic equation for which*[2] $T=0$ *(where* $432T$ *is defined to be* $72a_2a_4-27a_3^2-2a_2^3$*) can be reduced to the binomial form* $y^4+b_4=0$ *by means of a linear fractional transformation.*

To prove this theorem, we first note that, when $T=0$, equation (11) has the root $a_2/3$. Consequently equations (12) become

$$(13)\qquad c_2c_4 = -a_2/6, \qquad c_2 + c_4 = 3a_3/2a_2,$$

and c_2 and c_4 are the roots of the equation $6a_2c^2-9a_3c-a_2^2=0$. It is now a simple matter to substitute for (c_2+c_4), $(c_4^2+c_2^2+4c_2c_4)$, and $c_2^2c_4^2$ in equation (4′) and the left-hand side actually does reduce to zero provided $T=0$. We have thus proved the theorem and at the same time showed how to set up the necessary transformation.

Finally we consider reduction to the form $y^4+b_2y^2=0$. To obtain this form

[1] Mathesis, vol. 42 (1928), pp. 163–165.

[2] T is, of course, the cubic invariant of the given quartic.

we must make three coefficients of the transformed equation zero. This is not possible in general since we have but two essential parameters, c_2 and c_4, at our disposal, with which to satisfy the three equations (3′), (5′) and (6′). We may, however, state the following:

THEOREM IV. *Any quartic equation for which* $S^3-27T^2=0$ (*where* $S=a_4+a_2^2/12$, *and* T *is defined as above*[1]) *can be reduced to the form* $y^4+b_2y^2=0$ *by means of a linear fractional transformation.*

In the proof of this theorem it is convenient to use this

Lemma. *In case* $S^3-27T^2=0$, *the two equations*

$$4c^3+2a_2c-a_3=0, \tag{14}$$

$$2a_2c^2-3a_3c+4a_4=0 \tag{15}$$

have a common root.

The condition that these two equations have a common root is the vanishing of a certain fifth order determinant, and when the determinant is expanded the condition becomes simply that assumed in the lemma. We now choose c_2 as the common root of (14) and (15), and assume that these equations are re-written with c_2 substituted in place of c. If we now multiply (14) by c_2 and add (15), we obtain equation (6′). That is, this equation is satisfied by our choice of c_2. Similarly, if we multiply (14) by c_4 and add (15) we see that (5′) is satisfied by this same choice of c_2, without restriction on c_4. We may now choose c_4 as a root of the quadratic equation (7). From the way in which (7) was obtained we are now sure that(3′) is satisfied. Thus the proof of theorem IV is complete.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

DISCUSSIONS

I. STANDARD DEVIATION PROBLEMS WITH RATIONAL ANSWERS

By WALTER CROSBY EELLS, Stanford University

In teaching an elementary course in statistical method it is frequently desirable, when first studying the standard deviation and the Pearson product-moment coefficient of correlation, to have available problems for which the answers are known to be rational and perhaps integral as well. This is desirable for simple illustrative problems, for class assignments, for construction

[1] S is the quadratic invariant of the given quartic, and S^3-27T^2 is its discriminant.

of tests and examinations, for preparation of alternative equivalent forms of such tests—in all of which cases it is frequently desirable to illustrate the principles with simple problems in which computation is reduced to a minimum. Many elementary students experience difficulty with square root and vary greatly in the facility with which they handle problems in which it is involved. Therefore it is useful at first to have a few problems available in which the square root can be found by inspection.

One or two of the elementary textbooks have such problems, but they are not frequent. For example Jerome[1] gives the following (with slight change of notation) as his introductory problem:

X	d	d^2
3	3	9
5	1	1
6	0	0
7	1	1
9	3	9
Mean$=6$		20

$$\sigma=\sqrt{(20/5)}=\sqrt{4}=2$$

This is composed of X's of the type

$$X = x + d \qquad (d = 1,3)$$

or more generally of the type

$$X = x + kd \qquad (d = 1,3),$$

where $x=$ the mean, $k=$ any rational number, integral or fractional, $d=$ specially selected integers.

Numberless variations of this simple problem are possible. Typical ones are given in Table 1 for five selected values of x and k. For convenience a series of values like (1, 3) will be referred to as a *fundamental set.*

TABLE 1

Typical Variations of Fundamental Set (1, 3) *to Give Rational Values of the Standard Deviation.*

	$x=6$ $k=1$	$x=10$ $k=1$	$x=7.5$ $k=1$	$x=6$ $k=2$	$x=6$ $k=0.5$
$x-3$	3	7	4.5	0	4.5
$x-1$	5	9	6.5	4	5.5
x	6	10	7.5	6	6.0
$x+1$	7	11	8.5	8	6.5
$x+3$	9	13	10.5	12	7.5
	$\sigma=2k$ $=2$	$\sigma=2k$ $=2$	$\sigma=2k$ $=2$	$\sigma=2k$ $=4$	$\sigma=2k$ $=1$

It may thus be seen that there are a doubly infinite number of variations possible for the same fundamental set, obtained by varying x or k or both.

[1] Harry Jerome, *Statistical Method*, New York, 1924, p. 156.

But are there other essentially distinct fundamental sets of the same type? It is the object of this paper to record the existence of all such sets for values of $n<10$ and of $d\leqq 10$.

Let X = original scores,

x = mean of original scores, to be selected as any desired rational value, integral or fractional,

n = number of cases, or population,

k = any rational value, integral or fractional,

d = any positive integer.

Then in the expression

$$X_i = x \pm kd_j;\quad i = 1,2,3, \cdots, n;\quad j = n/2 \text{ if } n \text{ is even},\quad (n-1)/2 \text{ if } n \text{ is odd},$$

we seek values of the d's, such that

$$\sigma = \left(\frac{\sum k^2 d_j^2}{n}\right)^{1/2} = k\left(\frac{\sum d_j^2}{n}\right)^{1/2} = \text{a rational number}.$$

For $n<10$ and $d\leqq 10$ it can be shown that there are 385 essentially different combinations possible. By listing all of these it is found that the only sets of values of the d's having the desired property are the following:

$n=4$: One set of d's, 1, 7.
$n=5$: One set of d's, 1, 3.
$n=6$: One set of d's, 1, 5, 7.
$n=7$: Two sets of d's, 1, 2, 3; 1, 5, 10.
$n=8$: No sets of d's,
$n=9$: Three sets of d's, 1, 4, 8, 9; 1, 5, 6, 10; 2, 3, 7, 10.

The sets of measures or original scores (X's) corresponding to these d's, and the standard deviations resulting from them, are exhibited in Table 2. For convenience in this table, k has been taken as 1.

The three sets possible when $n=9$ are particularly interesting since they all lead to the same standard deviation, 6.

With the variations possible for any single fundamental set, as already illustrated in Table 1, these eight fundamental sets furnish an abundance of material for constructing problems involving the computation of the standard deviation which will "come out even." It is easy to memorize some of the combinations, especially those for $n=5$, and $n=7$. If it is desired to have the answer always integral, a sufficient condition (although not a necessary one) is that k shall be an integer. If it is desired to have the answer fractional, it is only necessary to so choose k that $2k$, $5k$, or $6k$ as the case may be shall be fractional.

In the same way in setting up simple problems for the computation of the Pearson product-moment coefficient of correlation, this table is of great value. Thus for a population of seven, one set of characteristics (x) can be assigned such numerical values that $\sigma_x=2k$, and the other set (y) such that $\sigma_y=6k$. Or with

a population of nine, various such sets are possible with the three fundamental sets given, although in this case both standard deviations will be 6 or some multiple of it.

TABLE 2

Sets of Individual Measures Having Rational Integral Standard Deviations ($k=1$; $n<10$; $d\leqq 10$)

$n=4$	$n=5$	$n=6$	$n=7$	$n=7$
$x-7$	$x-3$	$x-7$	$x-3$	$x-10$
$x-1$	$x-1$	$x-5$	$x-2$	$x-5$
$x+1$	x	$x-1$	$x-1$	$x-1$
$x+7$	$x+1$	$x+1$	x	x
	$x+3$	$x+5$	$x+1$	$x+1$
$\sigma=5$		$x+7$	$x+2$	$x+5$
	$\sigma=2$		$x+3$	$x+10$
		$\sigma=5$		
			$\sigma=2$	$\sigma=6$

$n=8$	$n=9$	$n=9$	$n=9$
none	$x=9$	$x-10$	$x-10$
	$x-8$	$x-6$	$x-7$
	$x-4$	$x-5$	$x-3$
	$x-1$	$x-1$	$x-2$
	x	x	x
	$x+1$	$x+1$	$x+2$
	$x+4$	$x+5$	$x+3$
	$x+8$	$x+6$	$x+7$
	$x+9$	$x+10$	$x+10$
	$\sigma=6$	$\sigma=6$	$\sigma=6$

It may be said that such problems in standard deviation or product-moment correlation are highly artificial. This is true, but the author has found them very useful in introducing these topics to classes unfamiliar with them when it is desirable to minimize the computational difficulties and to concentrate on the fundamental ideas involved. It was the feeling of need for such problems in his own classes that led to the working out of these eight fundamental sets (all of this type existing, with the limitations noted above) which furnish an abundance of such problem material. Numberless simple variations are possible, but in every case the instructor knows in advance that they will come out rational, and integral if desired, and he knows the answer without computation. Possibly other instructors in elementary statistics, in various fields, will be glad to have this source material for construction of problems in a form convenient for use.

II. Marginal Notes[1]

By T. H. Hildebrandt, University of Michigan

1. This is not a discourse on the desirability of making marginal notes in one's mathematical reading or teaching. It is simply a collection of notes of the type that every teacher who is alive to the subjects which he is teaching makes in the margins of his textbook. As a consequence there is no particular claim to originality or priority. In presenting these remarks, it is hoped that perhaps some other teacher of the first two years of collegiate mathematics may find something which will be helpful in his own teaching.

My first note refers to the area of a triangle, the coordinates of the vertices being $P_1:(x_1, y_1)$; $P_2:(x_2, y_2)$; $P_3:(x_3, y_3)$. The usual formula for this area is either a third order determinant, or the same expanded and arranged as follows:

$$\tfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)],$$

I do not understand why (if third order determinants are to be avoided) this latter is preferred to the equivalent expansion

$$\tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)],$$

which has the advantage of an easy geometric interpretation and proof and which is easily remembered and easily extended. For $\frac{1}{2}(x_1y_2 - x_2y_1)$ is the directed area of the triangle OP_1P_2O, a fact which is easy to prove. Then one finds that the directed area $P_1P_2P_3P_1$ is equal to the sum of the directed areas OP_1P_2O, OP_2P_3O, OP_3P_1O. Ease in remembering is of course increased by using second order determinants, which gives in reality a method for expanding the third order determinant form.[2] By way of extension, we note that it is an easy matter to write down the area of a closed polygon. For instance the directed area of the polygon $P_1P_2P_3P_4P_5P_1$ is

$$\frac{1}{2}\left\{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} + \begin{vmatrix} x_4 & y_4 \\ x_5 & y_5 \end{vmatrix} + \begin{vmatrix} x_5 & y_5 \\ x_1 & y_1 \end{vmatrix}\right\}.$$

The geometric interpretation of the answer would in general require a figure, especially where the sides intersect between the vertices (as for instance in a five pointed star). If we write

[1] Read before the Michigan Section of the Mathematical Association of America, March 31, 1927, at Ann Arbor, Mich.

[2] As a side remark, I should like to voice the opinion that it would be advantageous to drop from our teaching of determinants the so-called diagonal method of expansion of third order determinants and to replace it by the expansion by minors. The diagonal method is confusing because it is not applicable to determinants of order higher than the third, as one might have a right to expect. Expansion by minors is not only applicable to determinants of any order, and hence need not be unlearned, but even for determinants of the third order is in most cases simpler than the diagonal method. Moreover, it can be made (as any one can convince himself very easily) a logical foundation for the definition of the value of a determinant of any order and can be used as a basis for proof (by induction) of the properties of determinants.

$$x_1y_2 - x_2y_1 = x_1(y_2 - y_1) - y_1(x_2 - x_1) = x_1\Delta y - y_1\Delta x$$

it is easy to see that we are not far here from the rectangular equivalent of the polar element of area, which has to a large extent disappeared from our textbooks in calculus.

The extension of these considerations to three dimensions is not so simple. Obviously a cyclic performance such as

$$\tfrac{1}{6}[\,|x_1y_2z_3| + |x_2y_3z_4| + |x_3y_4z_1| + |x_4y_1z_2|\,],$$

in which $|x_1y_2z_3|$ expresses the third order determinant on the elements x_1, y_1, z_1; x_2, y_2, z_2; x_3, y_3, z_3, is not the volume of the tetrahedron $P_1P_2P_3P_4$, though $|x_1y_2z_3|$ is the volume of $OP_1P_2P_3$. The difficulty lies in the fact that a directed volume is slightly more complicated than a directed area. It can be shown that the result desired is:

$$\tfrac{1}{6}[\,|x_1y_2z_3| + |x_1y_3z_4| + |x_1y_4z_2| + |x_4y_3z_2|\,],$$

which shows a certain amount of symmetry especially if placed in juxtaposition with the fundamental equality which we used in connection with the area of a triangle, viz.,

$$OP_1P_2 + OP_2P_3 + OP_3P_1 + P_3P_2P_1 = 0.$$

The addition of a fifth point P_5 to the melee brings up the question what volume is defined by the points $P_1P_2P_3P_4P_5$. Obviously higher dimensions increase the difficulties. We are touching here upon the notion of directed volumes in higher dimensions.

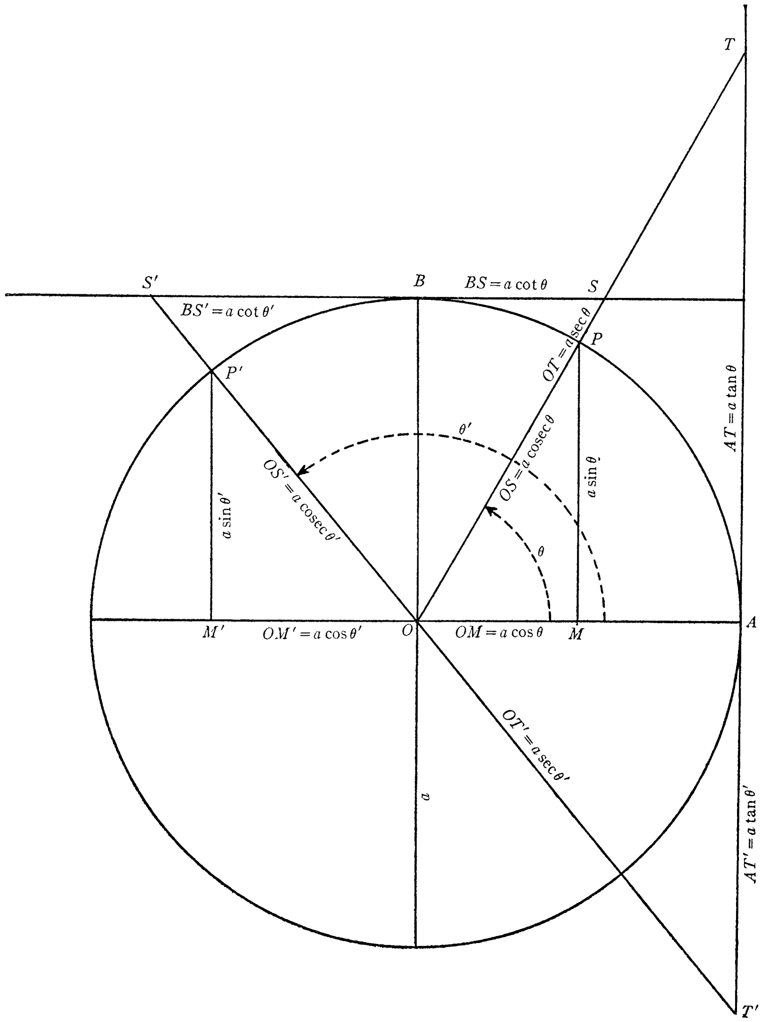
2. In the margin of the section on curve tracing in polar coordinates I find the adjoining well known (?) figure from trigonometry. It is being omitted from trigonometries to some extent, mainly because it is seldom put to use later on. If used in connection with polar coordinate paper and a pair of dividers, it yields a simple and elegant method for plotting without laborious computation enough points of curves with polar equations of the following types: $r=a\cos\theta+b$, $r=a\sec\theta+b$, $r=a\tan\theta$, $r=a\tan\theta+b\sec\theta$, to enable even freshmen to produce fine sketches. Methods may be developed for using this figure even for such curves as $r=a\cos 2\theta$, $r\cos 2\theta=a$, and others containing trigonometric functions.[1]

3. My next note occurs in connection with the remark that the tangent to a circle at a given point (x_1, y_1) on the circle can be obtained by writing down the equation of the point circle, center at (x_1, y_1) and finding the common chord of this circle and the given circle. The question arises whether this idea is extensible to conic sections and is to be answered in the affirmative. Thus the tangent to $a^{-2}x^2+b^{-2}y^2=1$ at (x_1, y_1) is the common chord of this conic and the degenerate conic $a^{-2}(x-x_1)^2+b^{-2}(y-y_1)^2=0$; the tangent to $a^{-2}x^2-b^{-2}y^2=1$

[1] More extensive remarks on the same subject are contained in a note by H. A. Bender in this Monthly, vol. 34 (1927), pp. 481–484.

is the common chord with $a^{-2}(x-x_1)^2-b^{-2}(y-y_1)^2=0$; and the tangent to $y^2=4ax$ is the common chord with $(y-y_1)^2=0$. This leads to the general statement that the tangent to

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$



at (x_1,y_1) on the curve is the common chord of this conic and the similar degenerate conic:

$$A(x-x_1)^2+B(x-x_1)(y-y_1)+C(y-y_1)^2=0.$$

In addition to giving us the equation of a tangent to a conic without the necessity of finding the slope by a limits process, i.e., the use of derivatives, it has the additional advantage of making it possible to write down the equation

of the tangent plane at a given point to a quadric in three-space. For example the equation of the tangent plane to the quadric $Ax^2+By^2+Cz^2+D=0$ at the point (x_1, y_1, z_1) on the quadric is the common plane of this quadric and the similar degenerate quadric: $A(x-x_1)^2+B(y-y_1)^2+C(z-z_1)^2=0$. I leave unanswered the question of what this process gives when the point is not on the conic or quadric.

4. At a point in the proof of the fact that the locus of points, the sum of whose distances from two fixed points is a constant, is an ellipse, I find the word "stop" in the margin. If for simplicity the fixed points are $F_1:(b, 0)$, $F_2:(-b, 0)$, and the constant is $2a$, then the first equation reads:

$$[(x - b)^2 + y^2]^{1/2} + [(x + b)^2 + y^2]^{1/2} = 2a.$$

Rationalizing[1] this equation and collecting terms ready for the second rationalization, one gets the expression to which the remark applies

$$a[(x - b)^2 + y^2]^{1/2} = bx - a^2.$$

Interpreting this equation we find that it says that the distance of (x, y) from $(b, 0)$ is b/a times its distance from the line $x=a^2/b$; i.e., we have arrived at the focus-directrix property of the ellipse, which is usually the basis of discussion in analytic geometry. Obviously this observation will give us the eccentricity and equations of directrices in any case where the foci and major axis are given.

5. In connection with the section on transformation of coordinates, I find the query, "why not use on curves other than just conic sections?" For instance, in calculus, one finds occasionally such equations as $x^3-3axy+y^3=0$, and $x^2y+xy^2=a^3$, having $x=y$ as an obvious line of symmetry. If this is made the new x-axis, the equations are of relatively simple type, solvable for y^2, and in the first case, it makes it even possible to find the area of the loop of the curve without resorting to the trick of polar coordinate representation.

6. In my calculus, I find such notes as the following:

(a) $1-\cos x$ can be rationalized by multiplying by $1+\cos x$, a remark which helps to avoid the introduction of half angles in proving that $\lim_{x\to 0}(1-\cos x)/x=0$, and in the integration of expressions such as $\int(1-\cos x)^{1/2}dx\cdot \int dx/(1-\cos x)$.

(b) For some curves it is simpler to apply the fundamental definition of curvature: $d\alpha/ds$, instead of the expression in terms of y' and y''. Examples $y=\log \cos x$; the circle $x=a \cos t$, $y=a \sin t$; the four cusped hypocycloid $x=a \cos^3 t$, $y=a \sin^3 t$; the cycloid; and the involute of the circle.

(c) The tangential and normal components of acceleration of a body moving in a plane curve are the components of velocity of a point on the corresponding

[1] Professor Shohat remarked in discussing this paper that this process could be simplified by noting that $(PF_1)^2-(PF_2)^2$ is a rational expression, so that by division by PF_1+PF_2 we get PF_1-PF_2 from which PF_1 and PF_2 can be determined at once, giving the same result as above. My point is that even the straightforward usual methods have interesting interpretations.

hodograph, along and perpendicular to the radius vector to the origin. Since this vector on the hodograph is v, the polar angle α (the angle between the tangent and x-axis on original curve), the tangential acceleration is dv/dt, and the normal acceleration is $v d\alpha/dt = v(d\alpha/ds)(ds/dt) = v^2/\rho$, where ρ is the radius of curvature of the original curve.

(d) A rational function which is a proper fraction can be uniquely expressed as the sum of proper fractions having the distinct factors of the original fraction as denominators. For example

$$\frac{4x^2 + x + 5}{(x - 2)^3(x^2 + 1)^2} = \frac{Ax^2 + Bx + C}{(x - 2)^3} + \frac{Dx^3 + Ex^2 + Fx + G}{(x^2 + 1)^2},$$

and the coefficients in the numerator are uniquely determinable. (They are no longer unique if the fractions on the right are not proper). The emphasis being upon the *proper* fraction, it follows that the numerator is always assumed to be one degree less than that of the denominator, and we have a method covering all possible cases. Incidentally this idea does not really increase the difficulty of integration. A term like the first, with a power of a linear expression in the denominator always yields to the substitution of a new variable for the linear expression. For terms of the second type, note that for instance $x^2/(x^2+1)^2$ is easier to integrate than $1/(x^2+1)^2$. Integration by parts, or substitution of $x = a + b \tan \theta$, for a term whose denominator is of the form $[(x-a)^2+b^2]$ will always work.

(e) An expression of the form

$$e^{\alpha x}[(A_n x^n + \cdots + A_1 x + A_0) \cos \beta x + (B_n x^n + \cdots B_1 x + B_0) \sin \beta x]$$

is a solution of a linear homogeneous differential equation with constant coefficients in which the associated algebraic equation (obtained by putting $y = e^{mx}$ and dividing by e^{mx}) has the roots $\alpha \pm i\beta$ repeated $n+1$ times. This remark enables us to write down the differential equations whose general solutions are similar to

$$y = (c_1 + c_2 x) + c_3 e^{-x} + e^{3x}(c_4 \cos 2x + c_5 \sin 2x).$$

It also suggests a method for writing down the form of the particular integral for a linear differential equation with constant coefficients, in which the terms which are functions of the independent variable only, are linear combinations of expressions like the above. A particular example will illustrate best. Consider the differential equation

$$y'' - 4y' - 5y = 3xe^{-x} + 6 \cos 2x + 5 \sin 2x + 4e^x \sin 3x.$$

Since the general solution of this differential equation will also be a solution of a linear *homogeneous* differential equation with constant coefficients but of higher order (at least eighth) the solution will be determined by an algebraic equation whose roots are 5 and -1 (from the left hand side of the equation), and -1, -1, $\pm 2i$, $1 \pm 3i$ (from the right). Note that $3xe^{-x} = (3x+0)e^{-x}$ and

$4e^x \sin 3x = e^x$ $(0 \cos 3x + 4 \sin 3x)$. As a consequence the complementary function is $y = c_1 e^{5x} + c_2 e^{-x}$ and the particular integral has the form

$$y = (c_3 x + c_4 x^2)e^{-x} + c_5 \cos 2x + c_6 \sin 2x + e^x(c_7 \cos 3x + c_8 \sin 3x),$$

the constants $c_3, \cdots, c_8$ being determined by the usual method of undetermined coefficients. Note that -1 counts as a triple root, as it actually is.

III. The Cheney Check Formulas

By William R. Ransom, Tufts College

After a quarter century of teaching college freshmen to solve and check right triangles, I was recently greatly pleased with the discovery by my colleague, Dr. William Fitch Cheney, Jr., of a much better check formula than any I had met during that time. From the identity

$$(c + a)[(c + b)/c] = (c + b)[(c + a)/c]$$

he obtains

$$(c + a)(1 + \cos A) = (c + b)(1 + \cos B)$$

which may be readily adapted to logarithmic computation by introducing half-angles. The logarithmic form is

$$\log (c + a) + 2 \log \cos (A/2) = \log (c + b) + 2 \log \cos (B/2).$$

Each of these formulas contains all five parts of the triangle; they contain no minus signs, are easily memorized, offer no troublesome peculiarity when the triangle is nearly isosceles or very slim, and they are equally available for all modes of solving since they use factors and logarithms not used in solving.

A Letter to the Editor

I have received a letter from Professor Julio Rey Pastor, formerly professor of mathematics in the University of Madrid and now professor at the University of Buenos Aires, in which he calls my attention to a paper "Caracteres de las formas cuadraticas definidas, con aplicacion a varias cuestiones" published by him in the Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales de Madrid, January, 1911. A reprint of the article accompanied his letter.

The paper contains a proof of the theorem on quadratic forms that appeared in my recent article in the Monthly,[1] using also the method of mathematical induction. Professor Pastor gives several interesting applications of the theorem. It seems unfortunate that the publications of mathematicians in Spain and South America are so nearly unavailable in this country.

Leonard M. Blumenthal

[1] Vol. 35 (1928), pp. 551–554.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College, New York, N. Y.

All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Introduction to the History of Science. Volume 1, from Homer to Omar Khayyam. By G. Sarton. Published for the Carnegie Institution of Washington by The Williams and Wilkins Co., Baltimore, 1927. Royal 8 vo. 12+839 pages. Price $10.00.

America's intellectual life has been enriched not a little as a result of the Great War. A notable instance is that of the author of this first volume of a work, monumental in conception, a Belgian scholar who arrived in the United States in 1915 and later adopted it as his country. For a decade he has been an Associate of the Carnegie Institution, and largely as a result of his inspiration the History of Science Society sprang into being in 1924, and the journal *Isis*, founded by Doctor Sarton in 1912, became its official organ. His great capacity for work is displayed not only by the large annual volumes of this publication and the completion of the work under review, but also by the fact that the manuscript for the second volume of the *History* is not far from being ready for the press. The period dealt with in the first volume is from the ninth century B.C. to the eleventh century A.D., and hence we do not find more than passing references to the very important ancient civilizations of Egypt, Babylon, and China.

The author looks upon the development of science as the development of systematized positive knowledge and thus one is impressed with his breadth of outlook even though the omission of discussion in some fields and the inclusion of others might at first be questioned. But because excellent works of reference are already available there are few references to such things as political history, economic history, and history of art. On the other hand there is discussion of the history of music, which for a millenium was regarded as a part of mathematics; and much space is devoted to the history of religion, since the author was not aware of any work giving in chronological order an account of the religious experiences of mankind. The early history of philology is also regarded as of considerable importance. For, as the author remarks, "the discovery of the logical structure of language was as much a scientific discovery as, for example, the discovering of the anatomical structure of the body."

In further defining the field of his inquiry Doctor Sarton contemplated at first its limitation to pure science but he met with difficulties in separating consideration of pure science from that of its applications. It was finally decided to refer, for examples, to "a physician, an engineer, or a teacher only if he added something definite to our knowledge, or if he wrote treatises which

were sufficiently original and valuable, or if he did his task in such a masterly way that he introduced new professional standards."

The work is divided into a long introductory chapter, and 34 other chapters each dealing with a particular epoch and labelled by a descriptive title including, for the most part, the name of an outstanding individual of the epoch: thus (I) "The Dawn of Greek and Hebrew Knowledge (ninth and eighth centuries B.C.)"; (V) "The Time of Plato (first half of fourth century B.C.)"; (VIII) "The Time of Archimedes (second half of third century B.C.)"; (XXII) "The Time of Proclos (second half of fifth century)"; (XXVII) "The Time of Bede (first half of the eight century)"; (XXIX) "The Time of al-Khwarizmi (first half of ninth century)"; (XXXIV) "The Time of Omar Khayyam (second half of eleventh century)". After the fourth chapter each chapter covers a half century so that chapters V to XXXIV deal with the period from the fourth century B.C. to the end of the eleventh century of our era.

To give an idea of the kind of material one may find in a chapter, some indication of the contents of chapters VIII and XXII will be given. In chapter VIII there are nine main sections: 1. Survey of Science in the second half of the third century B.C.; 2. Development of Buddhism in India; 3. Unification of China; 4. Chinese and Greek philosophy; 5. Greek mathematics, astronomy, and physics (the contributions of Archimedes, Eratosthenes, Conon, and Apollonios of Perga are considered); 6. Introduction of Greek medicine in Rome; 7. Chinese and Hellenistic technology; 8. Greek and Roman historiography; 9. Chinese writing. With each section there is a bibliography; in case of Apollonios this fills more than a page, and of Archimedes more than two pages. (Dr. Sarton attributes an edition of the works of Archimedes to Kliem who was in this connection only a translator of Heath's edition.) Throughout the different chapters the bibliographies, with many critical comments, constitute an exceedingly valuable feature of the work.

In chapter XXII, the time of Proclos, the nine subheadings are: 1. Survey of Science in the second half of the fifth century; 2. Religious background (the Talmud); 3. Hellenistic, Syriac, and Latin philosophy (Proclos, and half a dozen others); 4. Latin, Hellenistic, and Hindu mathematics (Domninos, Metrodorus, Āryabhata; Doctor Sarton will probably want to refer in a new edition to Datta's "Two Āryabhatas of Al-Biruni," *Bull. Calcutta Math. Soc.*, 17, 1926, p. 60-74; and to A. G. Laird's, "Plato's geometrical number and the comment of Proclus," Madison, Wis., 1918); 5. Latin Hellenistic, Chinese, and Hindu Astronomy; 6. Chinese Geography; 7. Latin and Singalese historiography; 8. Roman and barbarian law; 9. Chinese philosophy.

While such scholars as Steinschneider, Suter, Vincent, and Woepcke finely contributed to our knowledge of Muslim mathematics, the work which Doctor Sarton is publishing contains the first fairly complete account of mediaeval science as a whole. Why the Muslims were so far ahead of the Christians from the eighth to the eleventh centuries is set forth clearly. The Greek tradition

in connection with disinterested research was stifled by Roman utilitarianism followed by theological expediency, and "theological domination, which seemed for a long while to destroy every hope of a genuine scientific revival." On the other hand Persianized Muslims enthusiastically studied and imitated the intellectual structures reared by the Greeks, and from Spain to central Asia were developed centers radiating Muslim culture.

The student and teacher of the history of mathematics will find Doctor Sarton's very valuable work suggestive at almost every turn, and notably supplementing material available in other sources. It is heartily recommended both to the amateur and to the scholar.

R. C. ARCHIBALD

Algebra. By Oskar Perron. Walter de Gruyter and Co. Berlin and Leipzig, 1927. I: Die Grundlagen, VIII+307 pages. II: Theorie der algebraischen Gleichungen, viii+243 pages.

These two volumes, 8 and 9 of the Göschen series in pure mathematics, are from the pen of the author of the first volume. With each addition to the series these useful expositions seem to acquire a more substantial character. It is the express purpose of the author to write a book which, though designed primarily to meet the needs of advanced students, shall also be of service to the investigator. This double aim has been admirably accomplished for the book is more than a transcription, with pedagogic alterations or the more common pedagogic deletions, of existing treatises. A fair share of the important duty of putting modern mathematical research into connected exposition has been skillfully done.

In the view of the author algebra is that part of analysis which is based on the rational operations rather than on the notions of relative magnitude and of limit. The only important departures from this limitation are those inevitably involved in the proof of the fundamental theorem that every equation with coefficients in the field of complex number has a root belonging to that field, and in the numerical calculation of the roots. The notion of field is introduced at the outset and theorems are stated thereafter with reference to their field of validity.

The student of the subject will be attracted by the orderly and careful development, and by the sympathetic guidance afforded by the illuminating comment at various stages of progress. The historical references are adequate though not especially numerous.

In the first volume the existence of the real numbers as developed by the author in the initial volume of the Göschen series is assumed. After the introduction of complex number, number fields and number rings are defined. These concepts are then applied to polynomials. A second chapter on formal processes is followed by two of conventional type on determinants and symmetric functions. The last two chapters headed respectively "divisibility" and "existence of the roots" are of more novel character. The discussion of dependent polynomials

and of resultants, with finally a proof of Bézout's theorem on the number of solutions of k equations in k variables, based on relatively recent work of Mertens, is most satisfactory. Certain familiar topics such as elementary divisors, and the equivalence of pairs of bi-linear forms are not mentioned. For these topics, already well handled in current texts, the above substitutions are very welcome.

The second volume contains preliminary chapters on the numerical solution of equations and the solution of particular equations. A chapter on substitution groups is followed by an exposition of the Galois theory of equations. The volume concludes with a chapter on the equation of the fifth degree. The author first gives a method by which the general quintic equation can be transformed into the Brioschi normal form by the adjunction of square roots alone. According to a verification which he makes but does not reproduce this method fails only in trivial metacyclic cases. The sextic resolvent of Jacobi is then solved by "elliptic functions" rather than elliptic modular functions. The elliptic functions actually used are the values of the p-function for fifths of the periods, which in the terminology of Klein-Fricke are modular forms. From a function-theoretic standpoint such forms are not as simple as modular functions but, as Perron points out, they are more easy of access. Finally the roots of the Brioschi quintic are expressed in terms of the roots of the Jacobi resolvent.

ARTHUR B. COBLE

An Introduction to Linear Difference Equations. By Paul M. Batchelder. Published with the cooperation of the National Research Council. The Harvard University Press, 1927. viii+209 pages.

The development of the theory of linear difference equations for a long time lagged far behind that of the related field of linear differential equations. This fact was due to certain intrinsic difficulties in the former which called for the introduction of new ideas and methods. Some of these ideas and methods were introduced by Poincaré as far back as 1885 and 1886 (American Journal of Mathematics and Acta Mathematica) in connection with his development of the theory of asymptotic representation of functions by means of power series. But it was in the years 1909 to 1912 that the theory of difference equations began to take satisfactory form from the function-theoretic point of view. In these years comprehensive existence theorems were first developed and published almost simultaneously by workers in Denmark and America and France.

In recent years there has been a rapid development of the theory. Prior to the publication of the book under review only one treatise had appeared taking account of these recent advances, namely, *Vorlesungen über Differenzenrechnung* by N. E. Nörlund. Notwithstanding the fact that this book puts much greater emphasis upon some parts of the development than upon others of equal importance, it will probably remain for some time the most valuable treatment of the subject as a whole.

The work of Batchelder makes no attempt to be a comprehensive treatise;

in fact, its purpose is quite different. Batchelder contrasts his purpose with that of Nörlund in the following passage from the preface:

> Up to the present time no attempt has been made to provide the student with a convenient introduction to this new field, in which so many problems still await solution. The only book which deals with the theory from the modern standpoint is the recent one of Nörlund; while this is of great value to the advanced student in furnishing a general view of the literature of the subject, its failure to give a systematic presentation of the elements of the theory renders it unsuitable for the beginner. The aim of the present book is in part to fill this need by affording as simple and direct an approach as possible to the fundamental facts and ideas of the theory, and in part to extend the boundaries of the subject by studying certain important exceptional cases which have hitherto defied analysis.

Knowledge of the elements of the theory of functions of a complex variable is sufficient to enable the student to read the book of Batchelder. In the first chapter (pp. 1–31) several preliminary topics are treated and the reader becomes acquainted with some of the fundamental concepts and methods which he will need later. The second chapter (pp. 32–67) is devoted to equations of the first order. In particular, it contains a development and some applications of the theory of the gamma functions. In the first two chapters exercises are added for the use of the student. The remaining two chapters are devoted to the hypergeometric difference equation, that is, the linear homogeneous equation of the second order with linear coefficients. The third chapter (pp. 68–126) deals with the so-called general case. The irregular cases are treated in the fourth chapter (pp. 127–209). Unfortunately there is no index.

The treatment throughout, which is inspired principally by the methods of Birkhoff, is based mainly on the divergent power series which formally satisfy the difference equation; but use is also made (to a lesser extent) of certain methods of Nörlund. The first three chapters are the more valuable ones for the learner. The last chapter contains useful contributions to the theory. In the opinion of the reviewer, the book would have been more useful if the main content of the last chapter had been published in a memoir and the space thus saved had been devoted to other more general introductory aspects of the theory. But the first three chapters will probably furnish the reader with a sufficient introduction to the subject to enable him to pass with comfort to a study of the original memoirs. It is in this that the chief value of the book lies. Since it was not intended to be a comprehensive treatise on its subject, this is the most important use which it can serve; and it serves well such a purpose. By aid of this book it now becomes possible for the first time to obtain from the printed page (that is, from books and memoirs) a comfortable introduction to the theory of linear difference equations. On this account the book will be a very useful one.

This is not the place to give a detailed critical analysis of the exposition. In the main the treatment is satisfactory, though it is not always entirely free from criticism.

Notwithstanding the excellencies of this book and that of Nörlund and the fact that they supplement each other in useful ways, and notwithstanding the

great richness of content of Nörlund's book, there is still a distinct need for a comprehensive exposition of the theory of the difference calculus—an exposition which shall adequately present the principal researches already achieved and shall also fill up not a few lacunae now existing in the theory. The need of supplying these deficiencies makes it a difficult matter to prepare just the sort of book that is required; but the need remains as a challenge to the workers in this field.

R. D. CARMICHAEL

Plane and Spherical Trigonometry. By Jabir Shibli. Ginn and Company, 1928.

Professor Shibli has written a textbook on trigonometry that will make the subject a pleasure to the learner and a joy to the teacher. The matter contained in his book does not differ from that of dozens of other texts, but the manner of the presentation distinguishes it. The approach to the various topics is made in a fashion so attractive as to enlist the students' interest, and interest is the keynote of success in any study. With illustrations by simple graphic solutions of problems the learner is led on to understand the functional relations of the parts of a triangle and to proceed with confidence to the application of them in trigonometric analyses.

Some parts of his text, notably Chapter XI, "Complex Numbers," are rather sketchy; but the teacher can easily fill up the gaps if it is thought best to treat this matter in a freshman course in trigonometry. Long lists of problems of a nature to appeal to the student's interest are given at frequent intervals, from which the teacher may assign work suitable to all degrees of ability and industry of his students. The book is pedagogically sound and will no doubt meet with the favor it deserves. The publishers have given it a setting worthy of the book. It is a pleasure to see a textbook whose outward dress conforms so simply and yet so tastefully with its contents.

GEORGE A. HARTER

Mathematics in Liberal Education. By Florian Cajori. The Christopher Publishing House, Boston, 1928. 169 pages. Price $1.50.

It seemingly has been the habit of the centuries to challenge the value of mathematics in the school curriculum. Hence there have been many expressions of approval or disapproval from the greatest men of all ages. Professor Cajori, in his book, "Mathematics in Liberal Education," has collected this testimony and made it available for those who are sincerely interested in the subject. This book he has "dedicated to the advancement of sound education in America." On page 19, the author states his purpose thus: "We are endeavoring to take as complete a survey as possible of writings on the educational value of mathematics. We have taken great pains to collect judgments on both sides of the question. We have not selected our witnesses. We have given by name and have counted all writers known to us who have expressed themselves on the

mind-training value of mathematics. In no case have we suppressed the name of a writer whose testimony we found to be against mathematics."

Of course, the mathematician will be impressed by the obvious fairness of the plan and even the modern psychologist (if he believes that the witnesses were not "picked") will be forced to admit that Professor Cajori has made a very strong case for the great value of mathematics in a liberal education. The testimony of the witness is given in his own words and the comments of the author are brief and impartial, and cause one to feel that each witness is being given a fair chance to speak for himself.

The witnesses are divided into six groups: Greek, Roman and middle ages, fifteenth and sixteenth centuries, seventeenth and eighteenth centuries, nineteenth century, twentieth century. After each group is given a brief summary, the character of which may be illustrated by this quotation from the first one: "We find, therefore, that Protagoras opposed mathematics as necessary for mind training, while Plato, Aristotle, . . . , Plutarch, and Proclus stressed its great educational value. The count is 12 to 1."

The closing chapter gives a table, showing the statistical results of the preceding chapters, together with an excellent study of these results. Of those quoted, 603 favored the study of mathematics and 128 opposed it. Even if the testimony of the 195 teachers of mathematics be excluded because of bias, the vote would still be 413 to 123. It is interesting to note that though the average vote is about 5 to 1 in favor, the vote was closest in the seventeenth and eighteenth centuries when it was 103 to 30.

The book is attractive in appearance and the reader will find it most convenient for reference, as it is arranged chronologically and has a good index. Professor Cajori has done a fine piece of work that the young teacher will find particularly helpful at this time.

R. P. Stephens

Trigonometry. By A. W. Siddons and R. T. Hughes. Cambridge University Press, 1928. viii+320 pages.

If an instructor famed as a driver of students were allowed by his department and dean to give a hundred and twenty assignments from the English text under review to a selected class of American collegians, then that class would learn trigonometry as do few classes in this country. American colleges do not pretend to teach the amount of trigonometry contained in this book, written by two masters at Harrow and presumably designed primarily for use at the English public schools. By omitting four examples out of five, a class meeting fifty times could cover fairly adequately the first two parts of the text; the third part, dealing for some sixty pages with complex numbers, exponential and hyperbolic functions, and series, presupposes some mathematical maturity, and in several places a knowledge of the differential calculus.

Part I is concerned with the solution of triangles, and is hence called "numerical" trigonometry. The six trigonometric ratios are defined first for

acute angles, and later for obtuse angles. The pupil is supposed to be already familiar with logarithms. Tables, by the way, are not included in the text. "Algebraic" trigonometry forms the second part, running a hundred and sixty pages. The usual topics of general angle, compound angle, radians, identities and equations, periodic functions etc., are well handled; in addition there is some geometry of the triangle and quadrilateral beyond the powers of most of our freshmen. This second part of the text is thought by the authors to be indispensable to all students of science and engineering. Part III, whose contents we have mentioned, was intended for "serious" students of science and engineering.

The English believe in teaching elementary mathematics by problems. Think of 1870 problems, over a third consisting of several parts, all with answers, (the answers are in some thirty pages at the end of the book), and, praise Allah!, carefully grouped under such headings as right triangles, viva voce, heights and distances, traverses and projections, three dimensions, symbolical, sine and cosine rules, harder problems in three dimensions, etc., plus a large number of fully displayed illustrative problems, especially in the first part. There are twenty-three "Revision Papers," which turn out to be very well constructed two and three hour examinations. The reviewer's task of making out a final examination in trigonometry will be materially lessened for the next twenty three years.

Turning to minor details, the typography is excellent. There appear to be no misprints. The answers given seem reliable, for some five per cent of them, taken at random, were checked by the reviewer and several of his colleagues. The King's English occasionally sounds strange to American ears. Few freshmen girls could repress a snicker when they read on p. 17 "you would think of 12/c" or on p. 141 "you may be inclined to argue." The truth is that Messrs. Siddons and Hughes often write as they teach, somewhat informally. Frequently their footnotes are addressed directly to the student, and very helpful they are.

Opinions will likely vary about the possibility of using this English text with American freshmen; one would think that an experienced teacher could readily make the necessary modifications. As a collection of problems the volume is a masterpiece.

C. A. Rupp

Questions D'Arithmétique. Par B. Niewenglowski. Vuibert, Paris, 1927. viii+225 pages.

This book consists mainly of very clearly presented solutions of elementary problems and proofs of some elementary theorems in the theory of positive integers. As examples of these we may mention the following: to resolve in integers the equation $x^2+y^2=z^2$; to find the greatest common divisor of x^m-y^m and x^n-y^n when x, y, m, n are integers; to determine all even perfect numbers; if x^2+2y is a square then x^2+y is the sum of two squares; in order that a number p shall be a prime it is necessary and sufficient that there shall be one and just

one square x^2 such that $p+x^2$ is a square; if p is a prime and x belongs to the exponent 3 modulo p, then $x+1$ belongs to the exponent 6 modulo p. Sums of powers of consecutive integers are treated and the results are applied in the resolution of elementary problems. It is to such fragments that the book is mainly devoted. But it contains also a derivation of the law of quadratic reciprocity and a brief and incomplete account of the so-called Pell equation. The book makes no contribution to the theory of numbers; it seems to be intended only for amateurs. But it will afford a source for very easy problems suitable for use in an undergraduate course involving the most elementary parts of the theory of numbers.

R. D. CARMICHAEL

A History of Mathematical Notations. Volume I. Notations in Elementary Mathematics. By Florian Cajori, Ph.D. The Open Court Publishing Company, Chicago, 1928. xvi+451 pages. Price $6.00.

The aim of this work as stated in the Introduction is "to give not only the first appearance of a symbol and its origin (whenever possible), but also to indicate the competition encountered and the spread of the symbol among writers in different countries." The author states that he "believes that this history constitutes a mirror of past and present conditions in mathematics which can be made to bear on the notational problems now confronting mathematics."

In view of a present day movement to prepare for students of the history of the sciences, a series of "Source Books" which shall include the first appearance of epoch-making concepts, the publication of this work on mathematical notations is especially timely. It is a veritable source book for symbols in elementary mathematics. This aspect of the work may be illustrated by the three-page *Extract from Miscellanea Berolinensia*, due to Leibniz, followed by the translation. It may be illustrated also by the treatment accorded William Oughtred. This English mathematician used one hundred and fifty symbols, among them the present common sign for multiplication. Five pages are devoted to these symbols with an indication of the editions of the works in which they are found, all this followed by copious notes. *The History of Mathematical Notations* may be consulted for authors who used a symbol to represent any of the elementary notions of mathematics, for these individual writers are fully discussed; or the book may be consulted for the detailed history of any symbol.

There is a richness of explanation throughout the work which can only be touched upon. It is shown in the fascinating history of the symbol $\times$ in multiplication, a history which gives evidence of the thoroughness of the research here represented. It is shown again in the discussion of Pacioli as an individual writer. This is made from a direct study of the 1494 editions of Pacioli's great work. The author shows here a perfect familiarity with the ground that he is covering and displays his mastery of interpretation and his ability to coordinate. Such a treatment leads to an expression of the wish that the author had had

access to more original sources. A very complete study of the role of Pitiscus in the use of decimal fractions and the dot as a decimal separatrix is made from the 1612 edition of the *Trigonometria* with a presentation of the case leading to the conclusion that Pitiscus made extended use of decimal fractions but that the honor of introducing the dot as the separatrix between units and tenths must be assigned to others. Had the author been able to make the same convincing study of Edward Wright's 1616 edition of Napier's *Descriptio* and of Napier's *Rabdologiae . . . . Libri Dvo* of 1617, the reader would not experience the feeling of passing from a reliable exposition to an indifferent treatment of the places these two men take in the controversy; but it is doubtless impossible to avoid unevenness in a work of such magnitude as the one under consideration.

There is such a wealth of reliable material in this book that it is with some hesitation the matters which follow are noted. However, attention should be called to the fact that there are some misleading statements and illustrations. The statement that "these signs [*Herodianic signs*] appear on an abacus found in 1847, represented upon a Greek marble monument on the island of Salamis" does not square with the fact that the Salamis abacus is now in the Epigraphical Museum at Athens. Following this is a drawing (p. 23) with the title, "The computing table of Salamis," which departs from the original in several respects. On the latter there are no counters, as shown here, and the Greek number symbols do not appear alongside the set of lines containing the crosses but to the right of them and running over the shorter lines. The illustration (p. 79) from the Bakhshālī Manuscript taken from G. R. Kaye, *Indian Mathematics* (1915), is in an inverted position to the one given in that work. Again, since so much must be omitted from a treatise like this, the voluminous treatment of Hindu-Arabic numerals with several pages devoted to "Fanciful hypotheses on the origin of the numeral forms" might well have been shortened to make room for a more extended history of such a topic as rod numerals.

As page follows page in this work, there emerges the story of the gradual adoption of the symbols now in use. It is not an evolution but a survival for one reason or another of a particular form. The conclusions to be drawn are stated forcibly in the remarks made by the author at the close of the history of the symbol for subtraction. He says: "This study emphasizes the difficulty experienced even in ordinary arithmetic and algebra in reaching a common world-language. Centuries slip past before any marked step toward uniformity is made. . . . . The only hope for rapid approach of uniformity in mathematical symbolism lies in international cooperation through representative committees."

The contents are: "Numeral Symbols and Combinations of Symbols;" "Symbols in Arithmetic and Algebra" (Elementary Part), treated under "Groups of Symbols Used by Individual Writers" and "Topical Survey of the Use of Notations;" "Symbols in Geometry" (Elementary Part), treated under "Ordinary Elementary Geometry" and "Past Struggles between Symbolists and Rhetoricians in Elementary Geometry." The illustrations number 106 and include some entirely new ones such as the page from Klebetius and the page

from the Bolognetti manuscript, of possibly 1550, containing an equality sign almost identical with that of Robert Recorde. Innumerable footnotes throughout the work provide a bibliography of great value. There are also cross references in the text which add materially to the convenience of the reader and to the smoothness with which a study of it proceeds.

The Open Court Publishing Company has added another notable achievement to the long list of works made available through the courtesy of this unique house.

LAO G. SIMONS

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEM FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources,will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3372. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Given a point of a conic, the tangent at this point, an axis, and the center (at a finite, or infinite distance), the following two lines are constructed: (1) the symmetric of the given tangent with respect to the parallel to the given axis through the given point, and (2) the diameter through the symmetric of the given point with respect to the axis. Prove that if at the point of intersection of these two lines the perpendicular is erected to the first line, it will meet the normal to the conic at the given point in the center of curvature of the conic at this point.

3373. *Proposed by Paul Capron, U. S. Naval Academy.*

A plane cuts the lateral edge of a quadrangular pyramid so that the ratios of the edges of the whole pyramid to the corresponding edges of the smaller pyramid cut off are x_1, x_2, x_3, x_4 and the ratio of the smaller volume to the larger is r. Either diagonal of the base of the larger pyramid divides this base into triangles whose areas bear the ratio r_i, r_{i+2} to the whole area of the base, each of these ratios being numbered to correspond with the lateral edge opposite its triangle, so that $r_1+r_3=1=r_2+r_4$. Show that

$$x_1x_2x_3x_4r = x_1r_1 + x_3r_3 = x_2r_2 + x_4r_4.$$

3374. *Proposed by J. Rosenbaum, Milford, Conn.*

Given two triangles, one within the other, to construct a third triangle which shall be inscribed in the outer and circumscribed about the inner triangle. Also prove that if the two given triangles are equilateral and concentric the third triangle is equilateral.

3375. *Proposed by A. S. Levens, University of Minnesota.*

Show to find the angle between a line and a plane graphically, without first reducing the problem to that of finding the angle between two lines.

SOLUTIONS

3310 [1928, 154]. *Proposed by B. F. Finkel, Drury College.*

Find the envelope of a system of circles having for diameters the secants of constant length, $2r$, of a conic.

Solution by W. J. Patterson, The University of Western Ontario

Let equation of central conic be

$$b^2x^2 + a^2y^2 - a^2b^2 = 0. \tag{1}$$

Let equation of secant to conic be

$$y = mx + k, \tag{2}$$

and let P_1, P_2, be the points of intersection with conic. Then

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = 4r^2. \tag{3}$$

Therefore, the equation of the variable circle is

$$\{x - \tfrac{1}{2}(x_1 + x_2)\}^2 + \{y - \tfrac{1}{2}(y_1 + y_2)\}^2 - r^2 = 0. \tag{4}$$

Using equation (2) in equation (1), we get

$$\begin{aligned} (x_1 + x_2)/2 &= -a^2mk/(a^2m^2 + b^2), \\ x_1x_2 &= a^2(k^2 - b^2)/(a^2m^2 + b^2). \end{aligned} \tag{5}$$

Similarly,

$$\begin{aligned} (y_1 + y_2)/2 &= b^2k/(a^2m^2 + b^2), \\ y_1y_2 &= b^2(k^2 - a^2m^2)/(a^2m^2 + b^2). \end{aligned} \tag{6}$$

Combining (5) and (6) in (3), and clearing of fractional forms,

$$\begin{aligned} a^4m^2k^2 - a^2(k^2 - b^2)(a^2m^2 + b^2) + b^4k^2 \\ - b^2(k^2 - a^2m^2)(a^2m^2 + b^2) - r^2(a^2m^2 + b^2)^2 = 0. \end{aligned} \tag{7}$$

Also, combining (5) and (6) in (4), we have, simplified,

$$\begin{aligned} (x^2 + y^2)(a^2m^2 + b^2)^2 + 2a^2mk(a^2m^2 + b^2)x - 2b^2k(a^2m^2 + b^2)y \\ + k^2(a^4m^2 + b^4) - r^2(a^2m^2 + b^2)^2 = 0. \end{aligned} \tag{8}$$

Eliminating k between (7) and (8), arranging as a power series in m, and deriving the discriminant relation in the usual way, we obtain the equation of the required envelope.

The resulting equations are so complicated that it does not seem worth while to write them.

Also solved by William Hoover.

3311 [1928, 154] *Proposed by A. Pelletier, Montreal, Canada.*

Eliminate x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3 from the following equations:

$$a^{-2}x_1^2 + b^{-2}y_1^2 + c^{-2}z_1^2 = 1\,, \qquad a^{-2}x_2^2 + b^{-2}y_2^2 + c^{-2}z_2^2 = 1\,,$$

$$a^{-4}x_1x_2 + b^{-4}y_1y_2 + c^{-4}z_1z_2 = 0, \qquad a^{-4}x_2x_3 + b^{-4}y_2y_3 + c^{-4}z_2z_3 = 0,$$

$$a^{-2}x_1x + b^{-2}y_1y + c^{-2}z_1z = 1, \qquad a^{-2}x_2x + b^{-2}y_2y + c^{-2}z_2z = 1,$$

$$a^{-2}x_3^2 + b^{-2}y_3^2 + c^{-2}z_3^2 = 1,$$

$$a^{-4}x_1x_3 + b^{-4}y_1y_3 + c^{-4}z_1z_3 = 0,$$

$$a^{-2}x_3x + b^{-2}y_3y + c^{-2}z_3z = 1.$$

Solution by Frank Ayres, Jr., College Station, Tex.

Write the equations

$$a^{-2}x_i^2 + b^{-2}y_i^2 + c^{-2}z_i^2 = 1, \tag{1}$$

$$a^{-2}xx_i + b^{-2}yy_i + c^{-2}zz_i = 1, \tag{2}$$

$$a^{-4}x_ix_j + b^{-4}y_iy_j + c^{-4}z_iz_j = 0, \tag{3}$$

where $i, j = 1, 2, 3$; $i \neq j$.

(1) States that the three points (x_i, y_i, z_i) are on the ellipsoid, $a^{-2}x^2+b^{-2}y^2+c^{-2}z^2=1$.

(2) gives the equations of the tangent planes to the ellipsoid at the points.

(3) states that these planes are mutually perpendicular.

If we write

$$r_i = a^{-2}x_i(a^{-4}x_i^2 + b^{-4}y_i^2 + c^{-4}z_i^2)^{-1/2} \equiv a^{-2}x_i\omega_i^{-1}\,;$$

$$s_i = b^{-2}y_i\omega_i^{-1}\,; \qquad t_i = c^{-2}z_i\omega_i^{-1},$$

(2) becomes

$$r_ix + s_iy + t_iz = (a^2r_i^2 + b^2s_i^2 + c^2t_i^2)^{1/2}. \tag{4}$$

Squaring and adding, we have, since

$$\sum r_i^2 = \sum s_i^2 = \sum t_i^2 = 1\,; \qquad \sum r_is_i = \sum s_it_i = \sum t_ir_i = 0,$$

the required result,

$$x^2 + y^2 + z^2 = a^2 + b^2 + c^2.$$

This locus is the director sphere of the ellipsoid.

Also solved by F. L. Wilmer.

3313. [1928, 158] *Proposed by L. S. Johnston, University of Detroit.*

Consider the infinite sequence $[a_n]$ of real positive numbers with the recurrent relation $a^2_{k+1}=2a_k/(1+a_k)$ $(k=1, 2, 3, \cdots)$.

(1) Prove $L_{k\to\infty}a_k=1$ for every a_1.

(2) Prove $L_{n\to\infty}\prod_{k=1}^{n} a_k$ exists and is different from zero for every a_1.

(3) Express the limit in (2) as a function of a_1.

The problem is of course trivial for $a_1=1$.

Solution by the Proposer.

If $a_1=1$ it is evident that $a_k=1$ for all k's, and that $L_{n\to\infty}\prod_{k=1}^{n} a_k=1$.

Consider next the case for which $a_1>1$ and set $a_1=\sec\omega$, $0<\omega<\pi/2$. Then from the identity

$$\sec^2(\omega/2) = \frac{2\sec\omega}{\sec\omega+1},$$

we may set $a_2=\sec(\omega/2)$. Repeating this reasoning we obtain $a_n=\sec(2^{1-n}\omega)$. Hence it follows that a_n approaches the limit unity, and it decreases to this limit except in the trivial case $a_1=1$. We may now set

$$1\Big/\prod_{k=1}^{n}a_k = \cos\omega\cos 2^{-1}\omega\cos 2^{-2}\omega\cdots\cos 2^{1-n}\omega$$

$$= \frac{\sin 2\omega}{2^n\sin 2^{1-n}\omega} = \frac{\sin 2\omega}{2\omega}\,\frac{2^{1-n}\omega}{\sin 2^{1-n}\omega},$$

where the second form results by the transformation of each factor by means of the formula $\cos A=(\sin 2A)/(2\sin A)$. Hence we have

$$\lim_{n\to\infty}\prod_{k=1}^{n}a_k = \frac{2\omega}{\sin 2\omega} = \frac{a_1^2\sec^{-1}a_1}{(a_1^2-1)^{1/2}}.$$

For the case in which $a_1<1$ we may set $a_1=\operatorname{sech}\omega$, $\omega>0$. We have merely to replace the trigonometric formulae by the corresponding hyperbolic formulae, and the reasoning follows in a similar manner. We thus find that a_n approaches unity as a limit and it increases toward this limit, while

$$\lim_{n\to\infty}\prod_{k=1}^{n}a_k = \frac{a_1^2\operatorname{sech}^{-1}a_1}{(1-a_1^2)^{1/2}}.$$

Also solved by F. L. Wilmer.

3314 [1928, 154]. *Proposed by J. B. Reynolds, Lehigh University.*

A uniform flexible chain of length a, weight w, slides down an arch of a smooth inverted cycloid of total length $8a$ in a vertical plane, with its vertex lowest. If the chain is released from rest in a position on the arch in which one end is at one end of the arch, find the time until the middle of the chain is at

the lowest point of the cycloid, the speed of the chain at that instant and the tension at its middle point then.

I. *Solution by Wm. M. Borgman, Jr., College of the City of Detroit.*

Let the equations of the inverted cycloid be written in the form, $x=a(\theta-\sin\theta)$, $y=a(1+\cos\theta)$. Then the length of arc, measured from $(0, 2a)$, is $s=4a(1-\cos\frac{1}{2}\theta)$.

Let w = weight per unit length of the chain, K = kinetic energy of the chain and P = potential energy when one end of the chain is at the point $s=s$ and the x-axis is the datum line. Then

$$K = wav^2/2g, \quad P = w\int_s^{s+a} yds = -\frac{8}{3}a^2w\cos^3\frac{\theta}{2}\Big]_s^{s+a};$$

and

$$\frac{dP}{ds} = 4a^2w\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}\cdot\frac{d\theta}{ds}\Big]_s^{s+a} = 2aw\cos^2\frac{\theta}{2}\Big]_s^{s+a} = \frac{w}{8}(2s-7a),$$

since $ds/d\theta=2a\sin\frac{1}{2}\theta$. From the known fact that $dK/dt=-dP/dt$, we have $(wavg^{-1})dv/dt=-(dP/ds)v$, and hence $8a\ dv/dt=(7a-2s)g$, or $(d^2s/dt^2)+(g/4a)s=(7g/8)$.
The solution is

$$s = c_1\cos(g/4a)^{1/2}t + c_2\sin(g/4a)^{1/2}t + 7a/2.$$

When $t=0$, $s=0$, $v=0$. Hence

$$s=(7a/2)[1-\cos(g/4a)^{1/2}t] \text{ and } v=(7a/4)(ag)^{1/2}\sin(g/4a)^{1/2}t.$$

When the middle of the chain is at the vertex, $s=7a/2$, $t=\pi(a/g)^{1/2}$, and $v=(7/4)(ag)^{1/2}$.

Since the chain is moving with uniform velocity when the center is at the vertex of the cycloid, the chain may be considered at rest for the purpose of calculation of forces along its length.

Let T be the tension in the chain when its middle point is at the vertex of the cycloid and let ϕ be the angle between the horizontal and the tangent.

Then $dT=w\sin\phi\, ds=-aw\sin\theta d\theta$, since $\sin\phi=-\cos\frac{1}{2}\theta$ and $ds=2a\sin\frac{1}{2}\theta\, d\theta$. Hence $T=2aw\cos^2(\frac{1}{2}\theta)-(aw/32)$, where the constant of integration is chosen so that $T=0$ when $s=9a/2$.

Hence the tension at the middle of the chain (i.e., where $\theta=\pi$) is $-wa/32$.

II. *Solution by the Proposer.*

Let s measure the arc distance from the vertex of the cycloid to a variable point P on the chain and let z measure the arc distance from the vertex of the cycloid to the left end of the chain assuming that the right end is initially at the right cusp of the arch. Let T be the tension in the chain at the point P, ϕ the angle the normal at P makes with the vertical, and $v=dz/dt$ the speed of the chain.

Resolving forces along the tangent at P we find the equation

$$(1) \qquad dT/ds = (W/a)\sin\phi + (W/ga)vdv/dz,$$

in which T, s, and ϕ may vary independently of z and v. Now $s=4a\sin\phi$, and therefore we get from (1), upon integrating,

$$T = 2W\sin^2\phi + (4W/g)(vdv/dz)\sin\phi + C.$$

Now $T=0$ at the left end of the chain, so we have $T=0$ when $s=z=4a\sin\phi$; whence, determining the constant C, we have

$$(2) \qquad T = 2W\sin^2\phi - Wz^2/8a^2 + (4W/g)[\sin\phi - z/4a]vdv/dz.$$

Again $T=0$ at the right end of the chain where $s=z+a=4a\sin\phi$; whence by (2) we get

$$(3) \qquad vdv/dz = -(g/8a)(2z+a).$$

Integrating and determining the constant of integration by the condition that $v=0$ for $z=3a$, we find

$$(4) \qquad v^2 = (g/4a)[12a^2 - az - z^2] = (g/4a)(3a-z)(4a+z).$$

Equation (4) shows that the chain slides to the left until $z=-4a$, that is until the left end just reaches the left cusp. It therefore oscillates back and forth between the two cusps of the arch. The middle of the chain is at the lowest point when $z=-a/2$, and then $v=(7/4)(ag)^{1/2}$.

From equation (4) we have

$$dz/(12a^2 - az - z^2)^{1/2} = -\tfrac{1}{2}(g/a)^{1/2}dt,$$

and, upon integrating and noting that $t=0$ for $z=3a$, we have

$$(5) \qquad t = 2(a/g)^{1/2}\arccos[(2z+a)/7a].$$

When $z=-a/2$, $t=\pi(a/g)^{1/2}$; that is the chain reaches the lowest position in the same time as a particle starting from any point of the curve requires to reach the vertex.

Substituting v from (4) in (2) we find

$$T = (W/8a^2)(s-z)(s-z-a).$$

Where $s=z+\tfrac{1}{2}a$ (the middle point of the chain for any position), we find $T=-W/32$; that is, in all positions the chain at its middle point is under a compression equal to one thirty-second of its weight. Furthermore, since $(s-z)$ is the distance of P from the left end of the chain, we see that the chain at all points except its ends is under a compression which remains constant throughout the motion.

Also solved by William Hoover.

3315 [1928, 207]. *Proposed by Harry Langman, Arverne, N. Y.*

Show that

$$x^2 + y^2 + z^2 - yz - zx - xy + u^2 + v^2 + w^2 - vw - wu - uv \geqq 3^{1/2} \cdot \begin{vmatrix} u & x & 1 \\ v & y & 1 \\ w & z & 1 \end{vmatrix}$$

for all real values of the variables. Also find the conditions under which the equality obtains.

Solution by Frank Ayres, Jr., College Station, Texas

Let $A=(u, x)$, $B=(v, y)$, $C=(w, z)$ be three points determining a triangle for which sides will be denoted by a, b and c and area by D. Then, we are to show that

$$(a^2 + b^2 + c^2)3^{1/2}/4 \geqq 3D.$$

Of the triangles having a given perimeter $a+b+c=3p$, the equilateral triangle has the maximum area, $3^{1/2}p^2/4$. Also,[1] $a^2+b^2+c^2>3p^2$, unless $a=b=c=p$, in which case the two members are equal.

Hence the equality obtains when the triangle ABC is equilateral; otherwise, the inequality holds.

3317 [1928, 207]. *Proposed by Emma M. Gibson, High School, Springfield, Missouri.*

The coordinates of a point are expressed in terms of two parameters, α, β, by

$$x/a = (\alpha+\beta)/(\alpha-\beta), \qquad y/b = (1-\alpha\beta)/(\alpha-\beta), \qquad z/c = (1+\alpha\beta)/(\alpha-\beta),$$

where a, b, and c are constants.

Determine the locus of the point and determine the relations of the lines $\alpha=$constant and $\beta=$constant to the surface. Also show how to express the differential equation of the lines of curvature in terms of α and β.

Solution by J. H. Neelley, Carnegie Institute of Technology.

The locus of the point is obviously the hyperboloid of one sheet,

$$a^{-2}x^2 + b^{-2}y^2 - c^{-2}z^2 = 1 \tag{1}$$

The lines $\alpha=$constant and $\beta=$constant are the two systems of rectilinear generators of the surface with rectangular equations

$$\begin{aligned} b^{-1}y - c^{-1}x - \alpha(1 - a^{-1}x) &= 0, \\ 1 + a^{-1}x - \alpha(b^{-1}y + c^{-1}z) &= 0; \end{aligned} \tag{2}$$

and

$$\begin{aligned} b^{-1}y - c^{-1}z + \beta(1 + a^{-1}x) &= 0, \\ 1 - a^{-1}x + \beta(b^{-1}y + c^{-1}z) &= 0. \end{aligned} \tag{3}$$

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[1] Note by the Editors: See Theorem IV on p. 383 of vol. 34 (1927) of this Monthly.

The differential equation of the lines of curvature of the surface expressed in terms of α and β may be written

$$(4)\qquad \begin{vmatrix} d\beta^2 & -d\alpha d\beta & d\alpha^2 \\ E & F & G \\ D & D' & D'' \end{vmatrix} = 0,$$

where E, F, and G are the coefficients of $d\alpha^2$, $d\alpha d\beta$, and $d\beta^2$ respectively in the metric of the surface, and D, D', and D'' are the coefficients of the same expressions in the second fundamental quadratic differential form of the surface. Equation (4) is derived in any treatise on Differential Geometry. See Art. 50 of Eisenhart's text.

Also solved by H. L. Dorwart.

Note by Otto Dunkel. The differential equation of the lines of curvature may be expressed in another form, and, after some computation, it becomes

$$[4a^2\alpha^2 + b^2(\alpha^2 - 1)^2 + c^2(\alpha^2 + 1)^2]d\beta^2 - [4a^2\beta^2 + b^2(\beta^2 - 1)^2 + c^2(\beta^2 + 1)^2]d\alpha^2 = 0.$$

It will be found that $D = D'' = 0$, so that the equation (4) in the above solution becomes $Gd\beta^2 - Ed\alpha^2 = 0$, and this equation in turn reduces to the one above.

3318 [1928, 208]. *Proposed by C. N. Mills, Normal, Ill.*

For the parabola, $y^2 = -2p(x-k)$, find the length (in terms of p and k) of the shortest segment of a line tangent to the curve included between the coordinate axes.

Solution by Watson M. Davis, Albion College.

For every value of m, the line $2m^2(x-k) - 2my - p = 0$ is tangent to the parabola. Its x- and y-intercepts are, respectively, $(2m^2k+p)/2m^2$ and $-(2m^2k+p)/2m$.

Let D be the length of the segment of this line included between the axes. Then

$$D^2 = (2m^2k + p)^2(1 + m^2)/4m^4.$$

We may consider p and k as positive and the independent variable as m^2, varying from 0 to ∞. The derivative of D^2 with respect to m^2 consists of one factor which is never negative and another factor, $2km^4 - pm^2 - 2p$, which varies from $-$ to $+$. Hence there is a minimum given by the positive root m^2 of this factor or by

$$4km^2 = p + (p^2 + 16kp)^{1/2}.$$

This value of m^2 inserted in the above expression for D gives the minimum required.

Also solved by Frank Ayers, Jr., Nina M. Alderton, W. J. Patterson, E. G. Olds, and Fredrick Woods.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kbhn, Ohio State University, Columbus, Ohio.

On November 18, 1928, occurred the death of Alexander Ziwet, Professor Emeritus of Mathematics at the University of Michigan, one of the outstanding men in mathematics in this country during the last forty years.

The principal land marks of his life were as follows: born, February 8, 1853 at Breslau, Germany; graduated from the gymnasium, 1870; attended the University of Warsaw, 1871–3 and the University of Moscow, 1873–4; attended the Polytechnische Hochschule in Karlsruhe, 1876–1880; employed by the Lake Survey and the U. S. Coast and Geodetic Survey at Detroit, 1880–8; appointed instructor, then assistant professor, associate professor, professor, and finally head of the department of mathematics in the College of Engineering, University of Michigan, 1888–1925.

There was recognition of his ability in his being Associate Editor of the Bulletin of the American Mathematical Society, 1892–1920; Vice President of the Society, 1903; Vice President and Chairman of Section A, American Association for the Advancement of Science, 1905–6. He was a member of the first executive board of the Mathematical Association, and a member of its committee on relations with the Annals of Mathematics. The University of Michigan conferred on him the honorary degree of Sc.D. in 1927.

Professor Ziwet was outstanding as a scholar and a teacher. His range of knowledge was not limited to mathematics, especially from the applied point of view, but extended to many sections of pure mathematics, history of mathematics and the humanities. As a linguist he was perhaps unsurpassed by any member of the University faculty. As a teacher and lecturer especially during his prime he was unexcelled. Many an Engineering College graduate looks back with pride on the fact that he was taught his theoretical mechanics by Professor Ziwet, and many a graduate student and colleague was delighted and inspired by the clarity, accuracy, and balance of his presentations. He was a potent influence in the University, not only for high ideals in connection with engineering education, but also in the promotion of graduate work and research. His ideas on such matters were often far in advance of those of his colleagues and of the time. As an individual he was a gentleman in the fullest sense of the word, honest, just, upright, generous, faithful, modest, and retiring.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Thirteenth Summer Meeting of the Association, Boulder, Colorado, August 26-29, 1929.

Fourteenth Annual Meeting, Des Moines, Iowa, December 31, 1929, January 1, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled.

ILLINOIS, Carthage, Ill., May 3-4.

INDIANA, Culver Military Academy, May 3-4

IOWA, Fairfield, Iowa, April 26-27.

KANSAS, Topeka, Kansas, February 2.

KENTUCKY, Lexington, Ky., April 13.

LOUISIANA-MISSISSIPPI.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, May 4.

MICHIGAN, Ann Arbor, Mich., March 16.

MINNESOTA, St. Paul, May 8.

MISSOURI, Kansas City, Mo., November.

NEBRASKA.

OHIO, Columbus, Ohio, April 4.

PHILADELPHIA, University of Pennsylvania, November 30.

ROCKY MOUNTAIN, Greeley, Colo., April 12-13.

SOUTHEASTERN, Macon, Ga., April 19-20.

SOUTHERN CALIFORNIA, University of Redlands, March 9.

TEXAS, Houston, Texas, Jan. 26.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE AMERICAN

MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teachers' agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

THE DECEMBER MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION

The twenty-fourth regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held at the Johns Hopkins University, Baltimore, Md., on Saturday, December 8, 1928. Sessions were held in the morning and in the afternoon; Professor C. C. Bramble, chairman of the Section, presided at both sessions.

Forty-six persons attended the meeting, including the following thirty-three members of the Association: O. S. Adams, R. N. Ashmun, H. G. Avers, Clara L. Bacon, W. J. Berry, G. A. Bingley, L. M. Blumenthal, C. C. Bramble, G. R. Clements, A. Cohen, Tobias Dantzig, Alexander Dillingham, J. A. Duerksen, P. J. Federico, Michael Goldberg, Harry Gwinner, L. S. Hulburt, F. E. Johnston, H. P. Kaufman, L. M. Kells, W. D. Lambert, Florence P. Lewis, A. K. Mitchell, Frank Morley, W. K. Morrill, F. D. Murnaghan, C. H. Rawlins, Jr., W. F. Reynolds, A. W. Richeson, R. E. Root, John Tyler, Paul Wernicke, E. W. Woolard.

During the intermission between the morning and the afternoon sessions, those attending the meeting were entertained at luncheon by the University; a hearty vote of thanks was passed in appreciation of the generous hospitality shown by the University. Mr. Edgar W. Woolard, acting secretary for the meeting in place of Rev. E. C. Phillips, S.J., whose transfer to New York had necessitated his resignation, was appointed permanent secretary of the Section, and was requested to convey to Dr. Phillips the congratulations of the Section on his transfer to a larger field of usefulness, their appreciation of his past services as Secretary, and their regrets that he could no longer be with them.

An invitation to hold the next meeting at the George Washington University, Washington, D.C., in May, 1929, was accepted.

The following seven papers were presented:

1. "On Euclidean geometry," by Professor Frank Morley, Johns Hopkins University.

2. "Curves of pursuit," by Professor Tobias Dantzig, University of Maryland.

3. "Teaching limits to freshmen," by Professor R. E. Root, Postgraduate School, U. S. Naval Academy.

4. "Reiteration paths," by Professor John Tyler, U. S. Naval Academy.

5. "The theorems of Ceva and Menelaus and their extension (Second Paper)," by Dr. Paul Wernicke, U. S. Patent Office.

6. "A method of estimating tidal friction," by Mr. W. D. Lambert, U. S. Coast and Geodetic Survey.

7. "Correlation and causation," by Mr. E. W. Woolard, George Washington University.

The authors' abstracts of these papers follow:

1. Professor Morley reviewed the axioms of Euclidean plane geometry in the light of the theory of groups. The fundamental operation is paper-folding or reflection; the corresponding axiom is that of T. Hjelnvelev (Mathematische Annalen, vol. 64, 1907).

2. The problem in its original form was proposed by Leonardo da Vinci. It is capable of the following purely geometrical formulation: Between two curves (plane or skew) there exists a correspondence subject to the conditions (1) the tangent to (C') at P' contains the corresponding point P of (C); (2) the arc s' of (C') is a linear function of the arc s of (C); determine (C') when (C) is given. When both curves are plane the problem is reduced to a single differential equation of the second order by introducing the line PP' which envelops (C'). Some integrable cases were discussed. In the general case it was shown that the osculating plane to (C') at P' contains the tangent to (C) at P and that the distance $L=PP'$ is independent of the torsion of (C'). The radius of curvature is given by $R'=kL$ cosec b, where b is the angle between the tangents at P and P' and k is a constant. This formula can be utilized for the graphical integration of the problem.

3. Professor Root gave a critical analysis of certain definitions of a limit, typical of those appearing in some recent and some older textbooks for freshmen and sophomores. These definitions were held to be inadequate, either in failing to indicate the role of the independent variable or in failing to state conditions sufficient to properly define a limit. In some cases, it was stated, the authors not only define a limit inadequately, but further foster false notions of limits by their treatment of illustrative examples.

5. Ceva's and Menelaus's theorems in plane geometry are dual to one another. Their dualism is that of point and line (s_0-s_1) in S_2. In S_3 similar theorems prevail, the dualism of which is that of point and plane (s_0-s_2) and the self-dualism of the line (s_1-s_1). If the lines joining the vertices of the

tetrahedron $A = A_1A_2A_3A_4$ to a point P meet the opposite faces in A_1', A_2', A_3', A_4', and if we denote the "triangular parts" $A_2A_4'A_3$, $A_3A_4'A_1$, $A_1A_4'A_2$, by a_{41}, a_{42}, a_{43}, and correspondingly those of a_i by a_{i1}, a_{i2}, a_{i3} according as they have an edge pertaining to the pair $e_1 = (A_2A_3)(A_1A_4)$ or $e_2 = (A_3A_1)(A_2A_4)$ or $e_3 = (A_1A_2)(A_3A_4)$ for a side, then, as was proved in this paper, $a_{41}a_{32}a_{23}a_{14} = a_{31}a_{42}a_{13}a_{24} = a_{21}a_{12}a_{43}a_{34}$. Each product has as factors the triangular areas adjacent to one of the pairs e_j. The point P is given the barycentric coordinates $p_1:p_2:p_3:p_4$, etc. This corresponds to Ceva's theorem and is dualizable into one corresponding to that of Menelaus. By methods similar to those used in the first paper of this title (this Monthly, vol. 34, 1927, pp. 468–472), properties of the perspective of *three* tetrahedra in S_3 were established.

6. Mr. Lambert's paper recapitulated briefly the history of the problem of the apparent observed secular acceleration of the moon's mean motion, and explained the reasons for thinking that the portion of the apparent acceleration not explained by the ordinary lunar theory is due to the very gradual slowing down of the earth's rotation by tidal friction. The popular conception of tidal friction as analogous to the rotation of a wheel checked by a brake band, the earth being the wheel and the ocean water the brake band, is quite erroneous. The calculation involves simply the dissipation of energy and it is not always easy to form a mental picture of how the energy dissipated at a particular point contributes to the slowing down of the earth. The energy dissipated by tidal currents has been estimated by Taylor and Jeffreys, though the available data are rather uncertain, and found to be, for the mean lunar tides alone, about 1.1×10^{19} ergs/sec. Almost all of this friction occurs in shallow seas, the deep oceans contributing surprisingly little. The computations for a method of estimating tidal friction from an entirely different set of data were made by Heiskanen, though he did not at the time realize the significance of his result nor apply certain needed corrections. Heiskanen's work depends on the times and heights of high tide all over the world, that is on the vertical component of the tidal motion instead of on the horizontal component (current velocity) used by Taylor and Jeffreys. This second method has the advantage of permitting us to visualize to a certain extent the moment of the forces retarding the earth's rotation. The tides in the open sea are to a large extent conjectural, but, using the best available information, Heiskanen obtained a result which, when reinterpreted and approximately corrected, is 1.0×10^{19} ergs/sec; that is, work is being done by the tides at an average rate of about one and one-half (American) billion horsepower, and this energy comes out of the rotational energy of the earth. The close agreement of this result with the result obtained by a different method from data of a different nature is doubtless accidental in view of the uncertainty of both sets of data, and all we are justified in saying is that the quantity obtained is about the quantity required to explain the excess of the apparent observed secular acceleration over the acceleration computed by lunar theory. This seems to imply that there is little friction in the elastic body tides of the earth.

7. Mr. Woolard's paper dealt with the difficulties involved in the attempt to interpret correlation coefficients in terms of cause and effect, and was a résumé of several notes by himself and others published in the Monthly Weather Review, March and October, 1927, and March, 1928, and in the Meteorological Magazine, vol. 63 (1928), p. 12.

EDGAR W. WOOLARD, *Secretary*

THE SEVENTEENTH MEETING OF THE IOWA SECTION

The seventeenth annual meeting of the Iowa Section of the Mathematical Association of America was held in conjunction with the meeting of the Iowa Academy of Science at Grinnell College, Grinnell, Iowa, on May 4–5, 1928.

The attendance was about sixty including the following twenty-seven members of the Association: F. A. Brandner, E. W. Chittenden, Julia T. Colpitts, N. B. Conkwright, Marian E. Daniells, W. M. Davis, C. W. Emmons, Annie W. Fleming, Cornelius Gouwens, Dunham Jackson, Dora E. Kearney, Yetta V. Maizlish, R. B. McClenon, F. M. McGaw J. V. McKelvey, I. F. Neff, M. A. Nordgaard, J. F. Reilly, H. L. Rietz, Maria M. Roberts, C. C. Sherman, E. R. Smith, G. W. Snedecor, L. E. Ward, C. W. Wester, Roscoe Woods, C. C. Wylie.

Professor Roscoe Woods, the chairman of the Section, presided at both the Friday afternoon and the Saturday morning sessions. Dinner was enjoyed together Friday evening at the College Club. At the business meeting following the program a report from the executive committee was adopted relative to the program of the section; and the following were elected officers for 1928–1929: Chairman, C. W. Wester, Iowa State Teachers College; Vice-chairman, Julia T. Colpitts, Iowa State College; Secretary-treasurer, J. F. Reilly, University of Iowa.

The program consisted of twenty papers, as follows:

1. "Definition by geometric implication," by Professor E. R. Smith, Iowa State College.

2. "Supplementary reading in freshman mathematics," by Professor Edmund E. Ingalls, Iowa Wesleyan College.

3. "A differential method of determining the longitude or time from two altitudes of the sun," by Mr. C. C. Sherman, University of Iowa.

4. "Pacioli and his 'Sūma,' " by Professor R. B. McClenon, Grinnell College.

5. "The use of machine factoring in multiple correlation," by Professor A. E. Brandt, Iowa State College (by invitation).

6. "Arithmetical changes in statistical constants due to coding, and their correction," by Professor Brandt.

7. "Calculation and use of the standard deviation of a partial regression coefficient," by Professor Brandt.

8. "The almanac in a Lincoln trial," by Professor C. C. Wylie, University of Iowa.

9. "Notes on the orthocentric triangle" (by title), by Professor W. J. Rusk, Grinnell College.

10. "How can interest in calculus be increased?", by Professor Roscoe Woods, University of Iowa.

11. "An analogy between certain functional relations and Yule's nonsense-correlations in time-series," by Professor H. L. Rietz, University of Iowa.

12. "The graphs and tables used in the study of grades at Fletcher College," by Professor Paul C. Overstreet, John Fletcher College (by invitation).

13. "Are the freshmen at Fletcher getting a square deal in grades?", by Professor Overstreet.

14. "A few odds and ends of illustrations," by Professor Overstreet.

15. "Infinitesimal and finite integration of $1/x$," by Mrs. Yetta Maizlish, University of Iowa.

16. "On estimating freshman average grades," by Professor G. W. Snedecor, Iowa State College.

17. "Disabilities of the correlation ratio," by Professor Snedecor.

18. "Note on solving the linear homogeneous partial differential equation with constant coefficients," by Professor J. F. Reilly, University of Iowa.

19. "The cissoid," by Mr. W. M. Davis, University of Iowa.

20. "Note on hyperquadrics in Euclidean S_4," by Mr. C. S. Carlson, University of Iowa (by invitation).

Abstracts of these papers follow:

1. In this paper Professor Smith shows how some of the difficulties in the definition of certain mathematical terms may be avoided by geometrical devices. For example, an angle may be defined to be a mathematical element which may be put in a one-to-one correspondence with a figure made up of two co-initial rays and a curved arrow. Likewise a vector may be defined by a one-to-one correspondence with a directed line segment or stroke.

2. Professor Ingalls reported on his plan of assigning supplementary readings to the upper third of his freshman mathematics students. The purpose was to interest the better students in topics not assignable to the poorer ones, and to equalize the time spent by the two groups.

3. Mr. C. C. Sherman deduced an exact formula giving the correction to the mean of the times for two observations by allowing the hour angle, the declination, and the zenith distance to vary simultaneously.

For observations on the sun, the small change in declination permits simplifications, and the formula reduces to the sum of three corrections: first, for variation in declination; second, for difference in zenith distance; and a third which is a function of the other two. The first two have been put in simple form by previous workers who have investigated them separately. The third correction, which is a convergent infinite series, is easily obtained by means of a slide rule, but computations have shown that in practical work it may often be neglected altogether.

4. Professor McClennon gave a brief résumé of the life and work of Pacioli, and read literal renderings of a few extracts from the "Sūma" that had not previously been published in translation. A copy of the first edition of the "Sūma," (Venice, 1494) belonging to the Grinnell College Library, was exhibited.

5. The method of handling statistical series in multiple correlation studies, reported on by Professor Brandt, is made possible through the use of electrical sorting and tabulating machines and punch cards such as those described by Mr. Victor Johns in this Monthly, vol. 33 (1926), pp. 494–502. It has been designed to lessen as much as possible the labor of making a multiple correlation study and to make it possible to use original observations without coding or grouping. The method was described in general terms and then fully illustrated by a problem.

6. In this paper Professor Brandt defines the process of coding or grouping and quotes various authorities on the subject. Five rules and a caution are given for the coding of continuous series. The statement is made that coding is legitimate. The results indicate that Sheppard's correction should be applied if we assume that the values of the various statistical constants secured from the uncoded or raw data are the correct ones. However, if the more usual statistical view that the set of data at hand is merely a random sample from a population and that the values of the constants are approximations to the real values is taken, the arithmetical changes due to coding are not large compared to the errors of random sampling so that the use of Sheppard's correction is not imperative.

7. One of the fundamental concepts of statistics is reviewed and two distinct problems outlined. These problems are commonly solved for most of the statistical constants but usually are not for partial regression coefficients. Since the standard deviation is necessary, a formula given by Dr. Truman L. Kelley is rewritten in a form suitable for use with the tabular solution of the normal equations given by Wallace and Snedecor. The problem is illustrated by the calculation and interpretation of the standard deviations of the partial regression coefficients of a statistical set of 512 observations on four independent and one dependent variable. From the formula and the illustration, it is concluded that in large samples having moderate standard deviations for the variables the partial regression coefficients of the sample will not differ significantly from those of the population but that in small samples or in samples having great dispersion, each partial regression coefficient should be compared with its standard deviation.

8. The evidence indicates that in his defense of Duff Armstrong in 1858, Lincoln proved by an almanac that the moon was not shining at the time of the fight. Certain evidence also indicates, and astronomical computations confirm, that it was shining.

The explanation of the assistant counsel for the prosecution is that when Lincoln sent for an almanac two were brought in, one for 1857 (correct), and one for 1856, and that Lincoln accidentally used the wrong one. In this paper

Professor Wylie suggests that a mistake or misprint in the almanac might have been responsible. Several mistakes have been found in almanacs for 1857. Another suggestion, especially since the almanac was a year old, is that, in most almanacs, a blurring of certain headings due to poor printing or long use, would cause one to read the time of setting that night as rising, and hence to read that the moon did not rise that evening till after midnight.

10. Professor Woods's paper was published in the January, 1929 issue of this Monthly.

11. The paper of Professor Rietz considers the so-called nonsense-correlations of Yule with special reference to an analogy to certain functional relations involving time as a variable.

12. This paper was a discussion of a series of five charts prepared by Professor Overstreet, and used by him as a means of improving the grading of students.

13. In his second paper Professor Overstreet attempted to account for the fact that the average semester grade for his senior class was 68% higher than the average grade of his freshman class.

14. This paper consisted of a list of illustrations used by Professor Overstreet in a freshman geology-astronomy class.

15. In her paper Mrs. Maizlish developed a formula for the summation of $1/x$ in terms of gamma functions corresponding to the integral of $1/x$ in terms of the logarithm.

16. In his first paper Profesor Snedecor described and analyzed the prognostic tests given the freshmen at Iowa State College. The estimate for each individual is made by means of a nomographic chart instead of by direct computation from the regression equation. The standard error of estimate was found to be about 4% of a grade in a system using 75% to 100% as the passing range of grades. Tabulations were presented showing the close agreement of average estimated grades with attained grades, and the prognostic value in different grade levels.

17. Attention was directed by Professor Snedecor to R. A. Fisher's criticisms: (1) that the distribution of η does not tend toward normality, (2) that the distribution of $N(\eta^2-r^2)$ approaches the normal only with very large values of N, and (3) that no account is taken of the number of arrays although the mean value approached involves such number. The speaker's criticisms were, (1) that Blakeman's criterion affords no information as to character of the deviation from linearity in the sample, (2) that mere heterogeneity is adequately tested by the error of estimate, and (3) that the character of simple regression is of little practical significance in multiple regression studies.

18. In his paper Professor Reilly suggested using the substitution $z=\phi(ly+mx)$ in place of $z=\phi(y+mx)$ as is commonly done. This eliminates the difficulty the beginning student experiences in seeing that $\phi(y+mx)$ is a function of x only when m is infinite.

19. Mr. Davis showed that the general method used to obtain the Cissoid of Diocles may be used to obtain many other familiar curves. Interesting cases are the cissoid of two straight lines, of a straight line and a conic through the origin, and of two conics through the origin. The curves obtained in each case are, respectively, hyperbolas through the origin, cubics with double points at the origin, and quartics with triple points at the origin.

When the conics in this latter case intersect the infinite line in the same two points, the quartics break down into conics.

J. F. REILLY, *Secretary*

ON GENETIC EQUILIBRIUM[1]

By TOBIAS DANTZIG and WILLIAM KEMP, University of Maryland

1. *Introduction.* The question treated in this paper belongs to a constantly increasing group of problems in biology susceptible to mathematical formulation and treatment. It is hoped that it will prove of genuine interest to both mathematicians and biologists, to the former as an example of method, to the latter because well recognized biological facts are derived from a priori assumptions.

Genetics as a branch of biology is of very recent origin. Historically its inception dates back to the discoveries of Mendel, first published in 1865. In point of fact, however, it must be remembered that Mendel's contributions remained quite unnoticed until 1900. In the intervening period a great deal of experimental evidence accumulated confirming Mendel's laws, which was the more remarkable as the work was not undertaken with a view of verifying these laws.

This paper deals with the first Mendelian law in its generalized form. Its object is to establish criteria for the stability of a *genetic equilibrium under random breeding conditions* and to define and analyse the *reproductive capacity of a biological species capable of genetic equilibrium.*

2. *Definitions and assumptions.* For the sake of simplicity we shall deal in what follows with *cells.* The argument used can be readily extended to members of a species if it be admitted that each individual of the species possesses on the average the same number of cells.

We shall deal with two kinds of cells: the body-cell or *zygote* and the germ-cell or *gamete.* For an explanation of how the actual transformation from one to the other takes place and for a description of the mechanism of heredity in general, we refer the reader to any standard treatise on Genetics.[2]

Consider now any biological characteristic (C) such, for instance, as the tallness of edible peas in the classical Mendel experiment. Let us cross two

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[1] Read before the Maryland District of Columbia Section of the Mathematical Association of America, May 5, 1928, at Annapolis, Md.

[2] See, for instance, T. H. Morgan, *The Theory of the Gene.* Yale University Press, 1926.

pure varieties of the species of which one possesses the characteristic and the other does not. We say that (C) is a dominant characteristic if the hybrid variety possesses it, otherwise (C) is recessive. It is in this sense that we shall use the term *homozygous dominant* and *homozygous recessive* for the cells of pure strain and *heterozygous* for the hybrid cells. We shall speak of these three varieties of cells as *genetic types.*

The mathematical treatment presented in the following sections is based on the assumptions:

A. The population consists of an equal number of male and female cells.

B. *Random breeding:* All crossings are equally probable, regardless of the genetic type.

C. Successive generations do not overlap in the reproductive sense; that is, when a given generation is brought into play, all preceding generations have ceased to be reproductively active.

D. Mendel's first law is valid.

3. *Mathematical formulation of the problem.* Let us assume that the original population consisted of G_0 gamete cells of which A_0 were dominant and B_0 recessive with respect to the characteristic studied.

Under random breeding conditions the first filial generation of zygotes will be distributed as follows:

Homozygous Dominant	$A_0 \times A_0 = A_0^2$
Heterozygous	$A_0B_0 + B_0A_0 = 2A_0B_0$
Homozygous Recessive	$B_0 \times B_0 = B_0^2$.

The total zygote population is therefore $Z_1 = A_0^2 + 2A_0B_0 + B_0^2 = G_0^2$. Not all of these cells will reach the stage of reproductive activity, so that the gamete population will be less than Z_1. Now in the classical theory it is assumed that the survival rates of the three genetic types are equal. We shall assume them to be generally distinct and denote them by u, v, and w respectively.

The gamete population of the first filial generation will therefore consist of $A_1 = uA_0^2 + vA_0B_0$ dominants and $B_1 = vA_0B_0 + wB_0^2$ recessives, and the total number of gamete cells will be $G_1 = uA_0^2 + 2vA_0B_0 + wB_0^2$.

The argument used for the first filial generation is perfectly general and leads to the recurrence formulae

$$(1) \qquad A_{n+1} = uA_u^2 + vA_nB_n\,; \qquad\qquad B_{n+1} = vA_nB_n + wB_n^2$$

and

$$(2) \qquad G_{n+1} = uA_n^2 + 2vA_nB_n + wB_{n9}\,; \quad Z_{n+1} = G_n^2 = (A_n + B_n)^2.$$

4. *The dominance ratio.* We are interested here not in absolute numbers but in ratios. We therefore introduce the *dominance ratio* which is the ratio of dominant to recessive gametes. From (1) and the equation $r_n = A_n/B_n$ we derive the relation

$$r_{n+1} = r_n(\alpha r_n + 1)/(r_n + \beta), \tag{3}$$

where for simplicity we have set

$$u/v = \alpha;\ w/v = \beta.$$

As α and β remain constant throughout the hereditary process we can regard formula (3) as the equation of an *iteration* in one variable. The successive application of this iteration gives the sequence

$$r_0,\ r_1,\ r_2,\ \cdots,\ r_n,\ r_{n+1},\ \cdots, \tag{D}$$

which may or may not converge to a finite *limiting point.*

Before examining closer the conditions for the convergence of sequence (D) we shall define what we mean by genetic equilibrium.

A species is said to be in genetic equilibrium with respect to a certain biological characteristic if the dominance ratio does not change from generation to generation. If equilibrium is at all possible this critical ratio is obtained by setting in (3) $r_n = r_{n+1}$. Denoting the critical ratio by r we obtain

$$r = (1 - \beta)/(1 - \alpha) = (v - w)/(v - u). \tag{4}$$

As the critical ratio is essentially positive we see that *equilibrium is a priori impossible* unless the survival rates of the homozygous are both less than that of the heterozygous or both greater. This excludes the cases $u > v > w$ and $u < v < w$ and leaves for consideration the three remaining cases: (I) $u < v > w$; (II) $u > v < w$; (III) $u = v = w$.

The third case is that of *neutral equilibrium* because equilibrium takes place for any *initial value* of the dominance ratio. Equation (4) becomes indeterminate and there exists an infinite number of critical values.

In cases I and II the critical value is unique. Should the initial value be equal to the critical, equilibrium takes place; *for no other value of r is absolute equilibrium possible.*

5. *Criteria of stability.* To investigate the condition of convergence of sequence (D) we regard the iteration (3) as a continuous function:

$$y = x(\alpha x + 1)/(x + \beta).$$

The representative curve is a branch of a hyperbola with concavity downward when $\alpha < 1$ and $\beta < 1$ (Case I) and concavity upward when $\alpha > 1$, $\beta > 1$ (Case II). The two cases are shown in Figures 1 and 2 respectively.

It is obvious that the intersection of the curve with the line $y = x$ takes place for the critical value $x = r$. If D_n is any of the points $x = r_n$, the subsequent positions in the sequence (D) are obtained as follows. Draw the ordinate at D_n meeting the curve at R_n and the line at S_n; then carry the segment S_nR_n to the right of D_n if S_n is below R_n, and to the left if it is above. It is then

readily seen that since $D_nD_{n+1}=S_nR_n$, we must have $OD_{n+1}=r_{n+1}$. In this manner we construct graphically the sequence (D) from any initial position D.

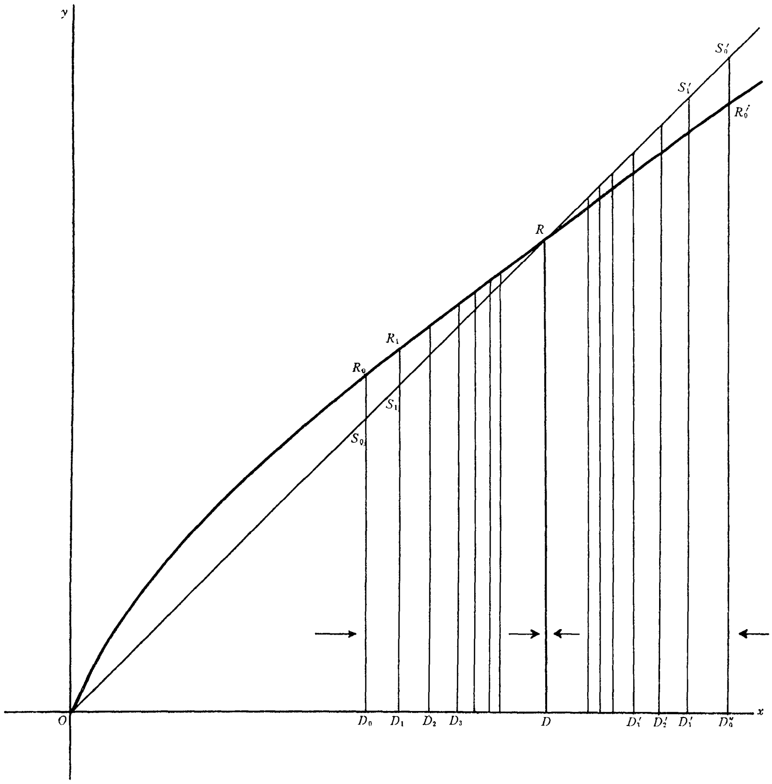

FIG. 1. Variation of the dominance ratio in a genetically stable population.

Case I. If the initial dominance ratio is less than the critical $r_0<r$, the point S_0 is below R_0 and consequently D_1 is to the right of D_0. Furthermore the segments S_nR_n decrease monotonically. The sequence (D) increases monotonically converging towards D as a limiting point.

On the other hand if $r_0>r$, a similar analysis shows that the sequence (D) decreases monotonically, still converging towards D. Therefore in either case:

When the survival rates of the heterozygous exceed those of the homozygous—the dominance ratio (regardless of its initial value) approaches the critical and the species tends towards a state of equilibrium.

Moreover, the equilibrium is stable. Indeed, should any unforseen factor disturb the equilibrium by making the dominance ratio deviate to either side of r, the process of natural selection will tend to restore the equilibrium of the species.

Case II. If $r_0<r$ the sequence (D) will approach zero as a limit. If $r_0>r$ the sequence diverges. Therefore *when the survival rate of the hybrids is below those of the homozygous natural selection will result in an increasing divergence from the state of equilibrium.*

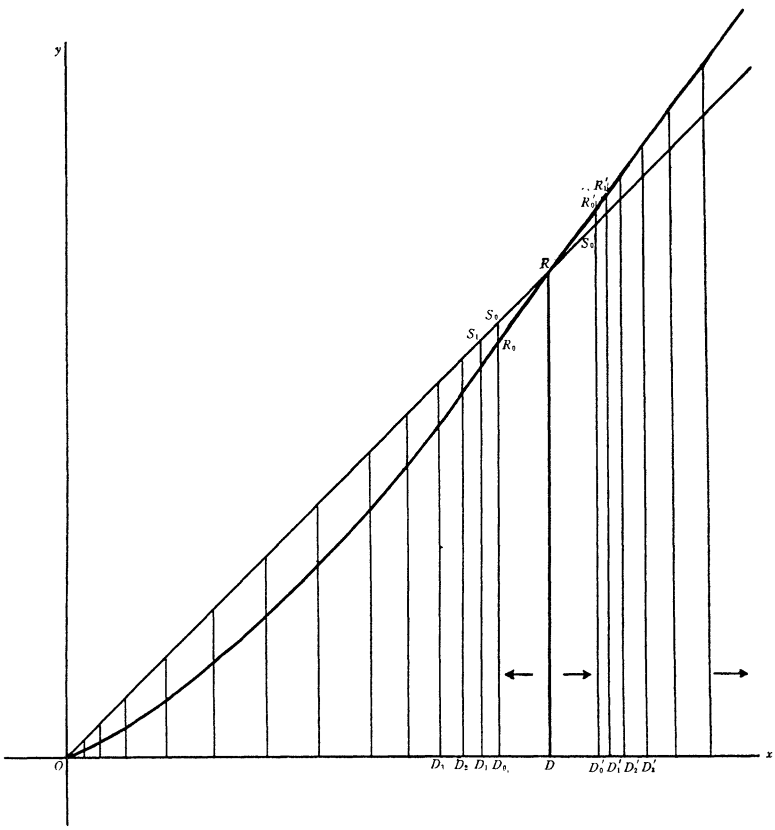

FIG. 2. Variation of the dominance ratio in a genetically unstable population.

Equilibrium is still possible if $r_0=r$. But should the slightest disturbance take place, the genetic process will further unbalance the distribution.

The equilibrium is unstable, the tendency of natural selection being a gradual elimination either of the dominant or of the recessive types.

6. *The reproductive capacity.* This for any generation is measured by the *ratio of the gamete to the zygote population:* $P_n=G_n/Z_n$. With the aid of formulae (2) this can be written:

$$P_{n+1} = (uA_n^2 + 2vA_nB_n + wB_n^2)/(A_n + B_n)^2.$$

Dividing through by B_n^2 and remembering that $A_n/B_n = r_n$, we have finally:

$$P_{n+1} = (ur_n^2 + 2vr_n + w)/(r_n + 1)^2.$$

We shall investigate the variation of this ratio in the case of stable evolution when $u < v > w$. With this in view we set $P_{n+1} = z$, $r_n = x$ and study the continuous function

$$z = (ux^2 + 2vx + w)/(x + 1)^2. \tag{5}$$

The graph of this function is shown in Figure 3. We find the sets of values

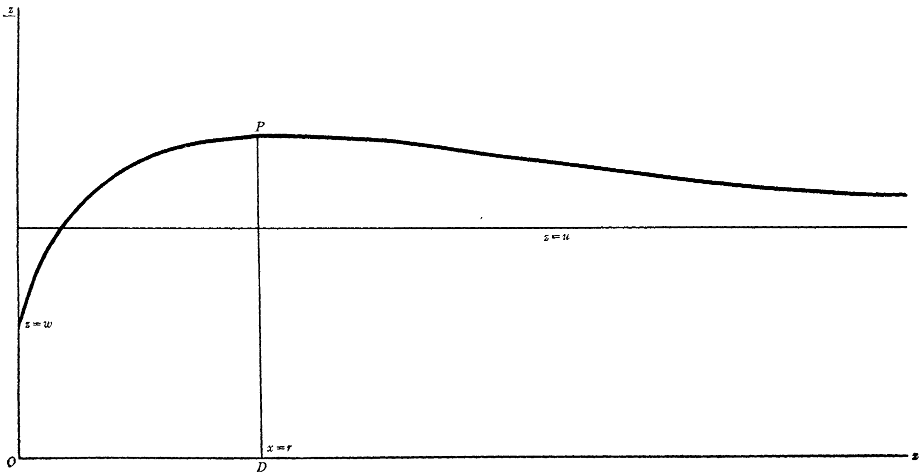

FIG. 3. Relation of reproductive capacity to dominance ratio in a genetically stable population.

$x = 0$, $z = w$ and $x = \infty$, $z = u$. By differentiating (5) and taking account of (4) we find

$$\frac{dz}{dx} = \frac{2[(v - w) - (v - u)x]}{(x + 1)^3} = \frac{2(v - u)(r - x)}{(x + 1)^3}.$$

The function is on the increase when $x_0 < r$. It therefore attains a maximum for the critical value $x = r$. Denoting by P this maximum value of x we have

$$P = \frac{ur^2 + 2vr + w}{(r + 1)^2} = \frac{u(v - w)^2 + 2v(v - w)(v - u) + w(v - u)^2}{(2v - u - w)^2},$$

which upon simplification becomes

$$P = (v^2 - uw)/(2v - u - w)$$

It remains to verify that this value is actually greater than any other. We find

$$P - P_{n+1} = \frac{(v - u)^2}{(2v - u - w)}\left(\frac{r_n - r}{r_n + 1}\right)^2,$$

and this difference remains positive as long as $2v > u+w$; this inequality is obviously satisfied in the case of genetic stability. Thus

A genetically stable species attains its maximum reproductive capacity at equilibrium.

The tendency of natural selection in a genetically stable species is towards a steady improvement of the reproductive capacity.

In the case of genetic instability it is proved in a similar way that the reproductive capacity is *minimum* at equilibrium.

7. *Summary.* If u, v, and w are the survival rates of the three genetic types, the species is genetically stable if $u < v > w$; unstable if $u > v < w$; neutral if $u = v = w$. When genetically stable the species will tend towards a final distribution of the dominant and recessive in the ratio, $v - w$ to $v - u$. When genetically unstable the tendency will be to annihilate one of the two types.

Natural selection has for effect a persistent increase of the reproductive capacity if the species is genetically stable, and a persistent decrease when it is unstable.

The reproductive capacity reaches an extremum at equilibrium: a maximum if the equilibrium is stable, a minimum if unstable.

8. *Conclusion.* This double tendency of natural selection towards genetic equilibrium on the one hand and towards maximum reproduction on the other is of considerable biological importance. The breeder in his effort to improve a plant or animal species will use *artificial* selection, i.e., he will endeavor to adjust the dominance ratio with a view of accentuating certain favorable characteristics and lessening the effect of others detrimental to his purpose. In so doing, however, he by necessity disturbs the equilibrium and consequently impairs the reproductive capacity. Thus arises the unavoidable conflict between *production* on the one hand and *reproduction* on the other. The existence of such a conflict is a well recognized fact but it is significant that it can be regarded as an a priori consequence of the first Mendelian law.

It is true that the results here obtained are based on simplified and therefore restrictive assumptions. In practice things are complicated by mutation and linkage. The effect of these latter phenomena on equilibrium and reproduction will be dealt with in a later paper.

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REMARKS ON A KNOWN EXAMPLE OF A MONOTONE CONTINUOUS FUNCTION

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In this note we are concerned with a well known example of a continuous monotone function. We have collected together a few properties of this function which is very well fitted for illustration of many important points of the theory of functions of a real variable. Some of these properties have been mentioned several times in the literature, some others, however, simple as they are, appear not to have been stated explicitly.

To simplify our formulas we shall consistently use the binary and ternary scales of notation. Thus ${}_3.101$ will mean $1/3+1/27=10/27$, while ${}_2.101 =1/2+1/8=5/8$.

To define our function[1] we first construct a perfect set of points nowhere dense on the interval (0, 1): Subdivide (0, 1) into three equal parts and remove the interior of the middle part (1-st stage of the process); subdivide each of the remaining two parts into three equal parts and remove the interiors of the middle parts of each of them (2-nd stage) and repeat this process indefinitely (the p-th repetition will be called the p-th stage of the process).

It is seen at once that the number of intervals removed at the p-th stage is 2^{p-1}. We denote them (ordered from left to right) by

$$\delta_{pk}\,(k = 1,2, \cdots, 2^{p-1}). \tag{1}$$

If we denote the length of the interval δ_{pk} by the same letter, then

$$\delta_{pk} = 3^{-p}. \tag{2}$$

With this notation we have

$$\delta_{11} = ({}_3.1,\ {}_3.2),\quad \delta_{21} = ({}_3.01,\ {}_3.02),\quad \delta_{22} = ({}_3.21,\ {}_3.22),\ \cdots .$$

The total number of the intervals δ_{pk} removed during the p first stages will be $1+2+\cdots+2^{p-1}=2^p-1$.

Let E be the set of points of (0, 1) which will not be removed. Then the complementary set $D=C(E)$ coincides with $\sum\delta_{pk}$ (where only the interior point of the intervals δ_{pk} are taken into account). The set E consists of all the end-points of the intervals δ_{pk} and of their limiting points. It is readily seen that E is identical with the set of points which are represented by infinite fractions

$${}_3.a_1a_2a_3\cdots a_n\cdots, \tag{3}$$

where only the digits 0 and 2 are admitted. Furthermore, the end-points of δ_{pk} are represented by the fractions (3) where all the digits after a certain place are all zeros or all two's, while the limiting points of the end-points will have infinitely many zeros and two's, except for the two extreme points

[1] Hobson, *The theory of functions of a real variable*, vol. 1, 3rd edition, 1927, pp. 123, 368. This is referred to as H in the sequel.

$$0 = {}_3.000 \cdots \quad \text{and} \quad 1 = {}_3.222 \cdots .$$

For instance the end-points of the interval δ_{34} are

$${}_3.221 = {}_3.220222 \cdots \quad \text{and} \quad {}_3.222 = {}_3.222000 \cdots .$$

Simultaneously with the intervals δ_{pk} we shall consider the intervals

$$\eta_{pk}\ (k = 1, 2, \cdots, 2^p) \tag{4}$$

which remain at the p-th stage. We assume the η_{pk} to be closed (while the δ_{pk} are open). The set E is always covered (in the large sense) by the intervals η_{pk}. All η_{pk} (for fixed p) are of the same length 3^{-p} and the sum of their lengths is $(2/3)^p$. Since this $\rightarrow 0$ as $p \rightarrow \infty$, the *set E is of measure* 0 (and even of Jordan content 0).[1]

Since all the numbers of the type (3) can be approximated as closely as we please by numbers of the same type, the set E contains all its limiting points, and also, each point of E is a limiting point, which shows that *E is perfect.* On the other hand each subinterval of (0, 1), no matter how small, contains parts which are free from points of E, whence *E is nowhere dense on* (0, 1).

We proceed now to the definition of our function $\omega(x)$. We agree once for all to use the letter a to indicate the digits 0 or 2 and to designate by b the number $a/2$, so that b assumes only the values 0 and 1. If $x = {}_3.a_1a_2a_3 \cdots a_n \cdots$ s a point of the set E, we define

$$\omega(x) = {}_2.b_1b_2b_3 \cdots b_n \cdots . \tag{5}$$

According to this definition $\omega(x)$ has equal values

$$\begin{aligned} \omega_{pk} &= \omega({}_3.a_1a_2 \cdots a_m0222 \cdots) = {}_2.b_1b_2 \cdots b_m0111 \cdots \\ &= {}_2.b_1b_2 \cdots b_m1000 \cdots = \omega({}_3.a_1a_2 \cdots a_m2000 \cdots) = (2k-1)/2^p \end{aligned} \tag{6}$$

at the end-points of each interval δ_{pk} and we take this value as the value of $\omega(x)$ at *all* the points of the corresponding δ_{pk}, with the result that the intervals δ_{pk} are intervals of constancy of $\omega(x)$. Now the function $\omega(x)$ is defined at all the points of (0, 1) and we may proceed to the enumeration of the properties of $\omega(x)$.

i. *$\omega(x)$ is monotone (non-decreasing) on* (0, 1) *and increases from* 0 *to* 1 *as x increases from* 0 *to* 1. *The intervals δ_{pk} are intervals of constancy of $\omega(x)$.*

Proof: In proving the inequality

$$\omega(x'') \geqq \omega(x') \quad \text{if} \quad x'' > x',$$

we may restrict ourself to the points of E, since $\omega(x)$ is constant on each δ_{pk}. Let

$$x' = {}_3.a_1'a_2' \cdots, \quad x'' = {}_3.a_1''a_2'' \cdots .$$

[1] H. p. 171. It follows then that meas. $D = 1$, which can be proved also by an immediate computation: meas. $D = \Sigma\delta_{pk} = 1/3 + 2/9 + \cdots + 2^{p-1}/3^p + \cdots = 1/3 \cdot 1/(1-2/3) = 1$.

If $x''>x'$, there will be a value of the subscript n for which

$$a_1' = a_1'', \cdots, a_{n-1}'' = a_{n-1}' \quad \text{but} \quad a_n'' > a_n',$$

whence

$$\omega(x'') = {}_2.b_1''b_2'' \cdots \geqq {}_2.b_1'b_2' \cdots = \omega(x').$$

ii. *$\omega(x)$ is continuous on* (0, 1).

Proof: We have to prove that

$$\omega(x') \to \omega(x) \quad \text{as} \quad x' \to x;$$

and again we may consider only the case where x is a point of E and x' assumes only the values belonging to E. It will suffice to give the proof only in the case where $x'>x$.

Let

$$x = {}_3.a_1a_2 \cdots, \quad x' = {}_3.a_1'a_2' \cdots.$$

If now $x'>x$ but $x'\to x$, then there will be a value of the subscript n (where $n\to\infty$ as $x'\to x$) such that

$$a_1' = a_1, \cdots, a_{n-1}' = a_{n-1} \quad \text{but} \quad a_n' > a_n;$$

whence

$$\omega(x') = {}_2.b_1b_2 \cdots b_{n-1}b_n' \cdots \to {}_2.b_1b_2 \cdots b_{n-1}b_n \cdots = \omega(x).$$

iii. *The function $\omega(x)$ is not absolutely continuous. Its λ-variation*[1] *on* (0, 1) *is constant and equals* 1.

Proof: To prove the last part of the statement it suffices to put $(\alpha_k, \beta_k) = \eta_{pk}$. The corresponding sum $\sum|\omega(\beta_k)-\omega(\alpha_k)| = \sum\{\omega(\beta_k)-\omega(\alpha_k)\} = 1$, while $\sum(\beta_k-\alpha_k) = \sum\eta_{pk}$ can be made as small as we please by taking p sufficiently large. The first part follows from the last one and the definition of the absolute continuity.[2]

iv. *The function $\psi(x) = (x+\omega(x))/2$ gives a continuous one-to-one correspondence between the segments* (0, 1) *on the X-axis and on the Y-axis, such that a set E of measure zero is transformed into a set of measure* $>0(=\frac{1}{2})$.[3]

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[1] Carathéodory, *Vorlesungen über reelle Funktionen*, 2nd edition, 1927, p. 511. By the λ-variation of a function $f(x)$ on (0, 1) is meant the upper limit of the sum

$$\sum_{k=1}^{m} |f(\beta_k) - f(\alpha_k)|$$

of absolute values of increments of $f(x)$ over any finite set of non-overlapping sub-intervals (α_k, β_k), $k=1, 2, \cdots, m$, whose total length

$$\sum_{k=1}^{m} (\beta_k - \alpha_k) \leqq \lambda.$$

[2] H. p. 291.

[3] See Carathéodory, loc. cit., p. 356.

Proof: Only the last statement of this property needs a justification. The transformation $y=\psi(x)$ makes to correspond to each interval δ_{pk} on the X-axis an interval of length $\delta_{pk}/2$ on the Y-axis, hence the set D of measure 1 is transformed into a set D_y of measure $\frac{1}{2}$. Then it is obvious that the set $E=C(D)$ (of measure 0) is transformed into a set $E_y=C(D_y)$ of measure $\frac{1}{2}$.

Remark: Since every set of measure >0 contains non-measurable subsets[1] the same function $\psi(x)$ gives an example of a continuous one-to-one transformation in which a measurable set (even a set of measure zero) is transformed into a non-measurable set.

v. *The derivative $\omega'(x)$ of the function $\omega(x)$ is zero almost everywhere on* (0, 1) (that is at all the points except for a set of points of measure zero).

Proof: This is obvious since $\omega'(x)=0$ at all the points of the set D, that is almost everywhere.[2]

Remark: Despite the fact that $\omega(x)$ has an integrable derivative almost everywhere, still

$$\int_0^x \omega'(x)dx = 0 \neq \omega(x) - \omega(0) = \omega(x).$$

vi. *The area under the curve $y=\omega(x)$ (that is the area limited by the curve, the the X-axis and the ordinates $x=0$, $x=1$) is $\frac{1}{2}$.*

Proof: Since the set E is of measure zero the area in question is

$$\begin{aligned} A &= \int_0^1 \omega(x)dx = \int_E \omega(x)dx + \int_D \omega(x)dx = \int_E \omega(x)dx = \textstyle\sum_{p,k}\omega_{pk}\delta_{pk} \\ &= \sum_{p=1}^{\infty} 3^{-p} \sum_{k=1}^{2^{p-1}} (2k-1)2^{-p} = \sum_{p=1}^{\infty} 6^{-p} \sum_{k=1}^{2^{p-1}} (2k-1) = \sum_{p=1}^{\infty} 6^{-p}2^{2p-2} \\ &= \tfrac{1}{4}\sum_{p=1}^{\infty} (\tfrac{2}{3})^p = \tfrac{1}{6}\cdot 1/(1-\tfrac{2}{3}) = \tfrac{1}{2}. \end{aligned}$$

This also follows from the skew symmetry of our curve with respect to the line $x=\frac{1}{2}$.

vii. *The length of the arc of the curve $y=\omega(x)$ between the points* (0, 0) *and* (1, 1) *is* 2.

Proof: Since $\omega(x)$ is monotone, hence of bounded variation, our curve has a finite length[3] which is defined in the usual manner, as the limit of the perimeter

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[1] Carathéodory, loc. cit., p. 268.

[2] It is readily proved by considerations of a general nature (H., pp. 601–602) that the set of points at which $\omega'(x)$ is $+\infty$, is not denumerable. It is not difficult to exhibit a continuum of such points (which necessarily are distinct from the end-points of the intervals δ_{pk}, where the left (right-)-hand derivative is $+\infty$ while the right (left-)-hand derivative is 0, according as the point in question is a left-hand or a right-hand end-point of δ_{pk}). But the question of a ***complete*** determination of *all* the points of E at which $\omega'(x)=+\infty$ requires more delicate considerations and undoubtedly is related to the arithmetic properties of fractions representing such points.

[3] H., pp. 338–339.

of an inscribed polygon. To prove property vii we shall show that the perimeter of any inscribed polygon (which does not cross itself) can not exceed 2, and, on the other hand, there exists an inscribed polygon whose perimeter is as near to 2 as we please. The first statement follows immediately from the fact that the perimeter of any inscribed polygon without double points can not exceed the sum of all the horizontal and vertical projections of its sides, which equals 2, provided the polygon starts from (0, 0) and ends at (1, 1). To prove the second statement take for the inscribed polygon the broken line whose vertices are at (0, 0), (1, 1) and at the end-points of the intervals δ_{pk} (n fixed, $p=1, 2, \cdots, n$; $k=1, 2, \cdots, 2^{p-1}$). The sum of the horizontal sides of this polygon is

$$\Sigma_{p,k}\, \delta_{pk} = \sum_{p=1}^{n} 2^{p-1}\cdot 3^{-p} = 1 - (\tfrac{2}{3})^n$$

while all the inclined sides are equal, their common length being

$$(2^{-2n} + 3^{-2n})^{1/2} = 2^{-n}[1 + (\tfrac{2}{3})^{2n}]^{1/2}$$

and the total number 2^n (the number of intervals η_{nk}). Hence the length l_n of our polygon is

$$1 - (\tfrac{2}{3})^n + [1 + (\tfrac{2}{3})^{2n}]^{1/2} \to 2 \quad \text{as} \quad n \to \infty\,.$$

Remark: It is interesting to observe that the length of our curve can not be computed by the familiar formula

$$\int_0^1 [1 + \omega'(x)^2]^{1/2} dx = 1 \neq 2\,.$$

The failure of this formula is due to the fact that $\omega(x)$ is not absolutely continuous.

viii. *The function $\omega(x)$ satisfies a Lipschitz condition of order $\alpha = \log 2/\log 3$.* In other words, if x and $x+h$ are in (0, 1),

$$|\,\omega(x + h) - \omega(x)\,| \leqq \Lambda\,|\,h\,|^{\alpha}\,; \qquad | \leqq \max\Lambda \leqq 2\,; \qquad \alpha = \log 2/\log 3.$$

Proof: If we set $\eta_{pk} = (x, x+h)$, then $\omega(x+h) - \omega(x) = 2^{-p}$ while $h = 3^{-p}$, and we see at once that the order α and the upper limit of the coefficient Λ can not be less than log 2/log 3 and 1, respectively. Let now x and $x+h$ be any pair of numbers in (0, 1); there will be no loss of generality in assuming $h>0$. We also may restrict the discussion to the case where both points x, $x+h$ belong to the set E, since, if x or $x+h$ is an interior point of an interval δ_{pk}, we can replace x by the right-hand end-point and $x+h$ by the left-hand end-point of the corresponding δ_{pk}, respectively. This will not change the difference $\omega(x+h) - \omega(x)$ but will reduce h; hence, if property viii is proved in the case where x and $x+h$ belong to E, it will hold true in the general case.

Let now

$$x = {}_3.a_1a_2 \cdots a_m \cdots\,; \qquad x + h = {}_3.a_1' a_2' \cdots a_m' \cdots\,,$$

and let n be determined by the condition

$$a_1' = a_1, \cdots, a_{n-1}' = a_{n-1}, \quad a_n' > a_n \quad \text{whence} \quad a_n' = 2, \quad a_n = 0.$$

We then have

$$\omega(x+h) - \omega(x) = {}_2.b_1b_2 \cdots b_{n-1}1 \cdots - {}_2.b_1b_2 \cdots b_{n-1}0 \cdots \leqq 2^{-n+1},$$

while

$$h = {}_3.a_1a_2 \cdots a_{n-1}2 \cdots - {}_3.a_1a_2 \cdots a_{n-1}0 \cdots \geqq 3^{-n}$$

and

$$[\omega(x+h) - \omega(x)]h^{-\alpha} \leqq 2^{-n+1} \cdot 3^{\alpha n} = 2 \quad \text{if} \quad \alpha = \log 2/\log 3.$$

Remark: The set of points x at which $\omega(x+h) - \omega(x) > h^\alpha$ for h sufficiently small is obviously a subset of E, hence it is of measure zero. This ought to be expected since otherwise the derivative $\omega'(x)$ would be $+\infty$ at a set of points of measure[1] >0.

The function $\omega(x)$ so far has been defined on the interval (0, 1). It can be extended outside this interval by setting

$$\omega(x) = \begin{cases} 0 & \text{if} \quad x \leqq 0, \\ 1 & \text{if} \quad x \geqq 1. \end{cases}$$

This will be assumed throughout the remainder of the present note. The difference $\omega(x+h) - \omega(x) = \phi_h(x)$ is of course $\geqq 0$, and, as a function of x, is of bounded variation, since it equals a difference of two monotone functions.[2] Let $T(h)$ be the total variation of the function $\phi_h(x)$ and let

$$\Omega(z) = \max T(h) \quad \text{for} \quad 0 \leqq h \leqq z.$$

It is plain that $\Omega(z)$ does not increase when z decreases and the natural question arises as to what is $\lim_{z\to 0} \Omega(z) = \Omega_0$? If $\phi_h(x)$ were absolutely continuous, then, since $\omega'(x) = 0$ almost everywhere, we would have[3]

$$T(h) = \int_{-\infty}^{\infty} |\phi_h'(x)|\, dx = \int_{-\infty}^{\infty} |\omega'(x+h) - \omega'(x)|\, dx = 0,$$

so that

$$\Omega(z) = \Omega_0 = 0.$$

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[1] H., p. 400.

[2] H., p. 329.

[3] H., p. 605. In the general case, where $\omega(x)$ is any absolutely continuous function, by a fundamental property of Lebesgue integrals (H., p. 636),

$$T(h) = \int_{-\infty}^{\infty} |\omega'(x+h) - \omega'(x)|\, dx \to 0 \quad \text{as} \quad h \to 0$$

which would yield the same result, $\Omega_0 = 0$.

The situation is entirely different, however, in our case (due to the non-absolute continuity of $\omega'(x)$), which is shown by the property[1]

ix. *The function $\Omega(z)$ as defined above is constant and equals* 2.

Proof: From the definition of the total variation of a function[2] it follows that $T(h)$, being a total variation of a difference $\omega(x+h)-\omega(x)$, can not exceed the sum of the total variations of the constituents $\omega(x+h)$, $\omega(x)$. Since $\omega(x)$ and $\omega(x+h)$ are monotone and increase from 0 to 1, they have the same total variation 1, whence

$$T(h) \leqq 2.$$

Take now $h=3^{-n}$. Since

$$(-\infty,\infty) = \delta_0 + \sum_{p=1}^{n} \sum_{k=1}^{2^{p-1}} \delta_{pk} + \sum_{i=1}^{2^n} \eta_{ni} + \delta_1\,; \quad \delta_0 = (-\infty, 0)\,; \quad \delta_1 = (1,\infty),$$

we have, by the additive property of the total variation, [3]

$$T(h) = T(\delta_0) + \Sigma_{p,k} T(\delta_{pk}) + \sum\nolimits_i T(\eta_{ni}) + T(\delta_1),$$

where each term of the right-hand member is the total variation of $\phi_h(x)$ over the corresponding interval. It is important to observe that our h equals the common length of the intervals η_{ni} and does not exceed the length of any of the intervals $\delta_{pk} (p=1, 2, \cdots n)$. Hence, when x ranges over an interval η_{ni}, $(x+h)$ ranges over a part of the interval δ_{pk} that is adjacent to η_{ni}. Under these circumstances, $\omega(x+h)$ remains constant while $\omega(x)$ increases by 2^{-n}. Hence

$$T(\eta_{ni}) = 2^{-n}.$$

To compute $T(\delta_{pk})$, let $x_1<x_2$ be the end-points of δ_{pk}. Subdivide δ_{pk} in two parts,[4] $\delta'=(x_1, x_2-h)$, $\delta''=(x_2-h, x_2)$ and denote by T', T'' the total variations of $\phi_h(x)$ in the intervals δ', δ'' respectively. When x ranges over δ' the functions $\omega(x)$ and $\omega(x+h)$ remain constant, so that $T'=0$. When x ranges over δ'', $(x+h)$ ranges over the interval η_{ni} that is adjacent to δ''. Then $\omega(x)$ remains constant but $\omega(x+h)$ increases by 2^{-n}, whence $T''=2^{-n}$ and

$$T(\delta_{pk}) = T' + T'' = 2^{-n}.$$

As to the terms $T(\delta_0)$ and $T(\delta_1)$, we find in exactly the same fashion that

$$T(\delta_0) = 2^{-n}, \quad T(\delta_1) = 0.$$

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[1] In the notation of the theory of Stieltjes integrals we can state this property as follows:

$$\Omega(z) = \max_{0\leqq h\leqq z} \int_{-\infty}^{\infty} |\, d\omega(x+h) - d\omega(x)\,| = 2.$$

In a recent important note [*Eine Kennzeichnung der totalstetigen Funktionen*, Crelle's Journal, vol. 160 (1929), pp. 26–32]. A. Plessner proved that the condition $T(h)\to O$ as $h\to O$ is necessary and sufficient for the absolute continuity of the function $\omega(x)$.

[2] H., p. 325.

[3] H., p. 330.

[4] The first part δ' exists only if $\delta_{pk}>h$, that is if $p<n$.

On combining all these facts and observing that the number of the intervals η_{ni} is 2^n while that of the intervals δ_{pk} is 2^n-1, it follows at once that

$$T(h) = 2^{-n}(1 + 2^n + 2^n - 1) = 2$$

whence

$$\Omega(z) = \Omega_0 = 2.$$

The last and perhaps the most interesting property of our function $\omega(x)$ is in connection with its "Fourier-Stieltjes coefficients." If $f(x)$ is any function given on $(0, 1)$ we may call its Fourier-Stieltjes coefficients the integrals[1]

$$f_n' = \int_0^1 e^{2\pi inx}df(x) \qquad (n = 0, \pm 1, \pm 2, \cdots). \tag{7}$$

If $f(x)$ is absolutely continuous, then integrals (7) reduce to the classical Fourier coefficients of $f'(x)$:

$$f_n' = \int_0^1 e^{2\pi inx}f'(x)dx ;$$

and, by the fundamental Riemann-Lebesgue theorem,[2] $f_n'\to 0$ as $|n|\to\infty$.

In a more general case where $f(x)$ is only of bounded variation, we still have right to integrate by parts:

$$f_n' = \int_0^1 e^{2\pi inx}df(x) = [f(1) - f(0)] - 2\pi in \int_0^1 e^{2\pi inx}f(x)dx, \tag{8}$$

so that

$$f_n = \frac{[f(1) - f(0)]}{2\pi in} - \frac{f_n'}{2\pi in} \tag{9}$$

is the Fourier coefficient of $f(x)$.

There is an essential difference between the two cases just mentioned, which is shown by the property

x. [3]*The Fourier-Stieltjes coefficient* ω_n' *of the function* $\omega(x)$ *does not tend to* 0 *as* $|n|\to\infty$.

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[1] We refer as to the definition and fundamental properties of Stieltjes integrals to H. Here we deal exclusively with the Riemann-Stieltjes integrals. A Riemann-Stieltjes integral of a function $g(x)$ with respect to the function $f(x)$ is defined as the limit (in case it exists) of the sum:

$$\int_0^1 g(x)df(x) = \lim \sum_{i=1}^{m} g(\xi_i)[f(x_i) - f(x_{i-1}); \quad x_0 = 0, \quad x_m = 1,$$

where (x_{i-1}, x_i), $i=1, 2, \cdots, m$ is any subdivision of the interval $(0, 1)$ such that the maximum length of the intervals $(x_{i-1}, x_i)\to 0$ as $m\to\infty$ and ξ_i is an arbitrary point of the interval (x_{i-1}, x_i), the end-points inclusive. The existence of this limit is assured if $g(x)$ is continuous and $f(x)$ is of bounded variation.

[2] H., vol. 2 (2nd edition), 1926, p. 514.

[3] This is a special case of an example of Carleman, *Sur les équations intégrales singulières à noyau réel et symétrique*, Uppsala Universitets Årsskrift, 1923, No. 3, pp. 223–226.

Proof: By definition we have

$$\omega_n' = \int_0^1 e^{2\pi i n x} d\omega(x) = \lim_{m\to\infty} \sum_{s=1}^{m} e^{2\pi i n \xi_s}[\omega(x_s) - \omega(x_{s-1})] = \lim_{m\to\infty} \Sigma_m .$$

In computing this limit we can take any special type of subdivisions of (0, 1); for instance, we may subdivide (0, 1) into $2^p = m$ equal parts. Then the set $\{(x_{s-1}, x_s)\}$ will consist partly of the intervals η_{pk} and partly of the intervals $\delta_{jk}(j=1, 2, \cdots, p)$ and their subdivisions. Since $\omega(x)$ is constant on each δ_{jk} this second part will give no contribution to the sum $\sum_m$. As to the points ξ_s we shall make them to coincide with the left-hand end-points of the corresponding intervals η_{pk}. They will be designated (in increasing order) by α_k, $k=1, 2, \cdots, 2^p$.

Since $\omega(x)$ increases by 2^{-p} when x ranges over an interval η_{pk},

$$\Sigma_m = \Sigma_{2^p} = 2^{-p} \sum_{k=1}^{2^p} e^{2\pi i n \alpha_k} .$$

It is readily seen that the set of points $\{\alpha_k\}$ consists of all the *finite* fractions of the form

$$_3 . a_1 a_2 \cdots a_p \qquad (a_j = 0 \quad \text{or} \quad 2).$$

The summation over all such values of a_j will be designated simply by $\sum_{(a)}$. Hence

$$\Sigma_{2^p} = 2^{-p} \sum_{(a)} \exp. \left[2\pi i n \sum_{j=1}^{p} a_j 3^{-j} \right] = 2^{-p} \sum_{(a)} \prod_{j=1}^{p} \exp. \left[2\pi i n a_j 3^{-j} \right]$$

$$= 2^{-p} \prod_{j=1}^{p} \{ 1 + \exp. (4\pi i n 3^{-j}) \} = \exp. \left(\sum_{j=1}^{p} 2\pi i n 3^{-j} \right) \prod_{j=1}^{p} \cos (2\pi n 3^{-j}) .$$

This yields the final result

$$\omega_n' = \lim_{p\to\infty} \Sigma_{2^p} = e^{\pi i n} \prod_{j=1}^{\infty} \cos (2\pi n 3^{-j})$$

since

$$2 \sum_{j=1}^{\infty} 3^{-j} = 1 .$$

In the preceding computation n was an arbitrary number (not necessarily an integer). Now we set $n = 3^q$ where q is a positive integer. Then

$$\omega_n' = \omega_{3^q}' = - \prod_{j=1}^{\infty} \cos (2\pi 3^{q-j}) = - \prod_{\nu=1}^{\infty} \cos (2\pi / 3^\nu) .$$

The infinite product of the left-hand member converges absolutely and contains no zero factor;[1] therefore it is different from 0. On the other hand it does not

[1] A necessary and sufficient condition for the absolute convergence of the infinite product $\Pi_\nu (1 + u_\nu)$ is given by the absolute convergence of the series $\Sigma \nu u \nu$. This condition is satisfied in the present case since $|u_\nu| = |1 - \cos (2\pi / 3^\nu)| = 2 \sin^2 (\pi / 3^\nu) < 2\pi^2 3^{-2^\nu}$.

depend on q. If now we make $q\to\infty$ the corresponding Fourier-Stieltjes coefficient ω_{3q}' of $\omega(x)$ will not tend to 0.

Remark: The function

$$\chi(x) = \omega(x) - x$$

gives an example of a periodic continuous function (of bounded variation) such that, if χ_n is the Fourier coefficient of $\chi(x)$, the product $n\chi_n$ does not tend to any limit as $|n|\to\infty$.

This follows immediately from (9) and property x.

The interest of this example lies in the fact that if $f(x)$ is continuous and periodic and if nf_n tends to a limit as $|n|\to\infty$ then this limit[1] is necessarily 0.

AN ALGEBRAIC METHOD OF DIFFERENTIATION

By ORRIN FRINK, JR., Pennsylvania State College

It is the purpose of this paper to present a method of obtaining the formulas of the differential calculus by purely algebraic means, without the use of limiting processes. The method is rather obvious, and is essentially equivalent to those used by the mathematicians of the eighteenth century, before the logical rigor which we associate with the name of Weierstrass came into favor.[2] The method here presented is rigorous, however, being based on the theory of analytic functions of a hypercomplex variable.

Consider the hypercomplex number system (or linear algebra), analogous to the ordinary complex number system, whose basal units are 1 and j, where $j^2=0$. Because of its many geometric applications, the function theory of this algebra has been much studied. It has been shown by Scheffers[3] that the most general analytic function of one variable in this algebra has the form

$$f(x + yj) = \phi(x) + [\phi'(x)y + \psi(x)]j, \tag{1}$$

where $\phi(x)$ and $\psi(x)$ are real functions of a real variable. (The terms real and imaginary will be used to distinguish x and yj, to keep the analogy with the theory of functions of a complex variable. It would be possible to allow x and y to be complex, and in this case the terms *scalar* and *nilpotent* would be less confusing.) If now f is a function which is real for real values of the argument, we have, setting $y=0$, that $\psi(x)=0$ and $\phi(x)=f(x)$, which gives us

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[1] Neder, *Über die Fourierkoeffizienten der Funktionen von beschränkten Schwankung*, Mathematische Zeitschrift, vol. 6 (1920), pp. 270–273; Steinhaus, *Bemerkung zu der Arbeit des Herrn Neder* . . . . , ibidem, vol. 8 (1920), pp. 320–322; Alexits, *Zwei Sätze über Fourierkoeffizienten*, ibidem, vol. 27 (1927), pp. 65–67. Another example of a continuous periodic function $f(x)$ for which lim (nf_n) does not exist was given by F. Riesz, ibidem vol. 2 (1918), pp. 312–315. Riesz's example, however, is of entirely different nature.

[2] See the interesting paper of Professor James Pierpont, *Mathematical Rigor, Past and Present* in the Bulletin of the American Mathematical Society, vol. 34 (1928), p. 23.

[3] Mathematische Annalen vol. 60 (1905), p. 529.

(2) $$f(x + yj) = f(x) + f'(x)yj.$$

It may easily be verified that this formula holds for the general f; in fact, it may be considered to be a Taylor's series expansion of f, all higher powers of j being automatically zero. If f is real for real values of the argument, however, this formula effects a separation of f into real and imaginary parts.

If we put $y=1$ in (2) we get

(3) $$f(x + j) = f(x) + f'(x)j.$$

This suggests the method of differentiation which we shall use: To find the derivative of a function $f(x)$, replace x by $x+j$, that is, give x the increment j. Then separate $f(x+j)$ into real and imaginary parts, and the coefficient of j in the imaginary part will be the derivative, $f'(x)$. The following examples will serve to illustrate the method:

I. $$f(x) = x^n;$$
$$(x + j)^n = x^n + nx^{n-1}j;$$
hence
$$f'(x) = nx^{n-1}.$$

The formulas for the derivative of the sum of two functions and of a constant times a function can be derived in the same simple way.

II. $f(x)=1/x$. We wish to separate $1/(x+j)$ into real and imaginary parts. Using the method of undetermined coefficients, we let $1/(x+j)=a+bj$. We wish to find b. Of course we know in advance that a will come out equal to $1/x$. Multiplying both sides by $x+j$ and making use of the fact that $j^2=0$, we have
$$1 = ax + aj + bxj.$$

Equating real and imaginary parts we get
$$a = 1/x, \qquad a + bx = 0, \qquad b = -a/x = -1/x^2 = f'(x).$$

III. $f(x)=\sqrt{x}$. Let $\sqrt{(x+j)}=a+bj$. Squaring, $x+j=a^2+2abj$. Hence $a=\sqrt{x}$, $b=1/(2a)=1/(2\sqrt{x})=f'(x)$.

IV. $f(x)=u(x)v(x)$. Let $u(x+j)v(x+j)=a+bj$. From (3)
$$[u(x) + u'(x)j]\,[v(x) + v'(x)j] = a + bj.$$
$$u(x)v(x) + u'(x)v(x)j + u(x)v'(x)j = a + bj$$
$$\therefore\ b = u'(x)v(x) + u(x)v'(x) = f'(x).$$

V. $f(x)=u(x)/v(x)$. Let $u(x+j)/v(x+j)=a+bj$. From (3),
$$[u(x) + u'(x)j]/[v(x) + v'(x)j] = a + bj.$$

Clearing of fractions,
$$u(x) + u'(x)j = av(x) + av'(x)j + bv(x)j.$$

Equating coefficents,

$$a = \frac{u(x)}{v(x)}, \quad u'(x) = av'(x) + bv(x)$$

and

$$b = \frac{u'(x) - av'(x)}{v(x)} = \left[u'(x) - \frac{u(x)}{v(x)}v'(x)\right] \Big/ v(x)$$

$$= \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2} = f'(x).$$

VI. $f(x) = g[h(x)]$. Let $g[h(x+j)] = a+bj$. Then $g[h(x)+h'(x)j] = a+bj$. From (2), we have

$$g[h(x)] + g'[h(x)]h'(x)j = a + bj.$$

Hence

$$b = g'[h(x)]h'(x) = f'(x).$$

VII. The formula for the derivative of an inverse function can likewise be derived. Let $y = f(x)$. We wish to prove that $dx/dy = 1/f'(x)$.

From (3)

$$f(x + j) = f(x) + f'(x)j.$$

Hence

$$x + j = f^{-1}[f(x) + f'(x)j].$$

From (2), we have

$$x + i = x + \left[\frac{d}{dy}f^{-1}(y)\right]f'(x)j$$

Hence

$$x + j = x + \frac{dx}{dy}\cdot f'(x)j, \quad \frac{dx}{dy} = \frac{1}{f'(x)}.$$

The derivatives of the elementary transcendental functions can be found by using the functional equations (addition theorems) which they satisfy. For example, e^x (and also a^x) satisfy the equation $f(x+y) = f(x)f(y)$. Making use of the principle of the permanence of functional equations, which can be proved here as in the case of the ordinary complex variable, we substitute j for y. We have then

$$f(x + j) = f(x)f(j) = f(x) + f'(x)j.$$

Dividing by $f(x)$,

$$f(j) = 1 + jf'(x)/f(x).$$

But since the other terms are constants, $f'(x)/f(x)$ must be a constant. The derivative of a^x must then be a constant times a^x, and we can define e to be that value of a for which this constant is unity. In particular if $(d/dx)(a^x) = ka^x$, we see that e must equal $a^{1/k}$. The formulas for the derivative of the logarithm

and of the other inverse transcendental functions can be found by using the formula in VII for inverse functions.

A similar method can be used to find second and higher derivatives directly. For second derivatives we use the algebra whose basal units are $(1, j, j^2)$, where $j^3=0$. For an analytic function of this kind of hypercomplex variable we find the formula: $f(x+j)=f(x)+f'(x)j+\frac{1}{2}f''(x)j^2$, which again may be considered a Taylor's series expansion of $f(x+j)$. Using generally the method of undetermined coefficients, we can find the first and second derivatives of a given function simultaneously.

Although this method may perhaps be of some interest in itself, its main point is the avoidance of limiting processes. The fundamental distinction between algebra and analysis is supposed to be that in algebra we make no use of order relations between the elements, such as are involved in the inequalities used in defining limit. From this point of view it is interesting to note that some of the results of analysis may be obtained algebraically, that is, without making use of order relations. In this connection it must be observed that in the above treatment the function theory of hypercomplex variables need not be considered as having been presupposed. We may define derivative to be the function obtained by the process laid down. The function-theoretic basis can then be considered as merely giving the proof that the derivative so obtained will coincide with the derivative as ordinarily defined.

DOUBLY HOMOGENEOUS FUNCTIONAL EQUATIONS

By JOSEPH D. GRANT, University of Illinois

Functional equations in one unknown function which have arguments linear and homogeneous in a set of variables and which are homogeneous in the unknown function may be referred to briefly as doubly homogeneous. The purpose of this paper is to consider the class of these equations having constant coefficients. As an example of a second degree equation of this class we have the following, a solution of which is known to be $\sin x$:

$$F(x+y)F(x-y)+F(y+z)F(y-z)+F(z+x)F(z-x)=0.$$

We shall consider only those solutions which are functions of one variable and which are analytic at zero except for a possible pole.

It will be convenient to prove the theorems to be stated here for a general second degree equation in two variables and to note that the methods apply without modification to equations of this form of any degree and in any number of variables. The general second degree equation in two variables and with constant coefficients is

$$\sum_{i=1}^{n} k_i F(a_i x+b_i y)F(c_i x+d_i y)=0$$

and the solution when arranged in ascending powers of t is to be of the form

$$F(t) = A_v t^v + A_{v+\delta} t^{v+\delta} + \cdots ,$$

where $A_v \neq 0$, $A_{v+\delta} \neq 0$, v and δ are integers, and $\delta > 0$.

The substitution of this solution results in an identity in the two variables x, y and from the theory of such identities it is known that all of the terms of a given degree taken together must vanish identically. Thus the first terms to be considered give rise to $A_v^2\phi_1(x, y) \equiv 0$ where $\phi_1(x, y)$ is of the form of a polynomial and is identically zero since $A_v \neq 0$. This calculation is seen to be independent of the subsequent terms of the series so that we have the following theorem.

Theorem I. *A necessary and sufficient condition that an equation of the class considered here have a solution is that it have a monomial solution.*

The terms of next higher degree give rise to $A_v A_{v+\delta}\phi_2(x, y) \equiv 0$ where $\phi_2(x, y)$ is of the form of a polynomial and is identically zero since $A_v \neq 0$ and $A_{v+\delta} \neq 0$. The existence of a δ such that $\phi_2(x, y)$ is identically zero is a necessary condition for an equation to have other than monomial solutions.

In general, continuing this process, a number of the first coefficients are left arbitrary and the subsequent ones are defined in terms of them. Such important equations as $F(x) = F(-x)$ and $F(x) = -F(-x)$ constitute an exception to this and do not serve to define particular functions as the general solution in each case would involve an infinite number of arbitrary elements.

Two points may be noted in this connection, first that every equation of this class having other than monomial solutions has solutions which depend upon two or more parameters and second that these parameters fall into two classes. A primary one is one which comes as the coefficient of the first term and can not be zero without the solution becoming the identically zero solution. A secondary one is one which leads on from the first term to the remainder of the series. Representing the primary ones as k and the secondary ones as m we have such functions as ke^{mt}, $k \cos mt$, $k \sin mt$ for possible solutions.

If an equation have a rational integral function of more than one term as a solution, the solution will have, when arranged in ascending powers of t, a last term which we may represent by $A_w t^w$. On substitution, the highest degree terms will give rise to $A_w^2\phi_3(x, y) \equiv 0$, where $\phi_3(x, y) = 0$ since $A_w \neq 0$. As before this calculation shows that $A_w t^w$ is a monomial solution so that a necessary condition for a polynomial solution is that the first and last terms be monomial solutions. It follows from this that if one could establish the existence of a set of first terms of a series, representing a solution, which would carry beyond the last monomial solution one would know that the series was infinite.

Another reason why there are at least two parameters connected with solutions other than monomials is given by the following:

Theorem II. *If $F(x)$ is a solution of an equation of this class, then $kF(mx)$ is also a solution.*

This follows directly from the two homogeneity properties.

A parameter may also enter into the solution of one of these equations by means of a multiplier e^{nx} as is the case for $F^2(x+y)=F(2x)F(2y)$. The general condition for the existence of such a multiplier is given by

Theorem III. *A necessary and sufficient condition that the solution $F(x)$ imply the solution $e^{nx}F(x)$ is that $a_i+c_i=p$ and $b_i+d_i=q$ where p and q are constants independent of i.*

The proof follows from the substitution of $e^{nx}F(x)$ which gives

$$e^{n(px+qy)}\sum_{i=1}^{n}k_iF(a_ix+b_iy)F(c_ix+d_iy)=0.$$

There also exists equations into whose solutions a parameter may enter by a multiplier e^{nx^2}. Such a one is

$$F(ax+by)\quad F(bx-ay)=F(ax-by)\quad F(bx+ay).$$

Theorem IV. *A necessary and sufficient condition that a solution $F(x)$ imply a solution $e^{nx^2}F(x)$ is that*

$$a_i^2+c_i^2=p,\quad a_ib_i+c_id_i=q,\quad b_i^2+d_i^2=r,$$

where p, q, r are constants independent of i.

The existence of one of these multipliers for the solutions of an equation does not imply that the multiplier is itself a solution nor does the existence of a solution e^{nx} or e^{nx^2} imply that it will multiply all other solutions. It is possible for equations to have both multipliers. Such a one is

$$F(x)F^2(x+3y)=F^2(x+y)F(x+4y).$$

Theorem IV is of further interest since each equation which satisfies its conditions implies a solution of a diophantine problem in sums of squares and conversely. For example, the four arguments which enter into the illustrative example above are a solution of the problem of finding the numbers expressible as the sums of two squares in two ways.[1]

Wilson[2] has shown that the only possible analytic solutions in the linear case are rational integral functions every term of which is independently a solution of the equation.

In non linear equations the existence of two solutions does not imply that their sum is a solution so that the solutions fall into distinct sets one for each monomial solution. One or more degenerate solutions are associated with each solution defined by two or more parameters. These degenerate solutions arise for the particular values of the secondary parameters

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[1] R. D. Carmichael's *Diophantine Analysis*, p. 25.

[2] American Journal of Mathematics, vol. 40, p. 263–286.

which cause a change of form of the solution. Such solutions as ke^{mx} and $k \cos mx$ become for $m=0$ merely k. Such solutions as $k \sin mx$ and $k(\cos mx-1)$ have degenerate solutions associated with them which are suggested by the limiting processes

$$\lim_{m=0} \frac{\sin mx}{\mathrm{m}} = x, \qquad \lim_{m=0} \frac{(\cos mx - 1)}{m^2} = - x^2/2.$$

These degenerate solutions are merely the monomial solutions required by theorem one. Other degenerate solutions are possible, however, as is shown by the following example.

We shall next show that the known analytic solutions for the three term equation[1] for the sigma function form a complete set. A set of analytic solutions will be said to be complete if every analytic solution (at zero) may be expressed in terms of the set by particularizing the parameters. We shall need the following known facts.[2]

1. $$\sigma(xg_2g_3) = x - \frac{g_2x^5}{2.5!} - \frac{6g_3x^0}{7!} - \frac{9g_2x^9}{4.9!} - \frac{18g_2g_3x^{11}}{11!} - \text{etc.}$$

2. If the cubic $4y^3-g_2y-g_3=0$ has two equal roots, $g_2{}^3-27g_3^2=0$ and $\sigma(xg_2g_3)$ degenerates to $e^{-x^2/6}\cdot \sin x$. If the cubic has three equal roots $g_2=0$, $g_3=0$, and $\sigma(xg_2g_3)$ degenerates to kx.

3. $$k\sigma\left(\frac{x}{k}, k^4g_2, k^6g_3\right) = \sigma(xg_2g_3).$$

4. $$\theta_1(x) = ke^{nx^2}\sigma(2wx), \quad k = \frac{\theta_1'(0)}{2w}, \quad n = 2\eta w.$$

5. $\sigma(xg_2g_3)$ and $\theta_1(x)$ are solutions of the three term equation:[3]

$$F(z+a)F(z-a)F(b+c)F(b-c) +$$
$$F(z+b)F(z-b)F(c+a)F(c-a) +$$
$$F(z+c)F(z-c)F(a+b)F(a-b) = 0.$$

It will be seen that x and $\sin x$ are solutions and that this equation satisfies the conditions of Theorem IV. The associated number theory problem is to express a number as the sum of four squares in three ways without using zero and having no number common to the three. That this does not give all numbers which are so expressible may be seen from the fact that the sum of the four

[1] This problem has been considered by Alfons Delisle in the Mathematische Annallen, vol. 30, p. 91–119, and by Hermite (see Whittaker and Watson's *Modern Analysis*, 3rd edition, p. 461).

[2] G. H. Halphen's *Traite des fonctions elliptiques*, vol. 1, pp. 169, 187, 251, 300.

[3] Another form of the three term equation involving five variables is given in Briot et Bouquet's *Théorie des fonctions elliptiques*, p. 486.

squares is $2(z^2+a^2+b^2+c^2)$ and that there exist odd numbers which are so expressible as

$$63 = 7^2 + 3^2 + 2^2 + 1 = 5^2 + 5^2 + 3^2 + 2^2 = 6^2 + 5^2 + 1 + 1.$$

It is to be shown that $ke^{nx^2}\sigma(xg_2g_3)$, $ke^{nx^2}\sin mx$, and $ke^{nx^2}x$ form a complete set of analytic solutions (at zero). We have seen that the solution $\theta_1(x)$ is expressible in terms of them. We shall consider the results of three substitutions upon the three term equation,

1. $z=a=b=c=0$ gives $f(0)=0$.
2. $b=c=0$ gives $f(-a)=-f(a)$; that is, any solution will be an odd function.
3. $z=3t,\ a=2t,\ b=t,\ c=0$ gives

$$F^3(t)F(5t) + F^3(3t)F(t) - F^3(2t)F(4t) = 0.$$

We note that in this equation the sum of the squares is 28 and that 28 is the smallest number satisfying the number theory conditions. It will be seen that every solution of the three term equation in four variables will satisfy the special form in one variable. The line of proof will be to show the converse, that is that the complete set of analytic solutions (at zero) for the equation in t is as given above.

By Theorem I, if the equation in t is to have solutions it will have monomial solutions. To ascertain the number of these, substitute $f(x)=A_vx^v$, which gives

$$A_v^4(5^v + 3^{3v} - 2^{5v})x^{4v} \equiv 0.$$

That is, we are to solve $(5/32)^v+(27/32)^v=1$ for v an integer.

Since the fractions are less than unity, increasing the exponent decreases their value; and for $v=2$ one has

$$(5/32)^2 + (27/32)^2 = 754/1024 < 1,$$

so that a solution will be a value of v less than 2.

If we replace v by $-w$, we are seeking integral solutions of

$$1 + (160/468)^w = (135/468)^w$$

where the fractions are less than unity, and for $w=0$ one has $2>1$, so that a solution will be a value of w less than zero or of v greater than zero. Since a solution is an integral value of v less than 2 and greater than zero, and since $v=1$ is solution, there is one and only one monomial solution of the t equation, and that is A_1x. It follows from this that there is only one set of solutions.

Any analytic solution (at zero) of the t equation will be odd and hence of the form

$$f(x) = A_1x + A_3x^3 + A_5x^5 + A_7x^7 + \cdots, \text{ where } A_1 \neq 0.$$

Substituting this series in the t equation, and equating to zero the coefficient of each power of t, one has

$$
\begin{aligned}
t^4 &: \; 0A_1^4 = 0,\\
t^6 &: \; 0A_3A_1^3 = 0,\\
t^8 &: \; 0A_5A_1^3 + 0A_3^2A_1^2 = 0,\\
t^{10} &: \; 0A_7A_1^3 + 0A_5A_3A_1^2 + 0A_3^3A_1 = 0,\\
t^{12} &: \; 2^5\cdot 3^4\cdot 5[28A_9A_1^3 - 28A_7A_3A^2 + 10A_5^2A_1^2 + 4A_5A_3^2A_1 - A^4_3] = 0.
\end{aligned}
$$

These equations leave $A_1A_3A_5A_7$ arbitrary and determine A_9 in terms of them. We need now to show that no fifth coefficient is left arbitrary by some equation not examined. We note that the equation for t^{2v} has in it A_{2v-3} and preceding coefficients. If A_{2v-3} is determined by that equation then there will be an equation to determine each A and none will be left arbitrary. To prove that A_{2v-3} is so determined uniquely for $v>6$ it is sufficient to show that the term $A_{2v-3}A_1^3t^{2v}$ has a non-zero coefficient for $v>6$. This coefficient is

$$42 + 5^{2v-3} + 3^{2v} - 3\cdot 2^{2v+1} - 2^{4v-3}.$$

By treating this expression in the same manner as the other exponential expression, the reader will verify, that it is never zero for $v>6$.

Expanding $ke^{nx^2}\sigma(xg_2g_3)$ and equating coefficients to the corresponding A's one has

$$
\begin{aligned}
A_1 &= k \neq 0, & k &= A_1,\\
A_3 &= kn, & n &= \frac{A_3}{A_1},\\
A_5 &= k\left(\frac{n^2}{2!} - \frac{g_2}{2.5!}\right), & g_2 &= \frac{5!(A_3^2 - 2A_1A_5)}{A_1^2},\\
A_7 &= k\left(\frac{n^3}{3!} - \frac{ng_2}{2.5!} - \frac{6g_3}{7!}\right), & g_3 &= \frac{7!(3A_1A_3A_5 - A_3 - 3A_1A_7)}{18A_1^3}
\end{aligned}
$$

Hence we have shown that the known solutions for the three term equation in 4 variables do form a complete set of solutions analytic at zero for the three term equation in one variable and will therefore be a complete set for the equation in four variables.

The following list of equations with their solution should be of interest as showing some of the possibilities of this class of equations. Those having more than one set of solutions are of particular note.

$$2F(2x)F(2y) - F(2x)F(x+y) - F(2y)F(x+y) = 0\,; \quad F(x) = k(1+mx)^{-1}.$$

$$F^2(x) - F^2(y) = F(y)F(2x+y) - F(x)F(x+2y) = 0\,; \quad F(x) = k\cos mx.$$

$$
\begin{aligned}
2F(x)F(y) + F(x)F(x-y) &= F(x+y)F(x) + F(x+y)F(y)\\
&\quad + F(x-y)F(y)\,; \quad F(x) = k\cot mx.
\end{aligned}
$$

$$
\begin{aligned}
F(x)F(3y-2x) + F(x)F(y) &= F(2x-y)F(2y-x) + F(y)F(2y-x)\,;\\
F(x) &= ke^{mx}, \quad k\cos mx, \quad k(1+mx), k\sin mx.
\end{aligned}
$$

$$F^2(y-x)+F(x)F(y-x)=F^2(y)+F(y)F(y-2x)\,;$$
$$F(x)=k\sin mx,\ k\cos mx.\ k(1+mx)\,ke^{mx}.$$
$$F^3(x+y)+F(2x+2y)F(x)F(y)=F(2x+y)F(x+y)F(y)+F(x+2y)F(x+y)F(x)\,;$$
$$F(x)=ke^{mx},\quad ke^{nx}\sin mx.$$
$$F^2(2x)+F^2(2y)+4F^2(x+y)+2F(2x)$$
$$-F(2y)4F(2x)F(x+y)-4F(2y)F(x+y)=0\,;$$
$$F(x)=k(1+mx).$$

Since the monomial solutions will exist in each case they are omitted.

UNPUBLISHED STEINER MANUSCRIPTS

By ARNOLD EMCH, University of Illinois

After the death of Steiner, in 1863, the personal and scientific papers of the great geometer were removed to the garret of the city library of Berne, where Professor Graf found them 30 years later in a box into which they had been carelessly dumped. The contents were turned over to Professor Bützberger for examination and eventual publication of worthy scientific material. He made a careful investigation of the posthumous works, had them cataloged and bound, and deposited ten volumes in the library of the University of Berne. The scientifically most important papers Bützberger kept. In the programs of the cantonal college of Zürich, 1913-14, he published some of Steiner's and some of his own investigations, including a particularly interesting article on the discovery of inversion.[1] Unfortunately Bützberger died before he had completed the task of editing the Steiner "Nachlass." Thus the matter stood again as before, since nobody seemed to take an interest in continuing Bützberger's valuable efforts.

In the summer of 1928 the writer had the opportunity to examine the ten volumes at the university library of Berne. They contain nothing of importance from a scientific standpoint, but the lecture notes on the elementary mathematical courses which Steiner followed at the Pestalozzi institution in Yverdon. Among them is however a monograph on the theory of forms and magnitudes (Formen- und Grössenlehre) by Steiner himself, in which he stresses Pestalozzi's idea that in the teaching of elementary geometry the knowledge of forms should precede that of magnitudes. All this covers the activities up to the year 1818 when Steiner went to the University of Heidelberg as a private teacher and as a student.

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[1] Über bizentrische Polygone, Steinersche Kreis-und Kugelreihen und die Erfindung der Inversion (Teubner, Leipzig, 1814). See also Emch, *The discovery of inversion*, Bulletin of the American Mathematical Society, vol. 21 (1914–15), p. 206.

The important manuscript on the geometry of the circle and the sphere to which Bützberger refers in his monograph was missing, a fact which the library authorities at Berne did not know. It is now in the possession of Mrs. Bützberger in Zürich, where I was given the opportunity to make a thorough examination of it. The complete title is:

"Allgemeine Theorie über das Berühren und Schneiden der Kreise und Kugeln mit vielen neuen Sätzen und Untersuchungen in einem systematischen Entwicklungsgang dargestellt," von Jacob Steiner, Privatlehrer in Berlin.

Steiner left Heidelberg in 1821 for Berlin, so that the conception of this theory took place in the early twenties. In 1824 he discovered inversion, which he described under the picturesque title of "Wiedergeburt und Auferstehung." In a fragment, of the same year, he refers to a paper on "Abspiegelung" (mapping) which has unfortunately been lost and in which he undoubtedly explained the principle of inversion.

That there were a number of other Steiner manuscripts in Bützberger's possession was new to me. He has written a rather extensive monograph on the most important mathematical results of these manuscripts, which is ready for publication; also a very careful and authoritative biography of Steiner's life and scientific accomplishments.

The most important of Steiner's posthumous works is, of course, that on the geometry of the circle and the sphere. Its early publication would be of great scientific and historic interest. It would establish the fact that Steiner is the originator and most important pioneer in this branch of geometry, not merely in an elementary sense, but even in its more advanced aspects. This is true because he did not limit himself to contact—and orthogonality—problems, but studied more general circular systems.

This, as is well known, leads in the end to the geometry of the 15-parameter group of homogeneous linear transformations of six variables leaving a certain quadratic form invariant.

Klein, whose attitude towards Steiner has never been very sympathetic, recognizes this, and he undoubtedly would have been even more emphatic in his acknowledgement of Steiner's merits had he known of the contents of the famous manuscript.

The first part deals with centers, lines, and planes of similitude of circles and spheres; the second with powers and locus of equal powers with reference to circles and spheres.

In the third part we find the theory of common powers between circles and spheres. The fourth part is by far the most extensive and also the most important and contains the theory of systems of circles and spheres intersecting at constant angles. In this theory Steiner has gone much farther than some of his successors, who merely studied the special cases of orthogonal and contact systems. Among the many important results which Steiner obtained I shall mention merely the theory of those cyclides which are obtained as the envelopes of spheres which cut three given spheres at a constant angle (0 in case of contact) and which are now known as Dupin cyclides. Steiner discovers many

properties of these surfaces and shows that the two systems of contact circles are lines of curvature.

When Fiedler received the Steiner prize from the Berlin Academy for his "Cyklographie," published in 1882 immediately after the publication of Steiner's collected works by Weierstrass, he was sure that the "lost" manuscript was not existing anymore. It is now apparent that many of Fiedler's important results were known to Steiner.

All these facts seem to make it clear that Steiner's posthumous works, especially that on the geometry of the circle and the sphere, should be made accessible to the mathematical public.

I have been fortunate in getting some members of the Swiss Mathematical Society interested in the possible publication of Bützberger's Steiner biography and the famous manuscript as a volume of its collection of mathematical monographs.

THE ALGEBRA OF FRANCESCO GHALIGAI

By SUZAN R. BENEDICT, Smith College

Questions as to when and how knowledge passes from one group of people to another are always important to historians. To scholars interested in the development of mathematical processes Francesco Ghaligai is known as one of the few early writers who assert that the algebra of Al-Khowarizmî was translated into Italian in the thirteenth century.[1] For these historians there is interest also in his attempt to symbolize the powers of an unknown quantity by rectilinear figures, nevertheless Ghaligai's *Summa De Arithmetica*, printed first in 1521, and afterward in 1548, and in 1552 with the title *Practica d'Arithmetica*, has been little noticed. Cantor mentions it however, as an excellent work,[2] and Libri[3] believed that it may have had a decided influence upon the study of mathematics—an opinion which he advanced because it is much simpler than such books as Pacioli's *Sūma*, and therefore could have been used more easily as an introduction to mathematical study. Unlike some of his contemporaries, Ghaligai does not preface his book with an autobiographical sketch, and we know only that he was a Florentine. Possibly because the *Sūma* was dedicated to Giulio de Medici, afterward Pope Clement VII, Libri thought that he may have been an ancestor of Leonora Ghaligai who went to France with Marie de Medici and became a prominent figure in the court of Louis XIII, but this unimportant suggestion seems to be all that has been written about him.

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[1] Folio 71r.

[2] Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. 2, p. 146.

[3] Libri, *Histoire de Sciences mathêmatiques en Italie*, vol. 3, p. 146.

Ghaligai makes no pretense of originality even in his use of the symbols generally connected with his name. He quotes freely from the writings of greater men, and having given credit to those whose work he appropriates, he condenses what he considers most important into a small treatise of 114 folios.

This treatise he divides into thirteen books, the first nine of which are wholly arithmetical. The last four are devoted to algebra, *Regula dell' Arcibra*, which as in other works of the time, includes long explanations of methods for the extraction of roots and for operations with binomial surds classified as they are found in Euclid's *Elements*. Usually his methods are those of Leonard of Pisa.

A theoretical treatment of the solution of equations appears on the last four of the forty-one pages comprising Book X. Here we find the equation with one variable in the six forms given by the Arab writer Al-Khowarizmî in his ninth century algebra, at least three of which were known also to Diophantus of Alexandria. It is interesting to note that even at so late a date as 1552, equations of the forms $ax^2+bx=c$ and $ax^2+c=bx$ were treated as different types, and that $12x=12x$ gave only one solution, namely $x=1$. In the latter case Ghaligai follows Pacioli, *secondo Maestro Luca*, and quotes *dice che* 12 *cose sieno equale a* 12 *cose & non* 12 *cose sieno equale a* 11 *cose, sempre parti el n*^{0} *delle cose pel numero delle cose, ne verrà sempre uno & tanto.*[1] The six types of equations are, in our symbols $ax^2=x$, $ax^2=b$, $ax=b$, $ax^2+bx=c$, $ax^2+c=bx$, $ax^2=bx+c$, where a, b, and c are always positive. Ghaligai recognizes only positive roots, and there is nothing unusual in his methods of solution; but he knows that the processes which he applies to the last three types will apply also to any equation of quadratic form. Among the illustrations[2] used are the equations $6x^6+(256)^{1/2}=50x^3$ and $2x^2=(8x^5+16x^4)^{1/3}$.

Book XI seems to give evidence of poor scholarship, for in his effort to find illustrative material, Ghaligai attempts to use the first eleven propositions of the second book of Euclid's *Elements*. The first ten of these are manifestly unsuitable, and lead to identities. In adapting Proposition XI, which lends itself to algebraic expression, Ghaligai says:[3] *To divide* 12 *into two parts so that the less multiplied by* 12 *shall equal the square of the greater* and obtains the correct results $(180)^{1/2}-6$ and $18-(180)^{1/2}$.

Book XII consists of twenty problems taken from the *Liber Abaci* of Leonard of Pisa[4] to whom acknowledgment is made. These problems are chosen with great care, as Ghaligai explains, and though varying slightly from the originals, are easily traced.

Book XIII is the last and most interesting in the text. Here we find forty-seven problems not accredited to any other author, though so similar to many

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[1] This appears to have been taken from the Sūma, where it is written *E. cosi habiamo che* 12 *co. sonno ēqli* 12 *co. Parti cose per cose: neuen.* 1. *per n*^{0}. *E tanto valse la cosa.* ed. 1523. f. 145r.

[2] f. 91r. art. 164, 165.

[3] Euclid's statement is: *To cut a given straight line so that the rectangle contained by the whole and one of the segments shall equal the square of the remaining segment.*

[4] *Scritti di Leonardo Pisano*, vol. 1, p. 410 et seq.

in the algebras of the period that we cannot claim originality for Ghaligai. They may be divided into three groups, advancing in each group from the simpler to the more difficult. A symbolism such as Ghaligai's does not encourage the use of more than one variable, and though many of the problems in this book might have been more easily solved by using simultaneous equations, he follows the custom of the century and adapts them to one.

The first sixteen problems are all of one kind. Typical of these, the fifteenth is as follows: *Two men found a purse. Said the first to the second, "Give me the purse and the cube root of your money, and I shall have as much as you." Said the second to the first, "Give me the purse and the square root of your money, and I shall have seven times as much as you." The question is, how much was in the purse, and how much had each man at first?* Though not so stated in the test, Ghaligai assumes a relation between the amounts held originally by the two men, i.e. that these were x^3 and x^2. He then shows that the purse held x^3-x^2-2x, and hence

$$x^3 + x + x^3 - x^2 - 2x = 7(x^2 - x).$$

The second group of problems includes 17–24. As typical of these, consider the twentieth: *Given three numbers, of which the second is double the first and the third is the product of the first by the square of the second. Knowing that the cube root of the sum of the product of the three numbers and the product of the third by twice the second is equal to twice the square of the first, find the numbers.*

The third group, including problems 25–47, deals with numbers in continued proportion. Some of these, without a better symbolism, might puzzle an abler mathematician, and we must admire the ingenuity with which Ghaligai reaches his results. Look, for example, at problem 47: *Find five numbers in continued proportion such that the sum of the second and fourth shall be* 10, *and the sum of the products of each number by each of the others shall be* 620. The solution is as follows:

First, 620 divided by twice the sum of the second and fourth will give the sum of the five, and since the sum of the second and fourth is 10, it follows that the sum of the first and fifth is 21 minus the third. Now if we have the sum and the product of two numbers, we can find the numbers.[1]
Then if $x=$ the third number,

$21-x=$ the sum of the first and fifth numbers,

$10=$ the sum of the second and fourth numbers,

$x^2=$ the product of the second and fourth numbers.

Hence $5-(25-x^2)^{1/2}=$ the second and $5+(25-x^2)^{1/2}=$ the fourth nnmber.
Since $x^2=$ the product of the second and fourth numbers and also the product of the first and fifth numbers,

---

[1] Ghaligai has already explained this process in Book III, article 67, as follows: *If* $x+y=a$, $x<y$ *and* $xy=b$, *let* z *be the absolute value of* $\frac{1}{2}a-x$ *or of* $y-\frac{1}{2}a$; *then* $(\frac{1}{2}a-z)(\frac{1}{2}a+z)=b$, *and* $z=(\frac{1}{4}a-b)^{1/2}$.

$$\frac{21}{2} - \frac{x}{2} - \left[\left(\frac{21-x}{2}\right)^2 - x^2\right]^{1/2} = \frac{21}{2} - \frac{x}{2} - \left[\frac{441}{4} - 10\frac{1}{2}x - \frac{3}{4}x^2\right]^{1/2}$$
$$= \frac{21}{2} - \frac{x}{2} - \left[\frac{441}{4} - \left(10\frac{1}{2}x + \frac{3}{4}x^2\right)\right]^{1/2}$$

is the first number and

$$\frac{21}{2} - \frac{x}{2} + \left[\frac{441}{4} - \left(10\frac{1}{2}x + \frac{3}{4}x^2\right)\right]^{1/2}$$

is the fifth number.

Now the sum of 620 and the sum of the squares of all the numbers is equal to the square of the sum of the five numbers; hence $541-(2x^2+42x)=961-620$ (since the sum of the numbers is 31), therefore $x=4$, and the numbers are 1, 2, 4, 8, 16.

The book itself, from the Giunti press, is printed with great care. The type is beautifully clear and the number of errors is remarkably small. The pages and the paragraphs are numbered, and frequent references are made, by number, to proofs already established.

It seems improbable that Ghaligai wrote primarily for merchants, as has been sometimes assumed. He seems rather to have been a teacher introducing mercantile problems for the sake of interest. The Pratica d'Arithmetica is in no sense one of the great books of the period, but if we may judge from the number of editions, it was a popular one. It is interesting from many points of view, and considered as a means by which the works of Fibonacci, Pacioli and many others were made accessible to students of the time, it is well worth some study.

RECENT PUBLICATIONS

EDITED BY PROFESSOR ROGER A. JOHNSON, Hunter College, New York, N. Y.

All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

The Theory of Determinants, Matrices and Invariants By H. W. Turnbull. Blackie & Sons, Limited, London, 1928. xvi+338 pages. 25 s. net.

This book aims to give an account of the salient features of invariant theory from its origins to the present. As an up-to-date treatment in English, it is alone in the field. Furthermore, it differs from other available works on the

subject in two respects. From the historical approach of the author, who traces many of the modern outgrowths of the theory to the roots in the earlier formal work of the English school, the reader gains a sense of the continuous development of the theory. On the pedagogical side, by the addition of numerous examples of varying difficulty, and the willingness to make lengthy digressions on the real significance of fundamental concepts when they are first introduced, the author has rendered an extensive and abstract theory relatively accessible to a beginner.

We proceed to outline, by chapters, the main topics discussed. The book begins with determinants and matrices (I–VII). The initial definitions of invariants and related ideas are illustrated for the binary form (VIII) and then extended to forms in general (IX, X). The "first fundamental theorem," establishing the possibility of representing every rational integral invariant (concomitant) of a set of ground forms as a polynomial in a limited number of typical expressions is treated for invariants in two restricted cases (XI), and then generalized (XII). A complete set of operations for these typical expressions is found, leading to the "second fundamental theorem" on the reduction of identities in concomitants (XIII). Seminvariants and their derivation from annihilators[1] are next briefly treated. The notions of complete and irreducible systems are introduced (XIV) and developed further in the next chapter (XV) where the Gordan-Hilbert theorem on the expressibility of the general rational integral concomitant in terms of a finite number is proved. The theorem on the limitation of the number of essential cogredient variables (XVI) is followed by the analogous one on the number of ground forms (XVII). The theorem on the connection between invariant equations (that is those expressing projective relations) and the technical invariant theory (XVIII) paves the way to applications to projective geometry (XIX, XX). The concluding chapter (XXI) on recent developments deals chiefly with the adjunction theory of Weitzenböck which extends the methods previously used for projective transformations to affine and orthogonal ones.

The author's remarks on the connection between the invariant theory under discussion and differential and relativistic geometry should prove helpful to the algebraist beginning to orient himself in relativity. A passing reference to the fundamental role of substitution groups and their representations with respect to algebraic theories in general, which follows these, is almost prophetic of the interest of this phase of algebra aroused among mathematical physicists by the recent work of Professor Weyl in the application of groups to quantum theory.

Throughout the book, the author has kept closely to his main theme. Thus the phases of matrix theory treated lean toward the symbolic side, and omit altogether some notions, such as elementary divisors, which are fundamental to the theory of matrices per se. This omission necessitates the author's later

[1] One might have expected a reference here to the work of Dickson connecting this theory with that of continuous groups.

confining himself to the "non-specialized" case in a few of the applications. He usually calls attention to the limitations of his discussion, and, as elsewhere where proofs or extensions are omitted, he gives adequate references to places where they may be found. The author has succeeded in giving in reasonable space, an outline of the theory of algebraic invariants, with the necessary introductory matter and a sufficient treatment of the applications to show the reader the importance of the subject. The book as a whole is clearly and carefully written, in a very readable style.

PHILIP FRANKLIN

Mathematisch-Naturwissenschaftlich-Technische Bücherei. Berlin, Otto Salle.

The booklets of this series present topics of mathematics, physics, the natural sciences, technology, and philosophy. The series started to appear in 1927, and is still being continued. At the moment, it comprises twenty-three volumes of 60 to 140 pages each.

The editors and most of the authors are men supervising and conducting instruction at such German schools as correspond to the American high schools and colleges. According to the editors, the booklets are intended primarily for students in what corresponds to the last years of high school and the first years of college. They offer an opportunity for acquiring a first insight into the historical development of the sciences, and becoming familiar with selected scientific topics of general interest which as a rule can be dealt with only superficially in the ordinary course of instruction.

The following gives a brief description of the series with stress on the booklets of mathematical content.

In a rough classification, the series is made up of volumes of four different types.

I. Volumes which are devoted each to the life and work of one great scientist. Archimedes (1), Galilei (4), Otto von Guericke (7), Euklid (8), Apollonius (13) —the numbers in brackets indicating the numbers of the volumes in the series. The first volume, for example, contains a discussion of Archimedes' most important works with excerpts from them in German translation. It is preceded by a short history of the development of mathematics up to Archimedes and biographical notes on his life. Among other details, it contains a review of the Greek number system. Volumes 4, 8, and 13 seem to be much on the same order. Volume 7, however, appears to lay more stress on the human and personal than on the scientific side.

II. Mathematische Quellenbücher: Rechnen und Algebra (3), Geometrie und Trigonometrie (11), Analytische und synthetische Geometrie (19). These three booklets contain excerpts from mathematical works of various ages and countries. The earliest is an Egyptian text of 1600 B.C., the latest a sample from Steiner's "Systematische Entwicklung . . . ." of 1832. Every excerpt is followed by the necessary mathematical and (in the case of old German texts) linguistic explanations. These volumes may be considered to some degree as an

introduction into the reading of original texts, for though all non-German texts are given in German translation, the original mathematical form is strictly preserved. They are appropriate to awaken in the reader some historic feeling for the development of certain mathematical disciplines and subjects, and for the development and importance of mathematical notations.

III. Volumes which are devoted each to one special scientific topic. This set of volumes forms the largest of the four. However, its subjects are on the whole chosen from sciences other than mathematics, the only mathematical volume being, *Einführung in die praktische Nomographie* (6). This booklet presents the essential principles of nomography in elementary form and on the basis of very elementary mathematical preparation. Numerous examples, mostly familiar from analytic geometry and physics, are worked out in great detail. As in many other volumes of this set, the stress is entirely on the practical side of the problem.

IV. This class contains three philosophical volumes (12), (17), (18), and two volumes, (20) and (21), entitled "Kulturgeschichte der Technik."

The getting-up of the entire series is pleasant and efficient; each booklet is equipped with numerous figures and a detailed index.

From the volumes I have had occasion to become acquainted with, it appears to me that the series is useful and stimulating and that additions to it are to be welcomed.

LULU HOFMANN

College Algebra. By N. J. Lennes. Harper and Brothers, 1928. xiv+301 pages.

Courses in college algebra are given for two reasons nowadays, the one administrative, the other scholastic. The wide diversity of subjects offered in high schools has led many colleges and universities to admit candidates who can count up the proper number of "units" in some fashion and then to require further work in certain subjects where preparation is short. Consequently there is a call for a course in algebra for students who have had only one year of it in high school. On the other hand, *calculus* may be named as the first objective of college mathematics; and to study that subject certain topics in algebra are needed even by students who have had more than one year in high school. The material in the gap between an elementary course and calculus may be organized into two parts: intermediate and college algebra.

The objectives of intermediate algebra are a thorough acquaintance with the quadratic equation, including "theory" as well as "solution," and logarithms as a tool for computation. These topics imply others, such as exponents in connection with logarithms; and with the necessary review of factoring and the solution of simultaneous systems, little time remains for further topics in a three hour course.

Besides a vigorous summary of intermediate algebra with drill in the formal manipulation of quadratic radical forms, the needs of calculus suggest specifi-

cally: the binomial theorem, partial fractions and enough of series and their convergence to evaluate e; and generally, as vivid a development of the concept of function as possible. This latter purpose may justify making the theory of equations a main topic. Since algebra is concerned with classes of numbers and the students are more mature, a discussion of the number system may be proper.

In the text before us the author proposes to take a middle road between a mere intermediate algebra and the stiff old books of a generation ago which were almost treatises, as indicated by the inclusion of Sturm's theorem. This book will be adequate for a satisfactory course of the first kind; teach the material as it comes and go as far as you can. It will be adequate for a course of the second kind as indicated in the preface; but as to how adequate it may be for use in engineering sections, for instance, there will be a difference of opinion. The reviewer finds that to use the book in the department where he teaches, supplementary material would necessarily have to be introduced.

Some features in the arrangement and treatment of material seem admirable; others undesirable. Separate chapters on "Literal equations" and on "Fractions and equations containing fractions" constitute a departure that should henceforth appear in every text, though we do not agree to the burying of the scant review of factoring in the latter chapter. Again, if a book is to appeal to a variety of people, we question the wisdom of taking various paragraphs from their normal positions and casting them into one chapter under the brand "miscellaneous."

It has been proposed as good pedagogy that the polynomial be treated as a simple kind of algebraic function whose values for various values of x are to be traced—say, by a graph—and that the theory of equations is concerned with finding those values of x for which the value of the polynomial is zero. In this book the word *function* does not occur in the chapter on the theory of equations, while for any treatment of the graphs of polynomials, beyond the scant half-page under Horner's method, one must go to the two last paragraphs of "Miscellaneous Topics." Isn't this treatment disregarding a good opportunity to unify and to prepare for calculus?

Experienced teachers have always used the principle of cumulative review. With this text there will be no need to prepare and to dictate material since the assignments are all made out. The collection of exercises in algebraic reductions which actually occur in analytic geometry and calculus furnishes a needed standard by which the capable students can be encouraged to measure themselves.

The story of algebra in the last chapter is actually interesting. With such a treatment at hand we imagine that many a teacher will make room for some recitations on it.

R. M. MATHEWS

The members of the Coast and Geodetic Survey of today are proud to point to Hassler as the founder of their Bureau and to acknowledge their debt of gratitude to Switzerland for giving us a man who was bold enough to insist on doing practical things in a scientific way.

OSCAR S. ADAMS

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3376. *Proposed by Harold Hotelling, Stanford University.*

A square array of n^2 cells is filled in by two players, each in turn putting any number he pleases into any empty cell. One player wins if when the cells are all filled the determinant is zero; the other tries to prevent this. Discuss the possible cases.

3377. *Proposed by T. S. Peterson, The Ohio State University.*

Evaluate the summation

$$\sum_{i=0}^{n} \frac{(2i+1)!(2n-2i+1)!}{i!(i+1)!(n-i)!(n-i+2)!}.$$

3378. *Proposed by Paul Wernicke, Washington, D. C.*

If a tetrahedron $A=A_1A_2A_3A_4$ has altitudes concurrent (in an orthocenter), then, denoting by (ij) the angle at A_j of the triangular face a_i (opposite A_i), the following products are equal: $\tan(41)\cdot\tan(32)\cdot\tan(23)\cdot\tan(14)$ $=\tan(31)\cdot\tan(42)\cdot\tan 13\cdot\tan(24)=\tan(21)\cdot\tan(12)\cdot\tan(43)\cdot\tan(34)$. The factors of each product are tangents of the four plane angles made by the edges which form a tetragram.

3379. *Proposed by J. H. Neelley and T. L. Smith, Carnegie Institute of Technology.*

Two men own jointly x cows which they sell for x dollars per head and with the returns buy sheep at \$12 per head. As their income from the cows is not

divisible by 12 they purchase a lamb with the remainder. Later they divided the flock so that each had the same number of animals. How much money was due the man with the lamb by the other man?

3380. *Proposed by Paul Wernicke, Washington, D. C.*

Consecutive rectangular triangles with the sides 3, 4, 5; 5, 12, 13, etc., each having for its sides the Pythagorean numbers h, $(h^2-1)/2$, $(h^2+1)/2$, where h is the hypotenuse of the preceding triangle, are laid off as follows: 3 as OA from the origin of rectangular Cartesian coordinates on the positive x-axis, 4 as AB, a positive ordinate, 5 as the hypotenuse OB; the next triangle with OB as one side and so as to contain the preceding one OAB within its area, then the third to contain the second (and also the first) within its area, etc.

Find the limiting directions of the hypotenuses.

3381. *Proposed by E. B. Escott, Oak Park, Ill.*

The following construction is taken from a book on mechanical drawing: *Problem.* To bisect the angle between two given lines whose point of intersection is beyond the limits of the drawing.

Construction. Let the given lines be CD and AB. With an arbitrary point E on CD as center and any radius strike an arc cutting CD in I and AB in G. With G as center and the same radius strike an arc cutting AB in L and CD in E. Draw the common chord of the two circles, HF. With I and L as centers and the same radius strike arcs cutting in J and K. Draw JK. The intersection of HF and JK is O, a point on the bisector.

To find a second point, with O as center, strike an arc cutting AB and CD in two points M and N. With M and N as centers and the same radius strike arcs cutting in P. Draw OP, the required bisector.

Give the proof of the construction.

SOLUTIONS

3085 [1924, 305]. *Proposed by E. T. Bell, University of Washington.*

Is there any simple formula, in terms only of n and its divisors, for the sum $\sum(-1)^{(x+y)/2}xy$ extended to all odd integers satisfying $x^2+y^2=n$?

Solution by C. F. Gummer, Queen's University.

Let S be the required sum, on the understanding that only positive x and y are used, and that when x and y are unequal both permutations of them are counted as separate solutions. Let s be the sum when such permutations are not reckoned separately. Let S' and s' be the corresponding sums when both positive and negative x and y are admitted as furnishing distinct solutions.

The equation $x^2+y^2=n$ has no solutions in odd numbers (and therefore $S=s=S'=s'=0$) unless n is of the form $2K^2P$, where K is either 1 or the product of one or more primes of the form $4m-1$, and P is either 1 or the product of primes $4m+1$. Let us suppose that n is of this form. Two cases may then be

distinguished. If P is not a perfect square, there is no solution in which x and y are equal. In this case $s=S/2$, and $s'=2S$. If P is a square, there is one solution with equal values, and on account of this $s=(S/2)-(K^2P)/2$, and $s'=2S-K^2P$. In either case, $S'=4S$. Moreover the theory of S is the same for both cases.

To every representation of n in the form x^2+y^2 corresponds the representation of $n/2$, or K^2P, in the form U^2+V^2, where U and V are $(x+y)/2$ and $(x-y)/2$. For the calculation of S, $(x+y)/2$ is positive, but $(x-y)/2$ may be positive, negative, or zero; and, if it is not zero, the change of its sign effects a permutation of x and y. Let U stand for the one of these two numbers which is even, so that V is odd. Conversely every such (U, V) leads to a suitable (x, y) appearing in S.

It is now seen that $(-1)^{(x+y)/2}xy=U^2-V^2$, so that $S=\sum(U^2-V^2)$.

Let $P=p_1^{r_1}p_2^{r_2}\cdots$, where $p_1, p_2, \cdots$, are distinct primes (of the form $4m+1$). In accordance with known theory, let $p_\alpha=u_\alpha^2+v_\alpha^2$, where u_α is even, and v_α is odd. We obtain a suitable (U, V) apart from an ambiguity of sign which is immaterial to our purpose, from the complex relation

$$\pm\ (U+iV) = K\ \prod_\alpha(u_\alpha+iv_\alpha)^{r_\alpha-h_\alpha}(u_\alpha-iv_\alpha)^{h_\alpha},$$

in which h_1 ranges from 0 to r_1, h_2 independently from 0 to r_2, etc. Moreover all the (U, V) required for S are given once each by this equation.

Now U^2-V^2 is the real part of the square of $U+iV$. Hence S is the real part of the sum $\sum(U+iV)^2$. But the above equation shows that this sum is real. Therefore

$$S = K^2\sum_{h_1h_2\cdots}\left[\prod_\alpha(u_\alpha+iv_\alpha)^{r_\alpha-h_\alpha}(u_\alpha-iv)^{h_\alpha}\right]^2 = K^2m_1m_2\cdots,$$

where

$$m_\alpha = \sum_{h_\alpha=0}^{r_\alpha}(u_\alpha+iv_\alpha)^{2r_\alpha-2h_\alpha}(u_\alpha-iv_\alpha)^{2h_\alpha} = \left[(u_\alpha+iv_\alpha)^{2r_\alpha+2}-(u_\alpha-iv_\alpha)^{2r_\alpha+2}\right]/(4iu_\alpha v_\alpha)$$

$$= \binom{r_\alpha+1}{1}(u_\alpha^2-v_\alpha^2)^{r_\alpha} - \binom{r_\alpha+1}{3}(u_\alpha^2-v_\alpha^2)^{r_\alpha-2}(2u_\alpha v_\alpha)^2$$

$$+\binom{r_\alpha+1}{5}(u_\alpha^2-v_\alpha^2)^{r_\alpha-4}(2u_\alpha v_\alpha)^4 - \cdots.$$

This discussion shows how S depends on the u's and v's belonging to the prime factors of form $4m+1$. It is clear that if we were able to express S directly in terms of the prime factors of n, we should then be able to express the u's and v's in terms of their corresponding primes, which is too much to expect. Incidentally we see that if S_1 and S_2 are the values found for $n=2K_1^2P_1$ and $n=2K_2^2P_2$, and if P_1 is prime to P_2, then S_1S_2 is the value of S for

$$n = 2(K_1K_2)^2P_1P_2.$$

3239 [1927, 98]. *Proposed by Alex S. Wiener, Cornell University.*

Solve the following simultaneous equations for u, v, w, x, y, and z:

$$(u - a_1)(x - b_1) - (v - b_1)(w - a_1) = 0,$$
$$(u - a_2)(z - b_2) - (v - b_2)(y - a_2) = 0,$$
$$(w - a_3)(z - b_3) - (x - b_3)(y - a_3) = 0,$$
$$u^2 + v^2 = w^2 + x^2 = y^2 + z^2 = r^2.$$

Solution by Otto Dunkel, Washington University

The unknowns when paired as (u, v), (w, x), (y, z) are the coordinates of points C_3, C_2, C_1 on the circle of radius r with the origin at the center, and such that the straight lines joining these points in the order written pass through the given points $A_1(a_1, b_1)$, $A_2(a_2, b_2)$, $A_3(a_3, b_3)$. It will be assumed that no one of these last three points lies on the circle. Certain trivial solutions are obvious. Thus the straight line A_1A_2 cuts the circle, in general, in two points, one of which may be taken as C_3 while the second may be taken as both C_1 and C_2. In order to eliminate more easily such a solution, rotate the axes until the new x-axis is perpendicular to A_1A_2; and suppose that the given points have now the coordinates (c, d_1), (c, d_2), (c_3, d_3), whereas C_1, C_2, C_3 have the coördinates (α_1, β_1), (α_2, β_2), (α_3, β_3).

The straight line C_3A_2 cuts the circle in the second point C_1 with coördinates $\alpha_1 = M_1/L_1$, $\beta_1 = N_1/L_1$, where

$$M_1 = 2c(r^2 - d_1\beta_3) - \alpha_3(c^2 - d_1^2 + r^2),$$
$$N_1 = 2d_1(r^2 - c\alpha_3) - \beta_3(d_1^2 - c^2 + r^2),$$
$$L_1 = c^2 + d_1^2 + r^2 - 2(\alpha_3 c + \beta_3 d_1).$$

By replacing the subscript (1) by (2) we shall have α_2 and β_2. The necessary and sufficient condition that C_1, C_2 and A_3 shall lie on a straight line is

$$\begin{vmatrix} M_1 & N_1 & L_1 \\ M_2 & N_2 & L_2 \\ c_3 & d_3 & 1 \end{vmatrix} = 0.$$

The determinant on the left may be carried through several reductions, using $\alpha_3^2 + \beta_3^2 = r^2$, so that $(d_2 - d_1)$ and $2(c - \alpha_3)$ may be removed as factors. Neither factor is zero if we discard trivial solutions. After collecting the terms we obtain an equation of the form

$$P_3\alpha_3 + Q_3\beta_3 + rR_3 = 0,$$

where

$$P_3 = c_3(c^2 + r^2 - d_1d_2) + cd_3(d_1 + d_2) - 2cr^2,$$
$$Q_3 = c_3c(d_1 + d_2) + d_3(d_1d_2 + r^2 - c^2) - r^2(d_1 + d_2),$$
$$R_3 = r[c^2 + r^2 + d_1d_2 - 2cc_3 - d_3(d_1 + d_2)].$$

By solving the pair of equations consisting of the linear equation above and $\alpha_3^2+\beta_3^2=r^2$, we obtain two solutions for the pair (α_3, β_3). Each solution determines uniquely the corresponding sets (α_1, β_1), (α_2, β_2) by aid of the equations above. A rotation transformation may then be made to restore the original variables and given constants. This requires merely the exercise of a little patience in computation. Excluding the trivial solutions there is no real solution if $P_3^2+Q_3^2$ is less then R_3^2; one and only one solution if they are equal; and two distinct solutions in the remaining case.

3315 [1928, 207]. *Proposed by Harry Langman, Arverne, L. I.*

Show that

$$x^2+y^2+z^2-yz-zx-xy+u^2+v^2+w^2-vw-wu-uv \geqq 3^{1/2}\begin{vmatrix} u & x & 1 \\ v & y & 1 \\ w & z & 1 \end{vmatrix}$$

for all real values of the variables. Also find the conditions under which the equality obtains.

II. *Solution by D. R. Curtiss, Northwestern University.*

The substitution

$$y = x + r_1 \sin\theta_1, \qquad v = u + r_1\cos\theta_1,$$
$$z = x + r_2 \sin\theta_2, \qquad w = u + r_2\cos\theta_2,$$

reduces the above inequality to

$$r_1^2 + r_2^2 - r_1r_2\cos(\theta_2-\theta_1) \geqq 3^{1/2}r_1r_2\sin(\theta_2-\theta_1),$$

which may be written

$$r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2-\theta_1-60^\circ) \geqq 0.$$

This is equivalent to the statement that the square of the distance between the points whose polar coordinates are (r_1, θ_1+60°) and (r_2,θ_2) is not negative. Other inequalities in which 60° is replaced by other angles are at once suggested. Equality obtains when and only when the two points coincide. In rectangular coordinates, the two points are

$$[(\tfrac{1}{2}(v-u)-\tfrac{1}{2}\sqrt{3}(y-x),\ \tfrac{1}{2}\sqrt{3}(v-u)+\tfrac{1}{2}(y-x)] \quad \text{and} \quad (w-u,\ z-x).$$

One easily verifies directly that the formula for the square of their distance apart is the left side of the inequality in the statement of the problem when all terms have been placed on that side. By equating the abscissas of the two points, and the ordinates, we express the conditions for equality in terms of the original variables.

3319 [1928, 261]. *Proposed by J. Rosenbaum, Milford, Connecticut.*

Given the sides of a quadrilateral inscribed in a circle, to construct the quadrilateral.

Solution by L. S. Johnston, University of Detroit.

It is assumed that the quadrilateral is to be convex. The construction for the non-convex quadrilateral is easily deducible from the first case.

Let AB, BC, CD, and DA be the sides. If two opposite sides are equal the quadrilateral becomes an isosceles trapezoid and the problem is trivial. It will therefore be assumed that no two opposite sides are equal, and, for definiteness, that $AB > CD$. The construction does not depend at all on the fact that AC and BD are different, but less elaborate methods of construction are available when two opposite sides are equal.

Divide AD by the point R into two segments $AR = a$, and $DR = d$, and divide BC by the point S into segments $BS = b$ and $CS = c$, such that $a/d = b/c = AB/CD$. Construct x from any one of the relations

$$x = \frac{a(d+c)}{a-d} = \frac{b(d+c)}{b-c} = \frac{c(a+b)}{b-c} = \frac{d(a+b)}{a-d}.$$

Construct the triangle CDO with $CD = CD$ given, $CO = x - c$, $DO = x - d$. Extend OC through C to B making $BC = BC$ given; extend OD through D to A, making $AD = AD$ given. Join A and B. $ABCD$ is the required quadrilateral.

Proof: Inscribe a quadrilateral $ABCD$ in a circle, with $AB > CD$. Then BC and AD are not parallel, and intersect at O, with DC between O and AB. Let P be the intersection of the diagonals. Now the triangles APB and DPC are similar, and $AP/PD = BP/PC = AB/CD$. Let the bisector of the angle APD intersect AD at R and BC at S. Then $AR/DR = AP/PD = BS/CS$, or $a/d = b/c$ where $AR = a$, $DR = d$, $BS = b$, $CS = c$. RS makes equal angles with AD and BC and the triangle RSO is ososceles. Triangles CDO and ABO are similar: whence $OB/OD = OA/OC = AB/CD = a/d = b/c$. If we set $OS = OR = x$, then

$$(x+b)/(x-d) = (x+a)/(x-c) = a/d = b/c,$$

from which are deducible the four relations already given between x and given quantities. All the steps of the above analysis are reversible, and the construction is proved valid.

If the problem sets the order of the sides, there is only one convex solution. If order be not specified, there are three convex solutions.

For the non-convex quadrilateral, inspection of a figure shows that two of the sides intersect inside the circle and are divided in the same ratio by their intersection point, and that the ratio of division is the ratio of the two sides which do not intersect inside the circle. The construction of a fundamental triangle is obvious, and, therefore, the construction of the quadrilateral is immediate.

Also solved by H. W. Brown, R. A. Johnson, I. N. Kagoro, E. G. Olds, W. J. Patterson, A. Pelletier, and Paul Wernicke.

3320 [1928, 261]. *Proposed by N. A. Court, University of Oklahoma.*

Through a given point to draw a line which shall form with the sides of a given angle a triangle of given perimeter.

Solution by D. Lawrence Barrick, Montezuma College, Montezuma, N. Mex.

Let D be the given point, EAF the given angle, and $2p$ the given perimeter. On AE take AZ equal to p and on AF take AY equal to p. Construct a circle tangent to AE and AF at Z and Y respectively. Through D construct a line tangent to the minor arc ZY meeting AE and AF at B and C respectively. This tangent is the required line.

Proof: The perimeter of ABC equals $2AZ$ equals $2p$.

Discussion: There is one and only one solution if D is outside of the given angle or its vertical angle, or if D falls on AZ, AY, or minor arc ZY not including points A, Z, or Y. There are two solutions if D falls within the area bounded by AZ, AY, and minor arc ZY. Otherwise there is no solution. It is assumed here that the sides of the triangle are not to lie upon the prolongations of EA and FA.

Also solved by Rufus Crane, R. A. Johnson, L. S. Johnston, W. J. Patterson, A. Pelletier, and Paul Wernicke.

3321 [1928, 261]. *Proposed by Roger A. Johnson, Hunter College of the City of New York.*

Through A', B', C' the mid-points of the sides of the triangle ABC lines are drawn perpendicular to the bisectors of the opposite angles. Show that these lines are sides of a triangle whose circumscribed circle equals that of the given triangle.

Solution by Nathan Altshiller-Court, University of Oklahoma.

The triangles ABC, $A'B'C'$ are homothetic; hence, the bisectors of corresponding angles are parallel. Thus the perpendicular QR from A' to the bisector of the angle A is also perpendicular to the bisector of the angle $B'A'C'$, that is, QR is the external bisector of the angle $B'A'C'$.

The vertices P, Q, R of the triangle formed by the three external bisectors of the triangle $A'B'C'$ are the excenters of $A'B'C'$, and $A'B'C'$ is the orthic triangle of PQR. The circumcircle of $A'B'C'$ is thus the nine-point circle of PQR, hence the circumradius of PQR is equal to the circumdiameter of $A'B'C'$. But the circumradius of ABC is also equal to the circumdiameter of $A'B'C'$, hence the proposition.

Also solved by Rufus Crane, Paul Wernicke, and the Proposer.

3322 [1928, 261]. *Proposed by Harry Langman, Brooklyn, N. Y.*

1. Let $ABCD$ be a quadrilateral inscribed in a circle such that the adjacent sides BC and CD are equal. Then $AC^2=AB\cdot AD+BC^2$.

2. Using this result, show that the ratio of the smallest diagonal of a regular heptagon to the side is given by a root of $y^3-y^2-2y+1=0$.

Solution by P. W. Stoner, Pasadena Junior College.

(1). $ABCD$ is an inscribed quadrilateral with $BC=CD$ and CX perpendicular to AB and CY perpendicular to AD.

Then $BX=DY$, $CX=CY$, the angle $BAC=$the angle CAD, and $AB=AC\cos BAC+BX$, $AD=AC\cos CAD-DY$ (if AD is less than AB).

Multiplying,

$$AB\cdot AD = AC^2\cos^2 BAC - BX^2 = AC^2 - AC^2\sin^2 BAC - BX^2$$
$$= AC^2 - CX^2 - BX^2 = AC^2 - BC^2.$$

Therefore $AC^2=AB\cdot AD+BC^2$.

(2). Let A, B, C, D, E be successive vertices of a regular inscribed heptagon. Then

$$\text{(a)}\qquad AC^2 = AB\cdot AD + BC^2 \quad\text{and}\quad (b)\qquad AD^2 = AC\cdot AD + BC^2,$$

for $AD=AE$ and $BC=CD$.

Solve (a) for AD and insert its value in (b). Then set $AC/BC=y$, after replacing AB by its equal BC; remove the factor y, and there results

$$y^3 - y^2 - 2y + 1 = 0\,.$$

Also solved by H. W. Brown, B. P. Hoover, R. A. Johnson, L. S. Johnston, J. H. Neelley, E. G. Olds, A. Pelletier, W. T. Short, Margaret M. Young, Paul Wernicke, and the Proposer.

Note by the Proposer.

Let A_1, A_2, A_3, $\cdots$, A_k be the consecutive vertices of a regular polygon with sides of unit length. Denote by y_n the length of A_1A_n. Then from (1) $y_n=(y_{n-1}^2-1)/y_{n-2}$, where $y_2=1$. By setting in turn $n=4$, 5, 6, 7 and replacing y_3 by y, there results, after reduction of the fractional forms on the right,

$$y_4 = y^2 - 1\,,\qquad y_5 = y^3 - 2y,\qquad y_6 = y^4 - 3y^2 + 1\,,\qquad y_7 = y^5 - 4y^3 + 3y.$$

Since $y_5=y_4$, we have at once the desired result. Also $y_7=1$ and the last equation gives the same result but with the factor (y^2+y-1).

The same process can be applied to all regular polygons. The coefficients of the powers of y in the expression for y_n follow the law obtained by taking Pascal's Triangle diagonally.

Note by Otto Dunkel.

The above equations by the proposer may be obtained more easily by a slightly different theorem which also yields a proof of the law of the coefficients.

Theorem: Let $ABCD$ be an inscribed quadrilateral such that the adjacent sides BC and CD are equal. Then $BC(AD+AB)=AC\cdot BD$.

From C draw the perpendiculars CX and CY to AB and AD, respectively. Then $CX=CY$, and $AY=AX=(AB+AD)/2$. Hence from the two congruent right triangles ACY and ACX we can form an isosceles triangle with base equal to $AB+AD$ and side AC which is similar to BCD. Hence $AC/(AB+AD)=BC/BD$, and the theorem is proved.

With the notation above take $AB=y_{n-2}$, $AC=y_{n-1}$, $AD=y_n$, $BD=y$, $BC=CD=1$, $y_3=y$, $y_2=1$, and we have $y_n=yy_{n-1}-y_{n-2}$. If we set $n=4$, 5, we obtain the first two equations. The following equations are then easily written down in turn by the general law, and it now follows by a simple proof that

$$y_n = \sum(-1)^i{}_{n-2-i}C_i y^{n-2-2i},$$

where the limits are $i=0$ and $i=[(n-2)/2]$, that is the greatest integer in $(n-2)/2$.

The other two roots of the equation in part (2) are $1/y_4$ and $-y_4/y$.

3323 [1928, 261]. *Proposed by C. N. Schmall, New York City.*

In elementary geometry it is shown that two circles will touch, cut each other, or have no point in common, according as the sum of their radii is equal to, greater than, or less than the distance between their centers. Prove this by analytical geometry.

Solution by B. P. Hoover, Carnegie Institute of Technology.

Let a and b be the radii of circles with centers at A and B, respectively, and let AB equal c. We consider each of the quantities a, b, and c as positive. Let A be the origin and the line from A through B the positive direction of the x-axis. Solving the equations of the circles, $x^2+y^2=a^2$ and $(x-c)^2+y^2=b^2$, we get $x=(a^2+c^2-b^2)/(2c)$, which is real, and, for y^2, we get

$$(1)\quad y^2 = a^2 - \left(\frac{a^2+c^2-b^2}{2c}\right)^2 = \frac{(a+b-c)(b+c-a)(c+a-b)(a+b+c)}{4c^2}.$$

The necessary and sufficient conditions for tangency, two real intersections, and no real point in common are respectively $y^2=0$, $y^2>0$, and, $y^2<0$.

I. The condition $y^2=0$ yields the three possibilities, $a+b-c=0$, $b+c-a=0$, $c+a-b=0$. These are mutually exclusive since a, b, and c are different from zero, and correspond, respectively, to external tangency, internal tangency with circle B inside A, internal tangency with A inside B.

II. For $y^2>0$, we must have all four of the factors of y^2 positive; since $a+b+c$ is positive, if two of the other factors, say, $b+c-a$ and $c+a-b$, are negative, we are led to the contradiction, $c<0$. Likewise the other two possible combinations of signs of the factors of y^2 are impossible since a and b are positive. Hence, we must have at the same time $a+b>c$, $b+c>a$, and $a+c>b$.

III. For $y^2<0$, there are four combinations of signs of the factors of y^2 to consider: 1) $++-+$; 2) $+-++$; 3) $-+++$; 4) $---+$. These signs are written in the same order as the factors of y^2 in (1) above. 1) is possible and corresponds to the case of circle A inside B, since $a+c>b$. 2) is possible and corresponds to the case of B inside A, since $b+c<a$. 3) is possible and corresponds to the case of circles A and B having no part of their areas in common since $a+b<c$. 4) is impossible since $a+b+c$ is the sum of the other three factors.

Also solved by R. P. Agnew, R. A. Johnson, F. A. Lewis, F. H. Miller, J. H. Neelley, E. G. Olds, A. Pelletier, Paul Wernicke, and F. L. Wren.

3324 [1928, 261]. *Proposed by H. Campbell, St. Johnsbury, Vt.*

Given the difference of the segments of the base made by the perpendicular to the base from the vertical angle, the difference of the base angles, and the difference of the sides including the vertical angle, to construct the triangle.

Solution by F. L. Wren, George Peabody College for Teachers.

Let s = difference of the two sides including the vertical angle; b = difference of the projections of these two sides on the base; and θ = difference of the two base angles, $0<\theta<80°$. Construct the triangle ABC with $AB=s$, $AC=b$, $\angle ABC=90°+\frac{1}{2}\theta$. Draw the perpendicular bisector of BC cutting AB produced in D, and then draw an arc of the circle with center D and radius DB cutting AC in C and E. Draw DE; then ADE is the required triangle. For $AD-DE=AD-DB=s$. If M is the foot of the altitude from D, $AM-ME=AM-CM=AC=b$. Also $\angle ADC=180°-2\angle DBC=\theta$. There are two cases I. $\angle ACB\geqq\frac{1}{2}\theta$, II. $\angle ACB<\frac{1}{2}\theta$. In either case $\angle ACD=\angle ACB+\angle BCD=90°-\frac{1}{2}\theta+\angle ACB$. In the first case $\angle ACD\geqq 90°$, and E is on AC produced or at C. In this latter case it is obvious that $\angle AED-\angle EAD=\theta$. In the other case of I, $\angle AED-\angle EAD=\angle ECD-\angle EAD=\theta$, and M falls within the base AE. In case II, $\angle ACD<90°$, and M falls on AE produced. Here b is the sum of the absolute values of the projections, since the direction of ME is the reverse of that in the second part of I. We now have $\angle AED$ = external angle at C of triangle $ACD=\angle EAD+\theta$; and hence $\angle AED-\angle EAD=\theta$. The construction is impossible if $b\leqq s$.

Also solved by H. W. Brown, William Hoover, L. S. Johnston, F. A. Lewis, A. Pelletier, W. R. Warne, and Margaret M. Young.

3375 [1929, 233] *Proposed by A. S. Levens, University of Minnesota.*

Show how to find the angle between a line and a plane graphically, without first reducing the problem to that of finding the angle between two lines.

Solution by the Proposer

The usual method given in textbooks of Descriptive Geometry for the solution of this problem is to drop a normal from some point of the given line to the given plane and then to find the angle between that normal and the given

line. The angle thus found is the complement of the required angle. That method, altho simple, is indirect.

A direct method for obtaining the view of the true angle by projecting the given figure upon a plane which is perpendicular to the given plane and parallel to the given line is described below. This method obviates the introduction of the normal and the projection of the given line upon the given plane or the addition of any element to the given figure.

Construction: (1) Project the line and plane upon an auxiliary plane which is perpendicular to the given plane and a reference plane. (2) Then project the line and plane upon a second auxiliary plane which is perpendicular to the first auxiliary plane and parallel to the given plane. (3) Project the line and plane upon a third auxiliary plane which is perpendicular to the second auxiliary plane and parallel to the given line. In this last view the true angle between the line and plane will be shown.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

THE THIRTEENTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The thirteenth summer meeting of the Association will be held at the University of Colorado, Boulder, Colorado, on Monday afternoon and Tuesday morning, August 26–27. A program of interest to college and university teachers of mathematics is being prepared. It will include the retiring presidential address of Professor W. B. Ford. Immediately following this meeting and lasting for the remainder of the week, there will be a colloquium and summer meeting of the American Mathematical Society. The colloquium will consist of a series of five lectures, beginning Tuesday afternoon, by Professor R. L. Moore of the University of Texas on "Point-set theory." On Thursday afternoon, Professor Virgil Snyder will give his address as the retiring president of the Society on "The problem of cubic variety in four-way space."

This meeting will offer an unusual opportunity to combine mathematical activities with a pleasant outing. Boulder is a beautiful city with a population of about 15,000, delightfully located at the entrance to the Rocky Mountains. Since it is not a summer resort, prices are reasonable. A fifteen minutes' walk from the campus takes a person into wooded canyons and foothills; an auto ride of less than one hour, into the high mountains. Boulder is one and one-half hours' bus ride from Denver, which has excellent railroad connections with all parts of the country.

Wednesday will probably be devoted to an all-day excursion into the elevated and scenic Estes Park and Rocky Mountain Park region. Other short trips and typical western outings are being planned. The committee on arrange-

ments is making thorough preparations for entertaining visitors, and will be glad to assist any who may wish to spend a part or all of their summer in the beautiful Rocky Mountain region. The University of Colorado maintains a recreation office during the summer session; this is to be continued during the meetings, and the director, Professor C. A. Hutchinson, places the resources of his bureau at the services of members and their families. He will be glad to give any information and advice concerning board and lodging to members who may consider bringing their families earlier than the meeting to enjoy a prolonged stay in Colorado, or who may wish to stay after the meeting. Addresses and prices of rooms or furnished houses in the city, or of cabins in the mountains, may also be secured from Professor Hutchinson. Lodging for the period of the meetings will be provided in the fraternity and sorority houses at a price not to exceed $1.00 per night per person, except that a limited number of single rooms will be available at $1.25 per night per person.

The full program of the Association meeting will be sent to the members in July with a post card for making reservations.

The original manuscript of Professor Albert Einstein's latest published research, "Zur Einheitlichen Feld-Theorie," is now in this country in the possession of Wesleyan University, Middletown, Connecticut, where it will be permanently kept in the Olin Library and exhibited to those interested to see it. The manuscript consists of eight pages of closely knit lines, all in Einstein's handwriting, together with mathematical calculations and interlineations. Certain portions have been crossed out and do not, therefore, appear in the published paper.

The first seven pages of the manuscript contain the scientific treatise, the results of six years of Einstein's deepest thought, and the mathematical statement of a scientific theory which it has been said that not more than twelve men understand at the present time. It is said that Professor Einstein considers this theory of more scientific significance than his famous theory of relativity and that he believes it will be years before the world of science will be able to grasp fully all the details and implications of his theory and check up on his calculations. The manuscript is autographed at the end of page seven. The eighth page contains expressions of thanks to Professor Einstein's coworkers.

The story of how Wesleyan University came into the ownership of the manuscript, as told by President James L. McConaughy, has many elements of human interest. The joint donors were George W. Davison, President of the Central Union Trust Company of New York, and Albert W. Johnston, financier, of 111 Broadway, New York. Mr. Davison and Mr. Johnston were college mates at Wesleyan and have long been active on its Board of Trustees, the former being now President of the Board and the latter Chairman of its Committee on Buildings and Grounds.

Immediately after the publication of the new theory which Professor Einstein had promulgated, Mr. Davison instructed his company's representative

in Berlin to enter into negotiations with Professor Einstein to discover if the manuscript could be acquired. These negotiations were carried on with Mrs. Einstein, who surrounds Professor Einstein's life with the greatest protection. Professor Einstein has been in poor health for a long time and his friends know that it has made him very happy to be able to complete the development and statement of his new theory in spite of the condition of his health and the strain of the arduous mental toil which the work has involved. Professor Einstein is an ardent Zionist and the manuscript of his relativity theory was sent to the Zionist University in Jerusalem. The manuscripts of Professor Einstein's works between the publication of his relativity theory and the publication of "Zur Einheitlichen Feld-Theorie" were purchased by Baron Rothschild of London who presented them to the Einstein Institute in Berlin.

When Mr. Davison's negotiations for the new manuscript were opened, no other approach had been made to Professor Einstein with a view to its purchase, and through Mrs. Einstein ready assent was given by Professor Einstein to the sale of the manuscript to Mr. Davison, whose representative explained that it would be permanently entrusted to the custody of an American university. The only interest which Professor Einstein had in the financial aspects of the transaction was that its sale should realize sufficient money to enable him and his wife to carry on the welfare work among university students in which both of them have long been very much interested.

The National Council of Teachers of Mathematics held its annual meeting at Cleveland, Ohio on February 22–23, 1929, the general theme being "The place of mathematics in education." The program included the following papers:

1. "The permanent nature of the necessity for mathematics in the secondary school," by Professor L. C. Karpinski.
2. "Current trends in mathematics teaching," by Doctor Vera Sanford.
3. "Mathematics and the future", by Professor C. N. Moore.
4. "For the good of the Council," by Professor W. D. Reeve.
5. "Mathematics plus," by Miss Florence Brooks Miller.
6. "The cultural value of mathematics," by Professor W. W. Rankin.
7. "The place of mathematics in junior high school education," by Agnes Grant Rowlands.
8. "Teaching geometry into its rightful place," by Professor J. O. Hassler.
9. "Informational mathematics versus computational mathematics," by Profsssor C. H. Judd.

The reports of the officers indicated a growth of the Council from about 4350 to over 5000 during the year. 230 were registered as in attendance at the meetings. A very cordial interest was indicated between the National Council and the Mathematical Association. This was evidenced in two ways, first, by the approval by the Directors of the Council of the affiliation between the two bodies whereby each is to contribute to the welfare of the other organization and whereby members of the Council may become members of the Association

without the payment of the initiation fee, and secondly, by the appointment of members of the Council to act with a committee of the Association in mapping out a one-year course in plane and solid geometry as a possible alternative to the year of plane geometry which is offered for college entrance. The joint committee thus formed will make a report to the two bodies after the necessary amount of consideration of this very important educational question. The committee for the Mathematical Association is composed of Professor Dunham Jackson, chairman, and Professors Ralph Beatley, J. O. Hassler, C. N. Moore, and W. D. Reeve.

The Chauvenet Prize

President J. W. Young has appointed the following committee on the Chauvenet Prize: Professor A. J. Kempner, chairman, and Professors D. R. Curtiss and W. A. Hurwitz. By reason of the gift made last year by ex-President Walter B. Ford,[1] the Association is able to award the prize at the end of this year, and this will doubtless be done at the time of the annual meeting at Des Moines, Iowa in December, 1929. This award will cover the four years 1925 to 1928 inclusive.

The U. S. Coast and Geodetic Survey is establishing mailing lists so that circulars giving information regarding publications issued by the bureau may be sent to those interested. Requests should be directed to the Director of the Survey, Washington, D. C. The following subjects are covered: Astronomic work, Base lines, Coast pilots, Currents, Geodesy, Gravity, Hydrography, Leveling, Nautical charts, Oceanography, Precise traverse, Seismology, Terrestrial magnetism, Tides, Topography, Triangulation, and Cartography.

Professor Earle Raymond Hedrick, of the University of California at Los Angeles, delivered the "Fifth Annual Faculty Research Lecture" at that University on April 26, 1929. His subject was, "Logical reasoning in mathematics and in science."

Professor H. E. Slaught, of the University of Chicago, delivered a lecture on "How mankind learned to count" at Lehigh University on March 11, 1929, at Lafayette College on March 12, at Haverford College on March 13, at Swarthmore College on March 14, and at Rutgers University on March 15. These lectures were usually sponsored by mathematics clubs in the respective institutions and the attendance varied from 100 to 400. The lectures were given for the purpose of stimulating interest in mathematics. Professor Tomlinson Fort of Lehigh University was responsible for the plan and he worked out the itinerary. Professor Slaught also delivered this lecture again on April 15 at the Open Court of Chicago before a group of people who come together occasionally to discuss philosophy, religion, and mathematics.

At the invitation of Sigma Xi and the departments of mathematics and physics, Professor J. H. VanVleck, of the University of Wisconsin, delivered two lectures at Iowa State College on March 7. The subjects were "Modern physics and molecular structure" and "The new quantum theory."

[1] See this Monthly, vol. 35 (1928), p. 457.

Professor G. H. Hardy lectured at the University of Iowa on March 23, 1929 on "Hilbert's logic."

The University of Oxford has conferred the degree of Doctor of Science on Professor Oswald Veblen, of Princeton University.

The American Mathematical Society has awarded its Bôcher Memorial Prize to Professor J. W. Alexander, of Princeton University, for his memoir entitled "Combinatorial analysis situs," in volume 28 of the Transactions of the American Mathematical Society.

Professor A. H. Compton, of the University of Chicago, has been awarded the gold medal of the Radiological Society of America for his work in X-rays.

The Alice Freeman Fellowship of Wellesley College has been awarded to Deborah May Hickey, of Houston, Texas. The fellowship, amounting to $1600, will enable her to continue her study of mathematics in Germany.

Associate Professor Elizabeth B. Cowley, of Vassar College, who is at present on leave of absence teaching in the Pittsburgh public schools, has been promoted to a professorship of mathematics at Vassar College.

Dr. Carl A. Garabedian, associate professor of mathematics at the University of Cincinnati, has been appointed associate professor of mathematics and organist at St. Stephen's College of Columbia University, Annandale-on-Hudson, N. Y.

At Hunter College the following promotions and appointments have been made: Dr. Lester S. Hill and Miss Evelyn Walker have been promoted from the rank of assistant professor to that of associate professor; and Dr. Louis Weisner from an instructorship to an assistant professorship. Dr. Martin Nordgaard has been appointed Lecturer and Mr. Aubrey Landers, Jr. has been appointed instructor.

Professor W. V. Lovitt, of Colorado College, has been appointed Dean of Men. He will continue as professor of mathematics.

Mr. Charles R. Scherer has been made head of the department of mathematics in Christian University.

The following additional courses in mathematics are announced for the summer of 1929:

University of California, Berkeley, intersession, May 20–June 29. By Professor T. M. Putnam: Theory of algebraic equations and of infinite series; Geometric introduction to the theory of functions. By Professor D. N. Lehmer; Metric differential geometry. Summer session, July 1–August 10. By Professor B. A. Bernstein: Elementary algebra for advanced students. By Professor E. R. Hedrick: Analytic geometry of space; Functions of a complex variable. By Professor James Pierpont: Non-Euclidean geometry. By Professor Hermann Weyl: Representation of groups; Applications and representations of groups to quantum physics.

University of California at Los Angeles, July 1–August 10. By Professor Harriet E. Glazier: Foundations of arithmetic. By Professor M. W. Haskell: Advanced geometry; Geometric introduction to the theory of functions. By Professor Sophia H. Levy: The teaching of mathematics.

Massachusetts Institute of Technology: First period, June 11 to July 23. Courses in calculus and differential equations, covering the prescribed work of the first two years. Second period, July 24 to September 4: Courses given in first period repeated. August 12 to September 14. Courses in algebra, solid geometry and trigonometry, in preparation for fall entrance examinations in those subjects. July 1 to July 30. Courses in methods of teaching mathematics in the Junior High School and the Senior High School. June 10 to July 12. Courses in advanced calculus and theoretical aeronautics. July 15 to August 9. Course in theoretical aeronautics continued. July 5 to August 5. Differential equations, intended primarily for Army officers.

University of Pittsburgh. In addition to the usual undergraduate courses the following advanced courses will be offered. By Professor K. D. Swartzel: Functions of a complex variable; Teaching of mathematics. By Professor F. A. Foraker: Modern synthetic geometry; solid analytic geometry. By Associate Professor Taylor: Advanced calculus; functions of a real variable. By Assistant Professor Culver: Theory of equations.

Syracuse University. In addition to the usual courses in mathematics through the calculus the following courses are offered. By Professor F. F. Decker: The teaching of algebra and geometry in secondary schools; Introduction to the theory of numbers or introduction to modern algebra. By Professor A. D. Campbell: Differential geometry or theory of functions of a complex variable.

University of Texas, first term, June 4 to July 15. By Professor R. L. Moore: Functions of real variables; Foundations of geometry. By Professor E. L. Dodd: Probability; Analytic functions. By Professor H. J. Ettlinger: Differential equations; Ruler and compass constructions. By Professor H. S. Vandiver: Number theory; Advanced calculus. By Professor A. E. Cooper: Advanced calculus; Theory of equations. By Professor P. M. Batchelder: Teaching problems in mathematics. By Professor Mary Decherd: Calculus and freshman courses, both terms. Second term, July 15 to August 26. By Professor R. L. Moore: Functions of real variables (continued); Non-Euclidean geometry. By Professor H. J. Ettlinger: Partial differential equations; Definite integrals. By Professor E. G. Keller: Advanced calculus. By Professor R. G. Lubben: Calculus.

University of Vermont. By Professor Millington: Courses in algebra, plane trigonometry, and solid geometry. By Professors Bullard and Butterfield: Courses in analytical geometry, differential calculus, integral calculus, and differential equations.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Thirteenth Summer Meeting of the Association, Boulder, Colorado, August 26-27, 1929.
Fourteenth Annual Meeting, Des Moines, Iowa, December 31, 1929, January 1, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled.

ILLINOIS, Carthage, Ill., May 3-4.
INDIANA, Culver Military Academy, May 3-4.
IOWA, Fairfield, Iowa, April 26-27.
KANSAS, Topeka, Kansas, February 2.
KENTUCKY, Lexington, Ky., April 13.
LOUISIANA-MISSISSIPPI, Lafayette, La., April 12-13.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, May 4.
MICHIGAN, Ann Arbor, Mich., March 16.
MINNESOTA, St. Paul, Minn., May 11.
MISSOURI, Kansas City, Mo., November 16.
NEBRASKA.
OHIO, Columbus, Ohio, April 4.
PHILADELPHIA, University of Pennsylvania, November 30.
ROCKY MOUNTAIN, Greeley, Colo., April 12-13.
SOUTHEASTERN, Macon, Ga., April 19-20.
SOUTHERN CALIFORNIA, University of Redlands, March 9.
TEXAS, Houston, Texas, Jan. 26.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teacher's agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to department or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

THE THIRD ANNUAL MEETING OF THE PHILADELPHIA SECTION

The third annual meeting of the Philadelphia Section of the Mathematical Association was held in Bennett Hall at the University of Pennsylvania on Saturday, December 1, 1928. There were 75 present including the following members of the Association: V. W. Adkisson, J. W. Alexander, A. A. Bennett, P. A. Caris, J. W. Clawson, E. S. Crawley, J. E. Davis, D. A. Flanders, O. Frink, T. Fort, M. Goldberg, G. A. Harter, F. H. Jackson, E. H. Johnson, J. R. Kline, M. S. Knebelman, P. A. Knedler, H. M. Lufkin, D. L. McDonough, H. H. Mitchell, J. A. Miller, R. Morris, F. W. Owens, H. B. Owens, E. A. Partridge, C. J. Rees, G. A. Rosengarten, L. L. Smail, W. M. Smith, M. B. Snyder, J. E. Stocker, J. M. Thompson, F. M. Weida, A. H. Wilson, R. R. Wood.

Professor F. W. Owens, the chairman of the Section, presided. A special luncheon was held at the end of the morning session for the mathematicians and their friends. At the business meeting the following officers were elected for the ensuing year: Chairman, Professor A. H. Wilson, Haverford College; Secretary, Professor P. A. Caris, University of Pennsylvania; Program Committee, Professor Tomlinson Fort, Lehigh University and Professor J. R. Kline, University of Pennsylvania.

The following program was presented:

1. "Errors in computation," by Professor F. M. Weida, Lehigh University.
2. "The geometry of the triangle," by Professor A. A. Bennett, Brown University.
3. "An algebraic method of differentiating," by Professor Orrin Frink, Pennsylvania State College.
4. "A mechanical theory of the solar corona," by Professor J. A. Miller, Swarthmore College.

5. "Knots," by Professor J. W. Alexander, Princeton University.

Abstracts of these papers follow:

1. The theory of errors is a branch of mathematics which belongs to practical analysis in applied mathematics. The scope of practical analysis, according to Felix Klein, covers the executive element in mathematics. In mathematical analysis, we are interested in obtaining the solution of a particular problem. To completely solve a problem, it is important (1) to know about the existence of a solution, (2) to be able to obtain the solution and ultimately express the desired results in numerical form, and (3) to examine the degree of reliability of the arithmetical results obtained as well as the accuracy of the given data.

In many problems we deal with the results of certain measurements. Measurements are only approximate. Hence there should be something to indicate the degree of accuracy of the measurement. It is well to write all measurements in such a form that they have one figure only before the decimal point and that they are multiplied by the appropriate power of ten. To illustrate: we write the number 357 as 3.57×10^2.

A computed result can be no more accurate than the least accurate of the given data. Among the errors that arise in computation we have the error in a product, the error in a quotient, error relative to an estimate, the error in a sum, the error in an average, the relation between the accuracy in measure of lengths and the measure of angles, probable error, etc.

Interpolation is also a source of error. We here assume that $f(x)$ possesses, in a certain interval, a continuous differential coefficient of a certain order k. Such a function may be written as the sum of a polynomial and a "remainder term." The remainder term may be found if two numbers are known between which the differential coefficients of order k are located. The function $f(x)$ is a real single-valued function, continuous in the interval $a<x<b$, possessing in this interval a continuous differential coefficient of the highest order necessary in deriving each formula under consideration.

A valuable asset for accuracy in computation is to be proficient in the methods available for the finding of the desired numerical results. To illustrate: it has been found that the real root of $x^6+2x^5+6x^4+17x^3+20x^2+2x-36=0$ may be found correct to six decimal places in two steps by using a method that resembles Horner's method. Also, it has been found that the complex roots of an equation of the form $a_nx^n+a_{n-1}x^{n-1}+ \ldots a_1x+a_0=0$ may be found in a fairly easy fashion by a method analogous to Horner's method for finding the real roots.

In all practical analysis, the number of significant figures and the unit used should indicate the accuracy of the given data. In the computation, we should demand the greatest accuracy possible in the obtained results which implies that we should never demand a higher degree of accuracy in the results than called for in the least accurate of the given data.

2. Professor Bennett reviewed briefly the various types of geometrical investigation traditionally used in the study of the geometry of triangles: elementary synthetic (Euclidean) methods, Cartesian analytic geometry, synthetic projective methods, analytical methods using general homogeneous co-ordinates, areal and normal trilinear co-ordinates, inversive geometry. In what might seem to be the most natural geometry of point triads, the vertices of the triangle are more fundamental than the sides, as is seen in considering inversive properties. The domain of rational points may be so taken as to include each point symmetrically determined by the triad of vertices but such as not to include in general the individual vertices themselves. The theory of binary forms affords two distinct methods of interpretation, both of them direct, analytic, and economical in contrast to the older methods but giving rise initially to different associated points. The first and most significant points obtained are the pair of isodynamic points and the symmedian point which play more important roles than the centroid, circumcentre, incentre, etc. in the extended group of transformations considered. The fundamental character of the circumfeuerbach point is touched upon.

3. Professor Frink's paper gives a method of obtaining the formulas of the differential calculus without the use of the limiting process. The method is similar to those of the early writers on the calculus, but differs from them in being rigorous. It is based on the theory of analytic functions of a hypercomplex variable. A modification enables one to obtain the first and higher derivatives simultaneously. It is shown that the method justifies considering the differential as an absolute infinitesimal.

4. Let us suppose that from a given point on the solar surface particles are ejected at successive intervals; that these particles are matter subject to the attraction of the sun and to a radiation pressure inversely proportional to the square of the distance of the particle from the sun's centre. Each particle describes a conic section. If now we assume that a given streamer in the solar corona is the projection on a plane perpendicular to the line of sight, of the locus of all these particles, then we can compute and plot the shape of these streamers as seen from the earth. The initial-velocity of the particle at the sun's surface is the resultant of the velocity of the particles due to the sun's rotation and the velocity of ejection. This theory has been applied to a great many solar coronas that have been photographed in the past twenty years with long focus telescopes. It has been found possible to determine constants in the equations that will reproduce any streamer that has ever been photographed. Most streamers have such a shape that they could not have been produced by ejection alone. The theory also has indicated that the ends of the streamers which are as much as three and one half radii of the sun from its centre should turn back there rather abruptly, and streamers of these types have been found on two or three of the coronas photographed. Of course this does not prove that the coronas are produced in this way but it does offer one explanation of their peculiarities.

5. Professor Alexander discussed the problem of finding sufficient invariants to determine completely the knot type of an arbitrary simple closed curve in space of three dimensions. He outlined the derivation of one of these invariants which takes the form of a polynomial $\Delta(x)$ with integral coefficients, where both the degree of the polynomial and the coefficients are functions of the curve with which it is associated. He pointed out that the problem was not completely solved and suggested several problems related to it.

J. R. KLINE, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The ninth regular meeting of the Southern California Section of the Mathematical Association was held at the University of Redlands, Redlands, California, on Saturday, March 9, 1929. Professor E. T. Bell presided. The attendance was forty-four, including the following thirty-one members of the Association: O. W. Albert, E. E. Allen, L. D. Ames, M. A. Basoco, Harry Bateman, Clifford Bell, E. T. Bell, Grace E. Berry, P. H. Daus, J. D. Elder, Iva B. Ernsberger, Raymond Garver, Harriet E. Glazier, W. L. Hart, E. R. Hedrick, G. H. Hunt, Glenn James, Mary M. Keith, G. R. Livingston, W. E. Mason, W. B. Orange, Lena E. Reynolds, W. P. Russell, H. M. Showman, D. V. Steed, H. C. VanBuskirk, Morgan Ward, L. E. Wear, Mabel G. Whiting, W. M. Whyburn, and Clyde Wolfe.

The meeting began with a luncheon at Grossmont Hall, and the Association was welcomed to Redlands by President Duke. The following officers were elected for the scholastic year 1929–1930: Chairman, E. E. Allen, Occidental College; Vice-Chairman, G. E. F. Sherwood, University of California at Los Angeles; Program Committee, Morgan Ward, California Institute of Technology and L. D. Ames, University of Southern California. The next meeting was tentatively scheduled for November 9, 1929, at the California Institute of Technology.

The following program was presented:

1. "Education in England," by Professor M. M. Keith, University of Redlands.
2. "Geometric fallacies due to incautious use of one-to-one assignment," by Professor E. R. Hedrick, University of California at Los Angeles.
3. "Three similar erroneous proofs of the fundamental theorem of algebra," by Professor Glenn James, University of California at Los Angeles.
4. "The classification of binary relations," by Dr. Morgan Ward, California Institute of Technology.
5. "Problems in road construction," by Miss Ethel Neily, University of Southern California (by invitation).
6. "On certain class number recurrences of the Kronecker type," by Mr. M. A. Basoco, California Institute of Technology.

Abstracts of these papers follow:

2. Using the well known one-to-one assignment between points on a line and points of a plane, Professor Hedrick constructed geometric examples which illustrated the fallacies which arise when the assignment is not continuous. The examples illustrated the inability to assign the order of infinity of the set of geometric elements so constructed.

3. Attempts to prove that every algebraic equation has at least one root have been for the most part of two sorts. In the field of the complex number theory the first and many later correct proofs have been developed. The earlier and less successful attempts have been to make the proof depend upon the solutions of equations of odd degrees. Three supposed proofs of this sort, by Clifford, Malet, and Walechi are shown to be fallacious. Up to the point where these proofs go wrong they parallel Gauss's proof although they appear quite different.

4. Binary relations are classified according as they are always, sometimes or never reflexive, symmetric, or transitive. Of the 27 cases apparently admissible under this classification, only 14 are actually possible, and these 14 cases must be further subdivided on the basis of the behavior of the contradictory relation to make the classification exhaustive. Simple examples of all the possible types of relations are given, and a few applications to the classification of various arithmetics indicated.

5. Several problems in road construction involving reverse curves were considered and solved by analytic geometry.

6. By making use of certain results due to G. Humbert (Journal des Mathématiques, 1907) and of the Fourier series expansions of certain quotients of the theta products recently obtained by the writer, several class number recurrences are derived. These are of the type given by Kronecker and may be found in Dickson's *History of the Theory of Numbers*, vol. 3, pp. 106–108.

P. H. Daus, *Secretary*

THE FIFTEENTH ANNUAL MEETING OF THE KANSAS SECTION

The fifteenth annual meeting of the Kansas Section of the Mathematical Association was held in the High School building, Topeka, February 2, 1929, Professor Homer S. Myers, Southwestern College, Winfield, chairman of the Section, presiding.

There were fifty-eight in attendance, among them the following thirty-one members of the Association: C. H. Ashton, Wealthy Babcock, Florence Black, A. S. Croom, L. T. Dougherty, E. F. Farner, W. H. Garrett, W. A. Harshbarger, J. O. Hassler, W. H. Hill, Emma Hyde, W. C. Janes, C. F. Lewis, O. B. Loewen, W. H. Lyons, Anna Marm, U. G. Mitchell, T. A. Mossman, H. S. Myers, O. J. Peterson, A. W. Phillips, T. I. Porter, Ethel Rumney, W. L. Remick, D. H. Richert, J. A. G. Shirk, Guy W. Smith, E. B. Stouffer, W. T. Stratton, J. J. Wheeler, A. E. White.

The following officers were elected: Chairman, Professor C. H. Ashton, Vice-Chairman, Professor Emma Hyde, Secretary-Treasurer, L. T. Dougherty.

The morning session was a joint meeting with the Kansas Association of Mathematics Teachers, Miss M. Bird Weimar, of Wichita, presiding. At this session, Professor J. O. Hassler, of Oklahoma University, spoke on the value of mathematical history to the teacher and to the pupil, and Professor E. B. Stouffer, of the University of Kansas, gave a report of the Mathematical Congress at Bologna in the Summer of 1928. At the joint luncheon of the two Associations, which followed the morning session, Professor U. G. Mitchell, of the University of Kansas, gave a very interesting talk and demonstration of "Mathematics and Poetry." In the afternoon, the Kansas Section met in separate session, the program consisting of two papers:

1. "The Gamma-function," by Professor Ashton.
2. "Some properties of Euler's phi-function," by Professor Richert.

Abstracts of these papers follow:

1. Just two hundred years ago, Euler introduced a new function, which has been the subject of many papers and a few entire volumes. Nearly a hundred years after its introduction by Euler, Legendre named it the Gamma-function. Comparatively little has been written about this function in this country, either in our books or in our journals. In this expository paper, it is defined by an integral, by infinite products, and by its difference equation, and some of its properties are discussed.

2. If m is any given positive integer, the number of integers not greater than m and prime to it, is called Euler's phi-function of m, (or indicator of m), and is denoted by $\phi(m)$. It is well known that this function is of frequent occurence in the theory of numbers. This paper deals with the fundamental properties of the phi-function.

LUCY T. DOUGHERTY, *Secretary*

CRYPTOGRAPHY IN AN ALGEBRAIC ALPHABET

By LESTER S. HILL, Hunter College

1. *The Bi-Operational Alphabet*

Let $a_0, a_1, \ldots, a_{25}$ denote any permutation of the letters of the English alphabet; and let us associate the letter a_i with the integer i. We define operations of modular addition and multiplication (modulo 26) over the alphabet as follows: $a_i+a_j=a_r$, $a_ia_j=a_t$, where r is the remainder obtained upon dividing the integer $i+j$ by the integer 26 and t is the remainder obtained on dividing ij by 26. The integers i and j may be the same or different.

It is easy to verify the following salient propositions concerning the bi-operational alphabet thus set up:

(1) If α, β, γ are any letters of the alphabet, $\alpha+\beta=\beta+\alpha$, $\alpha\beta=\beta\alpha$, $\alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma$, $\alpha(\beta\gamma)=(\alpha\beta)\gamma$, $\alpha(\beta+\gamma)=\alpha\beta+\alpha\gamma$.

(2) There is exactly one "zero" letter, namely a_0, characterized by the fact that the equation $\alpha+a_0=\alpha$ is satisfied whatever be the letter denoted by α. It should be observed that, by our definition of multiplication, if α denotes any letter of the alphabet, we have: $\alpha a_0=a_0\alpha=a_0$.

(3) Given any letter α, we can find exactly one letter β, dependent upon α, such that $\alpha+\beta=a_0$. We call β the "negative" of α, and write: $\beta=-\alpha$. Evidently, if $\beta=-\alpha$, then also $\alpha=-\beta$.

(4) Given any letters α, β we can find exactly one letter γ such that $\alpha+\gamma=\beta$. We write: $\gamma=\beta-\alpha$. It is obvious that $\beta-\alpha=\beta+(-\alpha)$; and also that if $\beta-\alpha=a_0$, then $\beta=\alpha$.

(5) Distinguishing the twelve letters, a_1, a_3, a_5, a_7, a_9, a_{11}, a_{15}, a_{17}, a_{19}, a_{21}, a_{23}, a_{25}, with subscripts prime to 26, as "primary" letters, we make this assertion, easily proved: If α is any primary letter and β is any letter, there is exactly one letter γ for which $\alpha\gamma=\beta$. We write: $\gamma=\beta/\alpha$. Each primary letter α has the "reciprocal" a_1/α, where a_1 is the "unit" letter; and the reciprocal is likewise primary. If α is primary, we shall call the "fraction" β/α "admissible." A table of the letters represented by the twelve particular admissible fractions a_1/α enables us, when used with the formula $\beta/\alpha=\beta(a_1/\alpha)$, to find immediately the letter represented by any admissible fraction.

(6) In any algebraic sum of terms, we may clearly omit terms of which the letter a_0 is a factor; and we need not write the letter a_1 explicitly as a factor in any product.

For the limited purposes of the present paper it will not be necessary to define exponential notations, etc.

2. *An Illustration*

Let the letters of the alphabet be associated with integers as follows:

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*m*
5	23	2	20	10	15	8	4	18	25	0	16	13

n	*o*	*p*	*q*	*r*	*s*	*t*	*u*	*v*	*w*	*x*	*y*	*z*
7	3	1	19	6	12	24	21	17	14	22	11	9

or, in another convenient formulation:

0	1	2	3	4	5	6	7	8	9	10	11	12
k	*p*	*c*	*o*	*h*	*a*	*r*	*n*	*g*	*z*	*e*	*y*	*s*

13	14	15	16	17	18	19	20	21	22	23	24	25
m	*w*	*f*	*l*	*v*	*i*	*q*	*d*	*u*	*x*	*b*	*t*	*j*

It will be seen that

$$c+x=t, \quad j+w=m, \quad f+y=k, \quad -f=y, \quad -y=f, \quad \text{etc.}$$

$$an=z, \quad hm=k, \quad cr=s, \quad \text{etc.}$$

The zero letter is k, and the unit letter is p. The primary letters are: $a\ b\ f\ j\ n\ o\ p\ q\ u\ v\ y\ z$.

Since this particular alphabet will be used several times, in the illustration of further developments, we append the following table of negatives and reciprocals:

Letter :	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
Negative :	u	o	t	r	l	y	i	x	g	p	k	e	m	q	b	j	n	d	w	c	a	z	s	h	f	v
Reciprocal :	u	v				n				j				f	z	p	y				a	b			q	o

The solution of the equation $z+\alpha=t$ is $\alpha=t-z$, or $\alpha=t+(-z)=t+v=f$.

The system of two linear equations: $o\alpha+u\beta=x$, $n\alpha+i\beta=q$ has the solution $\alpha=u$, $\beta=o$, which may be obtained by the familiar method of elimination or by formula (see Section 4).

3. *Concerning Determinants in the Bi-Operational Alphabet*

The determinant

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

where the a_{ij} denote letters of the bi-operational alphabet defined in Section 1, has the same definition, and the same properties, as the corresponding expression in ordinary algebra—except that additions and multiplications are, of course, effected in the modular sense.

We note explicitly these properties:

I. Let δ denote the value of the n-th order determinant D; let M_{ij} denote the value of the determinant of order $n-1$ obtained from D by striking out the row and column in which the element a_{ij} lies; and let $A_{ij}=\pm M_{ij}$, the positive or negative sign being used according as the integer $i+j$ is even or odd. Then each of the sums

$$\sum_{t=1}^{n} a_{it}A_{jt}, \qquad \sum_{t=1}^{n} a_{ti}A_{tj}$$

has the value δ if $i=j$, and the value a_0 if $i\neq j$.

II. The value of D is not changed: (1) if rows and columns are interchanged; or (2) if to each element of any row (column) is added α times the corresponding element of another row (column), where α denotes any letter of the alphabet.

III. The value of D is changed only in sign (1) if two rows (columns) are interchanged; or (2) if the signs of all elements in any row (column) are changed.

IV. The value of D is not changed if the elements of any row (column) are

multiplied by any primary letter β and the elements of another row (column) by the reciprocal, a_1/β, of β.

We shall call D a "primary determinant" if its value is a primary letter. We shall not have to deal, in this paper, with determinants that are not primary.

LEMMA: *By means of properties II and III,* we may obviously convert the determinant of n-th order;

$$
{}_nI_\alpha = \begin{vmatrix} a_1 & a_0 & a_0 \cdots a_0 \\ a_0 & a_1 & a_0 \cdots a_0 \\ \cdot & \cdot & \cdots\cdot \\ \cdot & \cdot & \cdots\cdot \\ a_0 & a_0 & a_0 \cdots \alpha \end{vmatrix} = \alpha ,
$$

into a variety of n-th order determinants, all of which have the value α, where α is any assigned letter of the alphabet.[1] In ${}_nI_\alpha$, all elements, except those of the principal diagonal, are equal to the zero letter a_0; and all elements of that diagonal, except the last, are equal to the unit letter a_1.

We have only to make α a primary letter if we wish to set up with great ease a wide variety of n-th order primary determinants.

4. *Normal Transformations and Polygraphic Cipher Systems*

The determinant D, of n-th order, which was written out in Section 3, fixes the linear transformation with coefficients a_{ij}:

$$
(T) \qquad \begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n , \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n , \\ &\cdot \quad \cdot \quad \cdots \quad \cdot \\ &\cdot \quad \cdot \quad \cdots \quad \cdot \\ &\cdot \quad \cdot \quad \cdots \quad \cdot \\ y_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n . \end{aligned}
$$

We call D the determinant of the transformation T; and we say that T is a "*normal*" transformation if its determinant is primary.

THEOREM: *A normal transformation T has an unique inverse T^{-1}, of which the equations are*:

$$x_i = D_i/D\,(i = 1,2, \cdots , n),$$

where D denotes the value of the determinant of T, and D_i denotes the value of the determinant obtained therefrom by replacing a_{ji} by $y_j(j=1, 2, \ldots , n)$. Moreover, T is the inverse of T^{-1}. The values of the determinants of T and T^{-1} are reciprocals (see Section 1), and therefore T^{-1} is a normal transformation.

[1] By means of II, III, IV, any assigned determinant which is of the nth order and whose value is any primary letter α can be obtained from ${}_nI_\alpha$.

Given any pair of inverse normal transformations T and T^{-1} in our bi-operational alphabet, we have a device which may be applied (1) to convert any message sequence of n letters into a corresponding cipher sequence of n letters, and (2) to convert the cipher sequence back into the message sequence from which it came. In other words, we have all the apparatus of an extraordinarily effective *polygraphic* (*n-graphic*) *cipher system.* We may regard $x_1x_2 \ldots x_n$ as the message sequence, and determine the cipher sequence $y_1y_2 \cdots y_n$ by means of T, using T^{-1} for decipherment; or we may encipher with T^{-1} and decipher with T, treating $y_1y_2 \cdots y_n$ as the message sequence and $x_1x_2 \cdots x_n$ as the cipher sequence. In either case, we begin by writing the message in sequences of n letters, as will be illustrated in Section 5.

A polygraphic cipher consisting of the inverse normal transformation of the literal sequences x_i, $y_i (i=1, 2, \ldots, n)$ may suitably be called a *linear cipher of order n*, and designated as a C_n.

5. *Illustration of Linear Ciphers*

Let us employ the particular bi-operational alphabet considered in Section 2.

EXAMPLE 1: To construct and apply a cipher of type C_3.

Selecting any primary letter, say y, we can immediately obtain from

$$_3I_y = \begin{vmatrix} p & k & k \\ k & p & k \\ k & k & y \end{vmatrix} = y$$

a host of different primary determinants all of which have the value y, as pointed out in the LEMMA in Section 3. One of these is the determinant

$$\begin{vmatrix} y & k & o \\ z & x & k \\ r & n & y \end{vmatrix}$$

of the normal transformation,

$$(T_1) \qquad \begin{aligned} y_1 &= yx_1 \qquad\quad + ox_3, \\ y_2 &= zx_1 + xx_2, \\ y_3 &= rx_1 + nx_2 + yx_3, \end{aligned}$$

of which the inverse,

$$(T_1^{-1}) \qquad \begin{aligned} x_1 &= xy_1 + zy_2 + dy_3, \\ x_2 &= vy_1 + ny_2 + qy_3, \\ x_3 &= fy_1 + qy_2 + xy_3, \end{aligned}$$

is easily found. It will be observed that the values of the determinants of T_1 and T_1^{-1} are y and q respectively, and that these letters are reciprocals.

Let the message to be enciphered consist of the word *Mississippi.* Writing

this message in 3-letter sequences, and filling the last sequence with any prearranged letter, say k, we have:

$$m\ i\ s \quad s\ i\ s \quad s\ i\ p \quad p\ i\ k.$$

Substituting $m\ i\ s$ for $x_1x_2x_3$ in T_1, we find $b\ q\ t$ for $y_1y_2y_3$, thus converting the message sequence $m\ i\ s$ into the cipher sequence $b\ q\ t$. Proceeding in like manner with the other message sequences, we obtain as the enciphered form of our message: $b\ q\ t \quad s\ e\ i \quad a\ e\ p \quad y\ f\ c$. We should probably send it in the customary five-letter grouping: *bqtse iaepy fc*.

To decipher, we substitute $b\ q\ t$ for $y_1y_2y_3$ in T_1^{-1} obtaining $m\ i\ s$ for $x_1x_2x_3$. Proceeding in the same way with the other cipher sequences, we regain the entire original message.

EXAMPLE 2: To construct and apply a cipher of type C_4.

Choosing any primary letter, say j, we may construct from

$${}_4I_j = \begin{vmatrix} p & k & k & k \\ k & p & k & k \\ k & k & p & k \\ k & k & k & j \end{vmatrix} = j$$

an enormous number of different primary determinants all of which have the value j. One of these is the determinant of the normal transformation:

$$(T_2) \qquad \begin{aligned} y_1 &= gx_1 + rx_2 + zx_3 + ax_4, \\ y_2 &= rx_1 + zx_2 + ax_3 + ex_4, \\ y_3 &= ax_1 + gx_2 + hx_3 + zx_4, \\ y_4 &= ex_1 + rx_2 + yx_3 + hx_4, \end{aligned}$$

the inverse of which is easily found to be:

$$(T_2^{-1}) \qquad \begin{aligned} x_1 &= by_1 + dy_2 + ay_3 + py_4, \\ x_2 &= cy_1 + yy_2 + iy_3 + py_4, \\ x_3 &= cy_1 + dy_2 + ry_3 + jy_4, \\ x_4 &= jy_1 + cy_2 + xy_3 + jy_4. \end{aligned}$$

We note that the determinants of T_2 and T_2^{-1} have the reciprocal values j and j, the letter j being its own reciprocal.

Let the message to be enciphered be *Delay operations*. Write it in the form:

$$d\ e\ l\ a \quad y\ o\ p\ e \quad r\ a\ t\ i \quad o\ n\ s\ u,$$

filling the last sequence with any prearranged letter, say u. Substituting $d\ e\ l\ a$ for $x_1x_2x_3x_4$ in T_2, we find $j\ c\ o\ w$ as the corresponding cipher sequence $y_1y_2y_3y_4$. Proceeding in this manner, we find the enciphered form of our message to be:

$$j\ c\ o\ w \quad z\ l\ v\ b \quad d\ v\ l\ e \quad q\ m\ x\ c.$$

To decipher, we substitute $j\ c\ o\ w$ for $y_1y_2y_3y_4$ in T_2^{-1}, etc.

6. *Concluding Remarks*

A great many other cryptographic constructions can, of course, be derived from the algebra, by no means fully developed in this paper, of the bi-operational alphabet. The purpose of the paper, however, will have been accomplished if the single construction described serves to emphasize sufficiently the circumstance that sets which fail to possess in full the character of algebraic fields may still admit a large measure of amusing, and possibly useful, algebraic manipulation. It need hardly be said that if full-fledged finite algebraic fields are employed, the opportunities of the cryptographer are greatly extended; he then has at his disposal a perfectly smooth algebra and its associated geometries. The writer hopes to submit a further communication on this subject. But the number of marks in a finite field is necessarily either a prime or a power of a prime. If our alphabet is to be converted into a finite field, the best that can be done is to omit one letter, say j, to obtain a field of twenty-five marks; or to adjoin an additional symbol so that a field of twenty-seven marks is available. The bi-operational alphabet[1] of twenty-six letters, and the further development of its algebra, should therefore be of some importance in cryptography.

If polygraphic ciphers based upon normal transformations (linear ciphers) prove to be of real interest, we shall indicate a surprising way in which these ciphers may be manipulated easily and quickly, even for fairly large values of n(say $n=8$, 9, or 10), and thus made effective in a distinctly practical sense. It should be remarked that a cipher of type C_n in which $n>4$, although easy to use, is extraordinarily difficult to "break," offering very high resistance to the methods of cryptanalysis.

THE LIFE INSURANCE ACTUARY AND HIS MATHEMATICS[2]

By RAYMOND V. CARPENTER, Metropolitan Life Insurance Co.

It is estimated that the amount of life insurance in force in United States companies at the end of 1928 is about \$95,000,000,000. The assets are about \$16,000,000,000 and the premium collections in 1928 were over \$3,000,000,000.

The employed personnel of a life insurance company consist of the field or agency force and the home office force. An important part of the home office force is the actuarial department.

The actuary has a wide range of duties. He must be reasonably familiar with the work of all departments of the company, and is sometimes called its "technical" man. His two main duties are, first, the calculation of the premiums

[1] The bi-operational alphabet employed in this paper is an example of a "ring." See the Bulletin of the National Research Council, *Report on Algebraic Numbers*, p. 59.

[2] This paper was read by invitation before the Mathematical Association of America at New York City on Dec. 29, 1928.

charged, according to plan of insurance and age, and second, the calculation of the reserves that must be held by the company to provide, with the aid of future premiums and interest earnings, for future obligations, which reserves must be covered by safe interest-earning assets. The $95,000,000,000 of insurance in force is mainly the result of solicitation by the agency forces, but for the sufficiency of the $3,000,000,000 yearly premiums and of the $16,000,000,000 of assets the actuaries are mainly responsible.

Bases of Calculation of Premiums and Reserves

In the calculation of premiums and reserves, the actuary must deal mainly with two factors, namely, the rates of mortality (death rates) according to age and the rate of compound interest. He must assume in his calculations a mortality table showing death rates which it seems safe to assume will not be exceeded and a rate of interest (usually 3 or $3\frac{1}{2}$ per cent.) on which he can safely rely. The necessity for conservatism in selecting these standards will be realized when it is considered that once a policy is issued, the premium is fixed for all time—perhaps for fifty years or more—and cannot be increased, even though it prove insufficient. In this respect life insurance differs from the usual commercial enterprise, where contracts are of short durations and prices can readily be changed.

A third factor entering into the calculation of premiums is that of management expenses and commissions. It is taken care of, however, by an additional charge termed the "loading," the calculation of which is usually relatively simple. The premium inclusive of loading is called the "gross" premium; exclusive of loading, the "net" premium.

Mortality and compound interest, then, are the two factors upon which most of the calculations in actuarial science depend. The combination of these two factors to meet various contingencies results in innumerable formulae, which become fairly complex when the probabilities of survivorship of more than one life are involved, as frequently happens.

Efforts to Discover A "Law of Mortality"

Various attempts have been made to discover a definite law of mortality. Benjamin Gompertz, about one hundred years ago, proposed "that death may be the consequence of two generally coexisting causes; the one, chance, without previous disposition to death or deterioration; the other, a deterioration, or increased inability to withstand destruction." In 1860, Makeham developed this theory to the point of obtaining the expression $l_x = ks^x(g)^{c^x}$, where x is the age, and l the number of lives in a given group of entrants which survive to age x, the other four symbols being constants. Makeham's law applies, with remarkable closeness, to many mortality tables from about age 20 to the end of life, and while it is not considered a true law of mortality, it forms a favorite basis for graduation of mortality tables, for the two reasons, first, that it affords a satisfactory method for continuing a mortality table through the late ages of life, where actual data are meagre, and second, that the special

properties of a table so graduated make it readily adaptable to the calcuation of many life insurance benefits involving more than one life. In 1924, W. P. Elderton presented a paper at the International Mathematical Congress at Toronto making certain suggestions as to the application of frequency curves to mortality experiences, but did not arrive at a definite "law of mortality."

It is usually conceded, then, that there is no true "law of mortality," and actuaries are generally agreed that the only way to obtain a reliable mortality table is to construct it from the actual record of the deaths and the number living among persons of the class to which the mortality table is applicable. For it must be remembered that mortality rates differ among different classes of persons. For example, persons insured under policies known as "industrial," which are for relatively small amounts, with premiums payable weekly, experience much higher rates of mortality than persons whose better economic conditions permit them to insure under so-called "ordinary" policies for higher amounts. Mortality tables based on insured lives differ from those based upon general population data. The selection of a proper mortality table to apply to a given class of risks is one of the problems of the actuary.

Elementary Principles

The more elementary calculations of actuarial science are set forth in several text books, such as the late Professor Dowling's *Mathematics of Life Insurance*. The more advanced calculations may be found in the two standard text-books—both in English—which are in universal use among English-speaking actuarial students, namely, Spurgeon's *Life Contingencies*, and *Actuarial Theory* by Robertson and Ross, and in various publications of the actuarial societies of America and Great Britain. For the study of compound interest, the (British) Institute of Actuaries' Text Book, Part I, by Todhunter, is a standard.

Time will not permit a discussion of the formulae used for the calculation of premiums. It will be sufficient to say that if we assume for convenience that premiums are payable at the beginning of each year and claims at the end of the year (which assumptions can be appropriately modified), and if we let v equal the present value of 1 due in one year, then the net single premium for a life aged x for insurance of 1 for the whole of life is

$$(vd_x + v^2d_{x+1} + \cdots \text{ to end of table})/l_x\ ;$$

that the present value of an annuity of 1 payable at the beginning of each year throughout life is

$$(l_x + vl_{x+1} + v^2l_{x+2} + \cdots \text{ to end of table})/l_x\ ;$$

and that the net annual premium is the net single premium divided by the present value of the annuity. This general principle, suitably modified, underlies the calculation of premiums and annuity values generally.

It might also be mentioned that by the simple algebraic device of multi-

plying numerator and denominator in the foregoing expressions by v^x we obtain a series of terms in which the exponents of v bear a constant relation, at all ages, to the subscripts of d and l, thus enabling us to avoid the large number of combinations that would otherwise occur, and to construct so-called "commutation columns" which greatly facilitate life insurance calculations.

"Select" Mortality Tables

Because of careful selection, medical or otherwise, insured persons usually show especially favorable mortality during the first few years of the policy. Hence so-called "select" tables are sometimes used. These consist of several "l" columns—the first column representing the number of persons entering the table at each age at the beginning of the insurance, the second column the number of them surviving one year, and so on. At the end of the "select" period, say five years, the various l's merge into a common "ultimate" table.

Reserves

The annual rate of mortality is very high in the first year of life. Then it decreases rapidly until about age 12. Thereafter it increases until it becomes very high at the older ages. If a group of persons each aged twenty became insured on the plan of simply paying the current mortality cost each year the premiums would be very low at first, but would increase each year, and in the later years would be very high. For this reason people prefer to pay premiums which will remain *level* throughout the period of the policy, except as the cost may be reduced by dividends. It is the level premium plan which, in the main, gives rise to the very large *reserves* that must be carried by life insurance companies and therefore to the huge volume of assets. Under the level premium plan, it is evident that in the early years of the policy the premiums largely exceed the current mortality cost, and that this excess will be needed in later years when the mortality costs exceed the premiums. The excess, with interest accretions, must therefore be held as a "reserve." Endowment features and plans under which payment of premiums is limited to a specified number of years, naturally require larger reserves than whole life policies. The laws of the several States fix the standards of mortality and interest on which the reserves must be based.

Disability Benefits

In recent years life insurance companies have been offering policies which provide not only for payment of insurance at death, but for certain benefits in event of the so-called "total and permanent disability" of the person insured. A common disability benefit is the waiver of the premiums and payment of a monthly income during the continuance of disability, without deduction from the life insurance benefit. In computing premiums and reserves under such policies the actuary meets added complications, for he must base his calculations upon a combined mortality and disability table, showing for each age (1) the

rate of mortality among *active* lives (i.e., those not disabled), (2) the rate of disability, and (3) the rate of mortality among "disabled" lives. If the disability benefit is such that recoveries from disability are frequent, then adjustment is needed for the recoveries.

Actuarial Societies

Naturally, actuaries have formed societies for the purposes of better acquaintance with one another and the interchange of ideas. The principal societies in Great Britain and America are the Institute of Actuaries of Great Britain, organized in 1848, the Faculty of Actuaries in Scotland (1856), the Actuarial Society of America (1889) and the American Institute of Actuaries (1909), the last being rather more closely identified with the western and southern companies, than the Actuarial Society, although its membership also includes many eastern actuaries. Many actuaries are members of both of the American societies. Both have many members from Canada. There is also the Casualty Actuarial Society, in America, which, however, is associated with casualty insurance more than life insurance.

These societies hold regular meetings for the presentation and discussion of papers, which are printed in their published transactions, and for the informal discussion of topics of current interest. Through committees they cooperate in compiling, from the joint experience of the various companies, tables of rates of mortality for standard lives or for various special classes of lives, especially those showing certain impairments. Occasionally they publish text books or reports on subjects of current interest. They hold annual examinations open to properly qualified actuarial students, and the passage of these examinations entitles the candidates to membership in the society. The great majority of present members have entered through examination.

Actuarial Examinations

The success of the actuarial student in his work, up to a certain point, is largely dependent upon passage of prescribed examinations, of which a brief description may be of interest. The subjects in the two American societies are quite similar—in fact, the examinations of the first two days will be identical next year. It will therefore be sufficient to treat of the Actuarial Society.

There are two grades of membership, Associateship and Fellowship. Candidates must be proposed by a Fellow and approved by the Society. The examinations are held in April, at a number of centers convenient for the candidates. The examinations for Associateship occupy four days—that is, two each year—but not more than two days' examinations may be taken in one single year. The examinations for Fellowship consist of two parts, and one or both may be taken in a single year. One can become an Associate, therefore, in a minimum of two years and a Fellow in three. Rarely, however, do students pass all of the examinations within the three years.

The first day's examinations for Associateship embrace arithmetic, elementary algebra, plane geometry, plane and spherical trigonometry and plane ana-

lytical geometry. The subjects for the second day are advanced algebra and the elements of the theory of probabilities, differential and integral calculus, the calculus of finite differences, and statistics. Thus far life insurance itself has not been touched. The third days's examinations include the subjects compound interest and annuities certain, the mortality table and its application, and the theory of life contingencies for one life only, including calculation of net premiums and reserves. The fourth day's examinations—the final for Associateship—cover the theory of life contingencies for more than one life, the use of tables involving more than one decrement, such as death and disability, calculations relating to such accident and disability benefits as are included in life policies, construction of actuarial tables, general nature of life insurance and annuity contracts, including statutory requirements, and the history of life insurance.

The examinations for Fellowship cover a wide range of subjects, both theoretical and practical. The subjects include the principles to be observed in making mortality and disability investigations and the methods of constructing and graduating such tables; the sources and characteristics of the principal mortality and disability tables and investigations; selection of risks and premiums for extra hazards, calculation of gross premiums; valuation of the assets and liabilities of life insurance companies; non-forfeiture values and changes in life insurance contracts; analysis and distribution of surplus; investment of life insurance funds; elements of banking and finance; insurance law; pension funds; general questions involving the application of actuarial principles; and current topics of actuarial interest.

To guide students in their studies, each of the American societies has an "Educational Committee," which publishes, for the guidance of students, a suggested course of reading in preparation for each of the examinations. Students and others interested can obtain information relative to actuarial examinations from the Secretary of the Actuarial Society at 256 Broadway New York City, or from the Secretary of the American Institute of Actuaries at 720 North Michigan Avenue, Chicago, Illinois.

Practical Work of the Actuary

The work of the actuary can be partially judged from the subjects of the examinations. His most important duty, as previously stated, is the calculation of premiums and reserves. The solvency of the company, at present and in the distant future, is largely dependent upon him. He has a large part in the preparation of the annual statements required by the Insurance Departments of the several States, including the highly analytical "gain and loss" exhibits, which analyze the gain or loss in surplus from such sources as mortality, interest, expense provisions, etc. In a mutual company, he assists in determining the total amount of so-called dividends (more properly, savings) which can be returned each year to policyholders and calculates the amounts that can equitably be returned to *each class* of policyholders, according to age, plan of insurance, and

duration of policy, using formulae which aim to allot the dividends in proportion to the share of surplus contributed by the respective classes. He calculates practically all of the data appearing in the rate book, including not only premiums but cash surrender and other non-forfeiture values. He fixes the terms under which policies may be changed from one form to another or otherwise adjusted. He cooperates with the legal department in the drafting of policy forms and in keeping in touch with current legislation; with the medical directors in fixing rules for the acceptance of various classes of risks involving additional risk by reason of occupation or impaired conditions of health, and with the agency department in fixing the compensation of agents. The investment department may call upon him for calculations relating to securities of a complicated or unusual nature. He carefully watches the mortality and disability rates experienced by his company. Either in his own company, or in cooperation with actuaries of other companies, he occasionally conducts mortality investigations relating to special classes of risks, or engages in the construction and graduation of new mortality tables. He may be called upon to devise plans for pensions. He must keep in touch with current developments in insurance. The list of his activities might be extended almost indefinitely, for he is constantly considering new problems of very great variety which arise in the business. On the whole he and his assistants are kept fairly busy. Most of his time is taken up in work of a very practical nature, and it must be confessed that he must often rely upon his assistants for the more complicated and laborious mathematical calculations required.

In most of their everyday work the actuary and his assistants employ no mathematics beyond relatively simple algebra and its application to insurance. Sometimes, however, as in complicated problems requiring the use of summation or interpolation formulae or in graduation, more advanced mathematics are needed. Always, however, the actuaries must have the capacity for analysis and must have an instinctive sense of proportion.

Preparation for Examinations

While most men with actuarial training are associated with life insurance companies, there are other opportunities. The various State Insurance Departments usually employ one or more actuarially trained men. Some actuaries, not connected wth any company, open offices of their own or enter into partnership as "consulting actuaries." Women are gradually entering the actuarial field.

Most of the preparation for actuarial examinations is done by the candidates while employed in company or other insurance work. Employees of the actuarial departments of the companies are increasingly being recruited from universities and colleges, efforts being made to select students with rather more than average mathematical talent. The man, however, who is purely a mathematical genius, and is not practical, will not develop into a successful actuary. Most of the students entering the companies have not prepared specifically for the actuarial examinations prior to graduation, but their mathematical

groundwork is helpful, and in fact almost essential. A number of students do undertake the early examinations while in college, and some are successful.

Actuarial Science in Universities and Colleges

While many universities and colleges offer insurance courses of various kinds, those making a serious effort to train students in actuarial science are few. A tabulation of the candidates who have recently passed the early examinations of the Actuarial Society and the American Institute of Actuaries while university students indicates that there are two universities in Canada, namely, Toronto and Manitoba, and two in the Middle West, namely, Iowa and Michigan, which give serious attention to actuarial training, but none in the Eastern States. Columbia University, however, is successfully conducting correspondence courses leading to the actuarial examinations, most of those enrolled being employees of insurance companies.

Opportunities

In the United States the university or college graduate is paid something like $1500 a year when he first enters a company for actuarial work. By the time he passes the Fellowship examinations, if he has the other qualifications needed, he should probably be able to command $3000 or $4000 yearly. After that his progress depends upon his individual ability and his opportunities.

For a number of years in this country, the demand for capable men of actuarial training and practical experience has exceeded the supply. This is not so true in Canada, and in England and Scotland the case is the reverse. As a result, many of the actuarial officers of life insurance companies in the United States, including some of the leaders in actuarial achievement, have come from England, Scotland, and Canada.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

DISCUSSIONS

I. ON THE RESOLUTION OF A FRACTION INTO PARTIAL FRACTIONS

By W. C. BRENKE, University of Nebraska

1. In this paper is explained a simple routine for obtaining the undetermined coefficients required to express a given fraction as a sum of partial fractions. It consists in applying Taylor's expansion in such a way as to require a minimum of numerical calculation. The method is useful in the calculus of residues, since each coefficient is determined independently.

Let the given fraction be $\phi(x)/p(x)$, where $p(x)$ is a polynomial containing

the factor $(x-\alpha)^r$, and $\phi(x)$ is analytic about $x=\alpha$. Then this factor will give rise to the sum of partial fractions:

$$\frac{A_1}{x-\alpha}+\frac{A_2}{(x-\alpha)^2}+\cdots+\frac{A_r}{(x-\alpha)^r}. \tag{1}$$

To determine $A_1, A_2, \cdots, A_r$, we first find the numerators of the partial fractions,

$$\frac{a_1}{x-\alpha}+\frac{a_2}{(x-\alpha)^2}+\cdots+\frac{a_r}{(x-\alpha)^r}, \tag{2}$$

in the development of $1/p(x)$. These will be the first r terms in the expansion of $f(x)=(x-\alpha)^r/p(x)$ in powers of $x-\alpha$; and so we shall have

$$a_{r-k} = f^{(k)}(\alpha)/k!\,; \quad k = 0,1,2,\cdots, r-1\,;$$

where the superscript indicates the kth derivative.

Multiplying (2) by the Taylor's expansion of $\phi(x)$, we have

$$A_k = \frac{\phi(\alpha)f^{(r-k)}(\alpha)}{0!(r-k)!}+\frac{\phi'(\alpha)f^{(r-k-1)}(\alpha)}{1!(r-k-1)!}+\cdots+\frac{\phi^{(r-k)}(\alpha)f(\alpha)}{(r-k)!0!}.$$

The calculation may now be reduced to a routine by forming a table of double entry in which the arguments at the top and side shall be the coefficients of the Taylor's expansions of $\phi(x)$ and $f(x)$ respectively. The entries in the table are to be the products of the arguments, and the coefficients A_k are to be obtained by summing these entries by diagonals.

EXAMPLE: To find the coefficients in the expansion,

$$\frac{x^3-2x+1}{(x-2)^4(x+3)^3} = \sum_{k=1}^{4}\frac{A_k}{(x-2)^k}+\sum_{k=1}^{3}\frac{B_k}{(x+3)^k}.$$

To find A_k, $k=1, 2, 3, 4$, we have $\alpha=2$, $r=4$, $\phi(x)=x^3-2x+1$, $f(x)=(x+3)^{-3}$. Also,

$$\begin{aligned} \phi(\alpha) &= 5, & f(\alpha) &= 1/5^3,\\ \phi'(\alpha) &= 10, & f'(\alpha) &= -3/5^4,\\ \phi''(\alpha)/2 &= 6, & f''(\alpha)/2 &= 6/5^5,\\ \phi'''(\alpha)/6 &= 1. & f'''(\alpha)/6 &= -10/5^6. \end{aligned}$$

We now form the following table, omitting the entries below the secondary diagonal:

$5^5 f \diagdown \phi$	5	10	6	1
25	125	250	150	25
-15	-75	-150	-90	
6	30	60		
-2	-10			

Summing by diagonals and dividing by 5^5, we have:

$$A_1 = -15/5^5, \quad A_2 = 30/5^5, \quad A_3 = 175/5^5, \quad A_4 = 125/5^5.$$

To find B_k, $k=1, 2, 3$, we have $\alpha = -3$, $r=3$, $\phi(x) = x^3 - 2x + 1$, $f(x) = (x-2)^{-4}$.

$5^5 f \backslash \phi$	-20	25	-9
5	-100	125	-45
4	-80	100	
2	-40		

$$B_1 = 15/5^5, \qquad B_2 = 45/5^5, \qquad B_3 = -100/5^5.$$

Thus we have:

$$\frac{x^3 - 2x + 1}{(x-2)^4(x+3)^3} = \frac{1}{5^4}\left[-\frac{3}{x-2} + \frac{6}{(x-2)^2} + \frac{35}{(x-2)^3} \right.$$
$$\left. + \frac{25}{(x-2)^4} + \frac{3}{x+3} + \frac{9}{(x+3)^2} - \frac{20}{(x+3)^3} \right].$$

Every figure that needs to be set down in the course of the calculations is shown here. The saving in labor and time over the purely algebraic method is obvious.

2. If the given fraction has in its denominator a quadratic factor with imaginary roots, and if a resolution into real fractions is desired, it will usually be much more expeditious to follow the plan of §1, and then to change to real form by uniting conjugate imaginary parts.

EXAMPLE: To resolve the fraction,

$$\frac{x^2 - 2}{x(x^2 - 2x + 2)^3} = \frac{A}{x} + \sum_{k=1}^{3} \frac{B_k}{(x-b)^k} + \sum_{k=1}^{3} \frac{C_k}{(x-c)^k}.$$

Here the roots of $x^2 - 2x + 2$ are $b = 1 - i$, and $c = 1 + i$.

Value of A: Take $\alpha = 0$, $r = 1$, $\phi(x) = x^2 - 2$, $f(x) = (x^2 - 2x + 2)^{-3}$.

$$A = \phi(\alpha) f(\alpha) = -1/4.$$

Values of B_1, B_2, B_3,: Take $\alpha = 1 - i$, $r = 3$, $\phi(x) = x^2 - 2$, $f(x) = x^{-1}(x-c)^{-3}$. Omitting some details of calculation, we find

$$\phi(\alpha) = -2(1+i), \qquad f(\alpha) = (1-i)/16,$$
$$\phi'(\alpha) = 2(1-i), \qquad f'(\alpha) = -(5+3i)/32,$$
$$\phi''(\alpha)/2 = 1. \qquad f''(\alpha)/2 = (-2+7i)/32.$$

Forming the table with the second set of numbers multiplied by 32, we have:

$32f \diagdown \phi$	$-2(1+i)$	$2(1-i)$	1
$2(1-i)$	-8	$-8i$	$2(1-i)$
$-5-3i$	$4+16i$	$-16+4i$	
$-2+7i$	$18-10i$	.	

$$B_1 = \frac{1-2i}{8}, \qquad B_2 = \frac{1+2i}{8}, \qquad B_3 = -\frac{1}{4}.$$

Values of C_1, C_2, C_3*:* These are evidently imaginaries conjugate to the values of B_1, B_2, B_3.

The fraction is now resolved into seven partial fractions of which the last six form conjugate pairs. Hence if we let $R(u)$ stand for the real part of the complex number u, we have:

$$\frac{x^2-2}{x(x^2-2x+2)^3} = -\frac{1}{4x} + 2R\left[\frac{1-2i}{8(x-b)} + \frac{1+2i}{8(x-b)^2} - \frac{1}{4(x-b)^3}\right]$$

$$= -\frac{1}{4x} + \frac{x-3}{4(x^2-2x+2)} + \frac{x^2+2x-4}{4(x^2-2x+2)^2} - \frac{x^3-3x^2+2}{2(x^2-2x+2)^3}.$$

The resolution by purely algebraic means would be extremely tedious in this case. It may be noted that the algebraic method would not give the form of result found above, inasmuch as it would give linear numerators in the last two fractions. These may be obtained by taking $-1/[4(x^2-2x+2)]$ from the second of our four fractions and adding it to the third, whose numerator would then be $4x-6$. Then by taking $(x-1)/[2(x^2-2x+2)^2]$ from this fraction and adding it to the fourth, its numerator would become $4x-4$.

3. The value of the definite integral,

$$I = \int_0^\infty \frac{\cos x\, dx}{(1+x^2)^{n+1}}$$

may be obtained from the contour integral

$$J = \int_C \frac{e^{iz}dz}{(1+z^2)^{n+1}},$$

where the path of integration is the part of the axis of reals, $-R \leqq x \leqq R$, and a semicircle in the positive half-plane drawn with this part of the axis of reals as its base. The value of I is πi times the residue at i of the integrand in J. To find this residue we calculate A_1 of §1, taking

$$\alpha = i, \quad r = n+1, \quad \phi(z) = e^{iz}, \quad f(z) = (z+i)^{-n-1}.$$

We find immediately:

$$\int_0^\infty \frac{\cos x\,dx}{(1+x^2)^{n+1}} = \frac{\pi}{n!2^{2n+1}e}\left[\frac{(2n)!}{0!n!} + \frac{2(2n-1)!}{1!(n-1)!} + \cdots + \frac{2^n n!}{n!0!}\right]$$

4. It may be useful to note here an interesting solution of the problem of resolving a given fraction published by Professor H. W. Turnbull.[1] If $p(x) = (x-a_1)(x-a_2)(x-a_3)\cdots(x-a_n)$, and $\phi(x)$ is a polynomial of degree less than n, the solution for the case where no two of the factors of $p(x)$ are equal is given by this ratio of two determinants:

$$\frac{\phi(x)}{p(x)} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_1^{n-2} & a_2^{n-2} & \cdots & a_n^{n-2} \\ \dfrac{\phi(a_1)}{x-a_1} & \dfrac{\phi(a_2)}{x-a_2} & \cdots & \dfrac{\phi(a_n)}{x-a_n} \end{vmatrix} \div \Delta,$$

where Δ is formed from the numerator determinant by replacing the last row by $a_1^{n-1}, a_2^{n-1}, \cdots, a_n^{n-1}$.

If $a_2 = a_1$, replace the a_2-column in both determinants by the first derivative with respect to a_1 of the a_1-columns in the respective determinants. If $a_3 = a_2 = a_1$, replace the a_2-columns as above, and also replace the a_3-columns by the second derivatives of the a_1-columns, and so on.

II. On the Approximate Division of a Circumference

By J. P. Ballantine, University of Washington

In recent issues of this Monthly, R. A. Johnson,[2] and T. Dantzig[3] have taken up the question of dividing a given circumference into any number of equal parts. They showed how to construct in one case the length of the chord and in the other case the length of the arc of one of the resulting segments of the circumference. From a practical standpoint, however, it might be troublesome to apply this chord a large number of times around the circumference because of a cumulative error. To avoid this difficulty, we have devised the following construction which determines the individual vertices of the regular inscribed polygon.

Let AOB be a diameter of the given circle with center at O. Let DOC be a line perpendicular to AOB, with D on the circumference and OC of length $1\frac{3}{4}$ radii. With D as center sweep off a small arc ECF.

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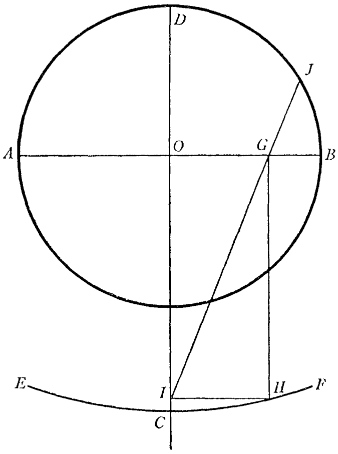
[1] Proceedings of the Edinburgh Mathematical Society, series 2, vol. 1 (1927), p. 49.

[2] Vol. 34 (1927), pp. 429–31.

[3] Vol. 35 (1928), pp. 185–87.

If G is any point on AB, then a corresponding point on arc ADB is constructed as follows: Draw GH parallel to DC with H on arc ECF. Draw HI parallel to AB with I on DC. Draw IG, and where it cuts arc ADB mark the point J, which is said to correspond to G on AB.



If G takes on the positions $G_0=O$, $G_1, \cdots, G_{10}=B$, eleven equally spaced points, then J takes on eleven positions, $J_0=D$, $J_1, \cdots, J_{10}=B$ on the arc DB. To state how nearly equally spaced these latter points are, we show in tabular form the approximate value of the central angle DOJ_i:

i	DOJ_i
0	0.°
1	9.0057°
2	18.0078°
3	27.0111°
4	36.0154°
5	45.0219°
6	54.0262°
7	63.0351°
8	72.0330°
9	81.0278°–
10	90.°

To divide the arc ADB into any required number of equal parts, simply divide the diameter AOB into the required number of equal parts and find the corresponding points of the arc ADB.

The present method has no theoretical interest, being entirely empirical. It was devised by dividing the arc ADB and the diameter AOB into the same number of equal parts, connecting the corresponding points, and studying the resulting family of lines. It was found that for G near O, and J near D, the length OI is $1/(\frac{1}{2}\pi-1)=1.75194$. For G and J near B, $OI=\frac{1}{2}\pi=1.57080$. For the construction to give best results for G and J near B, OD should be 1.09890. For the construction to be exact when G is midway between O and B, OD should be 1.05869.

We do not maintain that the decimals appearing above have any interest in themselves. The interesting point to our mind is that by rounding them off to $OC=1.75$ and $OD=1$, the resulting construction is at the same time simple and yet gives good values for a large range. We therefore suggest this method for the practical draftsman who does not worry about errors of the order of magnitude of the width of the lines he draws.

III. The Number of Arithmetical Operations Involved in the Solution of a System of Linear Equations[1]

By T. H. Gronwall, New York, N. Y.

It is proposed to determine the number A_n of additions (or subtractions), M_n of multiplications, and D_n of divisions which occur in the solution of a system of n linear equations in n unknowns:

$$\sum_{k=1}^{n} a_{ik}x_k = b_i \qquad (i = 1,2, \cdots, n), \tag{1}$$

under the assumption that the determinant of the system is different from zero, so that there is a unique solution.

This problem will be solved, first without assuming any special relations between the coefficients a, and then in the most important case in which there are such relations, namely the symmetrical case where $a_{ik}=a_{ki}$. The problem is of some practical importance in numerical work when n is large, as is the case in the application of the direct methods of the calculus of variations, of which Ritz's treatment of the elastic vibrations of a plate is a classical example.[2]

On the other hand, the equations giving the stresses in statically indeterminate structures require a much smaller number of arithmetical operations, since a large part of the coefficients vanish and there are highly special relations among the others.

In the numerical solution of the system (1), the use of determinants may be ruled out at once, since determinants of high order are among the most unpleasant objects to handle numerically. The best method appears to be that of substitution in the following form: Renumbering, if necessary, the equations (or the unknowns), we may assume $a_{11}\neq 0$, since otherwise the determinant of the a's would contain a column (or a row) of zeros and vanish contrary to hypothesis. The first equation (1) then gives

$$x_1 = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_1 - \cdots - \frac{a_{1n}}{a_{11}}x_n, \tag{2}$$

and the substitution of this in the remaining equations gives the system in $n-1$ unknowns:

$$\sum_{k=2}^{n}\left(a_{ik} - a_{i1}\cdot\frac{a_{1k}}{a_{11}}\right)x_k = b_i - a_{i1}\cdot\frac{b_1}{a_{11}}, \quad (i = 2,3, \cdots, n). \tag{3}$$

The divisions required to form equations (3) are $a_{1k}/a_{11}(k=2, \cdots, n)$ and b_1/a_{11}, their number being n, and the formation and solution of (3) thus requires

[1] Presented to the American Mathematical Society, March 30, 1929.

[2] Walter Ritz, *Über eine neue Methode zur Lösung gewisser Variationsprobleme der mathematischen Physik*, Journal für Mathematik, vol. 135 (1908), p. 1, and *Theorie der Transversalschwingungen einer quadratischen Platte mit freien Rändern*, Annalen der Physik, (4) vol. 28 (1909), p. 737.

$n+D_{n-1}$ divisions, while the calculation of x_1 by (2) introduces no new division. Hence:

$$D_n = n + D_{n-1}. \tag{4}$$

The multiplications required to form (3) are $a_{i1}\cdot(a_{1k}/a_{11})$ for $i, k=2, 3, \cdots, n$, in number, $(n-1)^2$; and $a_{i1}\cdot(b_1/a_{11})$, in number, $n-1$; the solution of (3) requires M_{n-1}, and the calculation of x_1 by (2) the $n-1$ multiplications $(a_{1k}/a_{11})x_k$, so that

$$M_n = n^2 - 1 + M_{n-1}. \tag{5}$$

In the above process, each multiplication is followed by an addition (or subtraction), and consequently

$$A_n = M_n. \tag{6}$$

We also note that for $n=1$, the system (1) reduces to $a_{11}x_1=b_1$, so that

$$A_1 = M_1 = 0, \quad D_1 = 1. \tag{7}$$

The integration of the difference equations (4) and (5) is immediately effected by the familiar formulas for the sums of the integers, and their squares, from 1 to n, and the constants of integration are determined by (7). We thus obtain:

$$\begin{aligned} A_n &= M_n = \tfrac{1}{6}n(n-1)(2n+5), \\ D_n &= \tfrac{1}{2}n(n+1). \end{aligned} \tag{8}$$

Passing to the symmetrical case,

$$a_{ik} = a_{ki} \qquad (i,k = 1,2, \cdots, n), \tag{9}$$

we note that the divisions to be performed are the same as before, and that each multiplication is followed by an addition, so that (4), (6), and (7) remain unchanged.

However, it follows at once from (9) that the coefficients to the left in (3) are also symmetrical, so that we have to calculate them only for $i\leqq k$, which reduces the number of multiplications $a_{i1}\cdot(a_{1k}/a_{11})$ from $(n-1)^2$ to $n(n-1)/2$; the remaining $2(n-1)$ multiplications are as before, so that (5) is replaced by

$$M_n = \tfrac{1}{2}(n-1)(n+4) + M_{n-1}. \tag{5a}$$

The integration gives, in the same way as above,

$$\begin{aligned} A_n &= M_n = \tfrac{1}{6}n(n-1)(n+7), \\ D_n &= \tfrac{1}{2}n(n+1). \end{aligned} \tag{8a}$$

Thus the number of additions and multiplications is almost cut in half by the symmetry of the coefficients.

The solution of (1) by substitution may of course be effected in other ways than the one described, but all modifications attempted thus far have increased the total number of operations.

For instance, (2) and (3) may be replaced by

$$a_{11}x_1 = b_1 - a_{12}x_2 - \cdots - a_{1n}x_n, \tag{10}$$

$$\sum_{k=2}^{n}(a_{11}a_{ik} - a_{i1}a_{1k})x_k = a_{11}b_i - a_{i1}b_1, \tag{11}$$

and the recurrence formulas now become, in the general case.

$$D_n = 1 + D_{n-1},$$
$$A_n = n^2 - 1 + A_{n-1},$$
$$M_n = (n-1)(2n+1) + M_{n-1},$$

whence

$$\begin{aligned} A_n &= \tfrac{1}{6}n(n-1)(2n+5), \\ M_n &= \tfrac{1}{6}n(n-1)(4n+7), \\ D_n &= n. \end{aligned} \tag{12}$$

In the symmetrical case, these expressions are replaced by

$$\begin{aligned} A_n &= \tfrac{1}{6}n(n-1)(n+7), \\ M_n &= \tfrac{1}{6}n(n-1)(2n+11), \\ D_n &= n. \end{aligned} \tag{12a}$$

The choice of method for solving (1) may be influenced by the type of calculating machine used; with a machine with automatic division, the first method is preferable, while with a machine such as the "Millionaire," where division is cumbersome but multiplication extremely rapid, the second method may have its advantages.

IV. Simplification of an Annuity Formula

By C. C. Wylie, University of Iowa

Consider an investor who receives a nominal annual interest rate per year, j, p payments of total amount R per year, and interest converted m times per year. The amount S of this annuity at the end of n years is given by the expression,[1]

$$S = R\frac{\left(1+\dfrac{j}{m}\right)^{mn} - 1}{p\left[\left(1+\dfrac{j}{m}\right)^{m/p} - 1\right]}. \tag{1}$$

If p is a multiple of m, this expression reduces by well known methods to the form

[1] Rietz, Crathorne, and Rietz, *Mathematics of Finance*, p. 37. (The notation has been modified slightly.)

$$S = \frac{p}{m} \cdot K \cdot S_{\overline{1}|}^{(p/m)} \cdot S_{\overline{mn}|}, \tag{2}$$

where K is the payment per period.

In this Monthly[1] for last August-September, Professor W. L. Hart gives the values of p/m (p in his notation) which he considers usual in problems in the application of annuities. For all of these p/m is an integer or m/p is an integer. For p/m (Professor Hart's p) an integer, formula (2) is a standard method of solution. The quantity $S_{\overline{1}|}^{(p/m)}$ is taken from the $S_{\overline{1}|}^{p}$ table and the quantity $S_{\overline{mn}|}$ is taken from the $S_{\overline{n}|}$ table, both for the interest rate j/m. Teachers in well equipped schools will no doubt prefer to multiply the four numbers of equation (2) on a calculating machine to obtain S; but they should keep in mind that many of their students will have occasion to solve these problems without such conveniences. If no calculating machine is at hand, the logarithms, instead of the numerical values, of the quantities $S_{\overline{1}|}^{(p/m)}$ and $S_{\overline{mn}|}$ can be taken from the table. Looking up also log p/m and log K, the addition of the four logarithms gives log S.

For the other common case, m/p an integer, Professor Hart suggests that the tables be extended to include "all the usual fractional as well as integral values of p." The extension has since been published by him, and formula (2) or its equivalent can now be used for most problems of this type. This case is, however, solved quite easily in another way, using only the standard $S_{\overline{n}|}$ tables. Here, as we have seen, m/p is an integer, and if the method by which formula (1) was reduced to formula (2) is applied, one obtains[2]

$$S = K \frac{S_{\overline{mn}|}}{S_{\overline{m/p}|}} \tag{3}$$

where the quantities $S_{\overline{mn}|}$ and $S_{\overline{m/p}|}$ are taken from the $S_{\overline{n}|}$ tables for the rate j/m. As stated, formula (3) involves a division, but this can be avoided if desired by using the tables for $1/S_{\overline{n}|}$; S is then obtained by a simple multiplication of three numbers. There is, however, some loss of accuracy in the reciprocal tables.

Professor Hart illustrates his method by two problems, which are, however, not solved. For the first problem, $j/m = 0.005$, $m/p = 6$, $mn = 66$, $K = \$100$. Let us solve first by formula (2) and the new tables.

$$S = 1/6 \times \$100 \times 0.98757273 \times 77.96497215 = \$1283.27.$$

By formula (3) the solution is $S = \$100 \times (1/S_{\overline{6}|}) \times S_{\overline{66}|}$ for the rate 0.005. Substituting from Glover's tables

$$S = \$100 \times 0.16459546 \times 77.96497215 = \$1283.27.$$

[1] Vol. 35 (1928), pp. 358–60.

[2] See W. L. Hart, *Mathematics of Investment*, supplementary note on page 51.

The $1/S_{\overline{n}|}$ tables are amply accurate for the solution of this problem. The arithmetical work is little different, except that formula (2) includes a division by 6.

The examples were, according to Professor Hart, to illustrate "problems with data chosen as inconveniently as possible." The second problem cannot be evaluated using formula (2) and the new tables. The data are $j/m=0.02$, $m/p=12$, $mn=144$, and $K=\$5000$. For this value of j/m, the new tables do not include $p=1/12$, and the usual $S_{\overline{n}|}$ tables do not extend to $n=144$. Formula (3) avoids the first difficulty; to meet the second, the substitution $S_{\overline{144}|}=S_{\overline{72}|}\{(1+i)^{72}+1\}$ can be made. Substituting the data of this problem in formula (3) we have, therefore,

$$S = \$5000 = S_{\overline{72}|} \times \{(1.02)^{72}+1\} \div S_{\overline{12}|} \text{ for rate } 0.02.$$

By Glover's tables,

$$S = \$5000 \times 158.05701875 \times 5.16114038 \div 13.41208973 = \$304{,}111.62.$$

The tables for $1/S_{\overline{n}|}$ were not used in the solution of this problem because $1/S_{\overline{12}|}$ is given to only seven significant figures, and the result for S, assuming it is desired to the nearest cent, requires eight.

The practice of the writer has been to advise that, in general, students substitute the numerical data of the problem in formula (1). If m/p is an integer, the problem is obviously solved as suggested in this paper, using only the $S_{\overline{n}|}$ tables. If p/m is an integer, it is solved by formula (2), and the use of the $S^{(p)}_{\overline{1}|}$ tables, in addition to the $S_{\overline{n}|}$ tables. Formula (1) is regarded as fundamental, and solved in the form of (2), or (3), as appears most convenient. The students are also supposed to notice that, as is stated by Professor Hart, for many problems formula (1) assumes a simpler form if the payment interval, rather than the year, is used as the unit.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College, New York, N. Y.

All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Loria, Gino. Storia delle matematiche. Volume 1. Turin, Societa tipografico editrice nazionale, 1929. 427 pages.

Bush, Vannevar. Operational Circuit Analysis. New York, John Wiley and Sons, 1929. x+392 pages. \$4.50.

Cajori, Florian. A History of Mathematical Notations. Volume 2. Notations in Higher Mathematics. Chicago, The Open Court Publishing Co., 1929. xvii+367 pages. \$6.00.

Garabedian, Carl A. and *Winston, Jean.* Plane Trigonometry. New York, The McGraw-Hill Book Company, 1929. xviii+306 pages.

Fraenkel, A. Einleitung in die Mengenlehre. Berlin, Julius Springer, 1928. xiv+424 pages. RM 22.60.

Reynolds, J. B. Analytic Mechanics. New York, The Prentice-Hall Company, 1929. x+348 pages.

Maclean, John. Graphs and Statistics. Bombay, Ramchandra Govind and Son, 1928. xiii+200 pages.

Prasad, Ganesh. Six Lectures on Recent Researches in the Theory of Fourier Series. Published by the University of Calcutta, 1928. xiv+240 pages.

Jordan, H. H. and *Porter, F. M.* Descriptive Geometry. Boston, Ginn and Company, 1929. xii+350 pages. $3.00.

Keyser, Cassius J. The Pastures of Wonder. The Realm of Mathematics and the Realm of Science. New York, Columbia University Press, 1929. xii+208 pages. $2.75.

Walker, Helen M. Studies in the History of Statistical Method. Baltimore, The Williams and Wilkins Company, 1929.

Fowler, R. H. Statistical Mechanics. The Properties of Matter in Equilibrium. Cambridge University Press, 1929. 570 pages.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Functions of Real Variables. By E. J. Townsend. Henry Holt and Company, New York, 1928. ix+405 pages.

The present volume is a companion volume to the author's well-known text on the *Functions of a Complex Variable.* According to the author it is based upon a course in "real variables" given at the University of Illinois, the purpose of which is "to discuss those topics which will enable the student to have a better grasp and understanding of the fundamental principles of the calculus of real variables and know something of the more recent developments of this branch of analysis." The books in the English language which have been written with this purpose in view are few indeed, and not all of these are usable by the beginner on account of generality of treatment. Undoubtedly the work under review will be welcomed by teachers and students alike.

As a basis for the theory of functions of real variables, the author has given in Chapter I a short and very readable account of the real number system. The two definitions of irrational number given by Cantor and Dedekind are explained and shown to be equivalent, and the fundamental operations are defined for the system of real numbers. The geometric interpretation of real numbers is then taken up. By means of the axiom of Archimedes and the axiom of continuity, a one-to-one correspondence is established between the totality of real numbers and the totality of points of a straight line, and thus

is suggested the possibility of a geometric interpretation of the problems of analysis. A brief article on the base of a system of notation concludes the chapter.

Chapter II, which is devoted to the theory of point sets, is an outstanding feature of the book and one that will appeal to many. Here is given a systematic account of linear point sets in-so-far as point sets constitute a tool in the subsequent developments. Attention is called to the validity of some of the theorems for two-dimensional space, and a very short article on non-linear sets is given. The Heine-Borel theorem is proved for linear sets and its planar generalization is stated. A brief account of Jordan measure is followed by a discussion of Lebesgue measure and the proofs of some important theorems on measurable sets.

Chapter III deals with the continuity and discontinuity of functions. In addition to a treatment of the laws of operations with limits and the properties of continuous functions of one and of two variables, the notion of absolute continuity is introduced and illustrated, and semi-continuous functions receive some consideration. The chapter concludes with a long article on discontinuous functions which includes an account of Baire's classification of functions.

Derivatives and their properties form the subject matter of Chapter IV. A method of constructing the successive approximations to a curve which is continuous in a given interval, but which has no definite direction at any point, is followed by a treatment of the properties of derivatives and derivative numbers. Partial derivatives and their properties, total differentiability, and the interchange of the order of differentiation are discussed.

Chapter V contains an account of the Riemann theory of integration. Properties of infinite integrals and the manner of convergence of such integrals receive consideration. In Chapter VI the Lebesgue integral is discussed and compared with the Riemann integral. At the end of this chapter the reader's attention is called to several other theories of integration. The integrals of Stieltjes and Hellinger are defined and Denjoy's generalization of the Lebesgue integral is treated briefly. Some indication is also given of the nature of the integrals of Perron, Borel, Young, and Pierpont. This is one of many instances in which the author leads the reader to a point from which considerable territory can be seen dimly and then supplies him with road maps in the form of a few select references.

Chapter VII deals with infinite series. An article on the laws of operation with series is followed by a discussion of uniform convergence, which is illustrated by means of several interesting geometric figures. Simply-uniform convergence and quasi-uniform convergence are also treated. The Hankel and Cantor methods of condensation of singularities are illustrated. A brief but interesting introduction to divergent series concludes the volume.

The book is elementary in style and presupposes only a "year's course in calculus and perhaps an elementary course in differential equations or a course in advanced calculus." The author has been careful to refrain from generalizing

the treatment. However, here and there is pointed out the possibility of extending the theory to functions defined for a set of points or to functions of several variables. Undoubtedly thus restricting the treatment will better serve to realize the author's purpose as stated above.

The value of the work as a text-book is enhanced by numerous illustrative examples, by a select list of problems at the end of each chapter, and by well-drawn geometric figures and good general appearance. The manner in which the book is written with its not infrequent statement of more general results with appropriate references constitutes an invitation to the reader to delve more deeply into the subject.

ALBERT W. RAAB

Partielle Differentialgleichungen. By Dr. G. Hoheisel. Sammlung Göschen, Leipzig and Berlin, Walter de Gruyter & Co., 1928. 159 pages.

This little book contains an excellent treatment of a large amount of material. The order of presentation is: The differential equation of the first order with two independent variables, the differential equation of the first order with n independent variables, systems of equations, the differential equation of the second order with two independent variables. The methods of involutions, characteristics, and transformations are discussed for the systems considered. The author also considers solution manifolds and boundary conditions in the form of given curves on the integral surfaces.

An outstanding feature of the presentation is the effort that is made to state the exact mathematical conditions under which the methods and proofs are valid. This is especially gratifying when one notes that many of the past and present writers on partial differential equations do little more than to give formal developments of methods of treatment. However, the book under review is not entirely immune to this criticism, as is seen by examining page 15. Extensive use is there made of the boundedness of $f_y(x, y)$ for each x in a neighborhood of x_0 and for *all* real values of y, yet there is nothing in the text to point out this requirement.

The author is quite careful to recall useful theorems of analysis. Pages 9–13 summarize important properties of Jacobians and give implicit function theorems.

The book contains a collection of material that is not readily available and I am sure that it will prove a valuable addition to the private libraries of many mathematicians.

WILLIAM M. WHYBURN

New Analytic Geometry. By Percey F. Smith, Arthur Sullivan Gale, and John Haven Neelley. Ginn and Co., Boston, 1928. x+326 pages.

Analytic Geometry. By R. L. Borger. The McGraw-Hill Book Co., New York, 1928. xii+334 pages.

The first-named book, which is a revision of the *New Analytic Geometry*

by Smith and Gale published in 1912, opens with a statement of such formulae and tables from elementary mathematics as the student will need. The other book does not contain such formulae and tables. The print in the Smith-Gale-Neelley book is such as to make important statements, conclusions, and theorems stand out more distinctly than in Borger's book. Incidentally, in the book by Borger, the important discussion (pp. 145, 146) of the significant relation between the nature of the roots of a quadratic and the solvability by means of ruler and compasses, and the relation between the nature of the cubic and the possibility of the trisection of an angle can not be located by means of the alphabetical index. When we observe that this index is the more complete of the two it is perhaps evident that we need more complete alphabetical indices in some of our mathematical books.

The content as well as the order of topics and the nature of their treatment in the Smith-Gale-Neelley book is such as one would expect in a traditional textbook. In general the statement of the theorem is followed by its demonstration, then by drill exercises and problems—a deductive procedure. The book by Borger is less rigorously logical and more psychological in its treatment. It is to be hoped that a number of our mathematics books in the lower division will be written more from the point of view from which the student will attack his problems in later life. Outside the class-room walls the student's work will usually be a more inductive procedure than the rigorously deductive method of some of our books, elegant as the latter may be. It is to be hoped also that our authors may show how a knowledge of analytic geometry is applicable in activities of life.

One good feature among others in the Smith-Gale-Neelley text is the placing of groups of difficult problems at the close of a number of the sets of problems. These are intended to challenge the mettle and ability of the stronger students. While the Borger book has sufficiently difficult problems to stimulate the brilliant students, such problems are not grouped separately. A better plan than to group such more difficult problems separately would perhaps be to leave the various difficult problems among the rest of their kind, but to mark them with asterisks.

The discussions of problems and solutions are very good in both books, those in the book by Borger being perhaps the more stimulating. The same feature in the latter book may be observed in the problems. It seems that the author took pains to have each exercise and problem a thought-provoking exercise. The question might arise whether there are enough drill problems of the various types to insure that the student will develop sufficient technique.

The text by Borger treats determinants with a degree of completeness that compares favorably with that of some of our books on college algebra. No separate chapters are devoted to polar co-ordinates and to parametric equations, these topics being taken up at various times when they would be most naturally suggested by their related topics. The Smith-Gale-Neelley book devotes a chapter or a part of a chapter to each of these topics.

The book by Borger treats the tangent in a somewhat unusual way, treating it from the point of view of the calculus and differentiating algebraic equations in order to determine the slope of the tangent to the corresponding curve. Another unusual feature in this book is the treatment of the ellipse. Instead of using a directrix the author uses a director circle which has its center at either focus and a radius equal to the major axis of the ellipse. This book has no chapter on empirical equations, in which it differs from the Smith-Gale-Neelley text.

For so-called self-study a student will fare better by using the Smith-Gale-Neelley book. A teacher teaching the course for the first time will be safer in using the same book as a text. One who has taught the course and who is supplementing from outside sources will find the Borger text suggestive in its attack and more stimulating.

J. CALVIN FUNK

Les espaces abstraits et leur théorie considérée comme introduction a l'analyse générale. By Maurice Fréchet. Gauthier-Villars, Paris, 1928. (Borel series.) xi+296 pages.

The subject of abstract spaces has attracted the attention of many mathematicians ever since the publication of Professor Fréchet's thesis in 1906. Most of them have found it difficult, however, to learn much about abstract spaces, on account of the great number of articles which have been written on this subject, each one representing an advance on the previous knowledge of some phase of the subject, but no one of them summarizing the existing knowledge of all phases of the subject.

Hence until the appearance of the present volume, it has been necessary to study all of Fréchet's papers in their chronological order in order to familiarize oneself with the present status of the subject of abstract spaces. In studying them one can observe incidentally the evolution of ideas and of terminology as Fréchet has developed the subject from its beginnings. This evolution is interesting enough in itself but of no interest to one who desires to know, for his own use, only the final results of Fréchet's research. As an example of this evolution, what was called *écart* in Fréchet's thesis is now called *distance*, and *écart* is now used in a different sense. Also a space (V) as defined in Fréchet's thesis and as defined now are quite different types of spaces. There is thus an element of danger in attempting to prove a theorem by combining theorems from two different papers by Fréchet, as there is always the possibility that a word has been used in different senses in the two papers. Fréchet has realized for some time the need for a volume which would enable the reader to start with the elements of the subject of abstract spaces and to become familiar in a general way with the present state of the subject, a volume furthermore which would preserve a uniform notation and which could safely be referred to in order to settle vexatious questions concerning the logical relations between abstract spaces of various types.

The present volume fulfills these requirements admirably, in the opinion of the reviewer. It is not a textbook, in the sense that a textbook should contain a complete detailed exposition of the subject matter. It is rather to be classified as a reference book, as it contains an outline of the principal ideas and theorems, with references to the literature for the reader interested in the details of any particular phase of the subject.

The book can be read with profit by anyone with a knowledge of the theory of functions of a real variable. For those only slightly interested in abstract sets, we especially recommend the "Introduction." Very few will fail to obtain some new ideas from it. Those who wish to make a more detailed study of abstract sets will appreciate the copious references to the literature and the frequent remarks concerning unsolved problems and possible directions for further research.

After a short preface, the book proper consists of four main divisions, which we shall describe in turn. The first main division, the "Introduction," is devoted principally to describing that branch of mathematics known as general analysis and to explaining why a study of abstract spaces must precede a study of general analysis. General analysis is defined as the study of relations between two elements of any nature whatsoever, one of which plays the part of the independent variable, the other that of the dependent variable, or functional. In other words, it is the study of the relation $y=f(x)$, where x and y are arbitrary elements and not necessarily numerical quantities as in classical analysis. (For example, x may stand for a differentiable function of a real variable t, and y for the derivative of this function with respect to t.) The functional y is defined for all values of x lying in a certain *range* or *space*, and we are thus led to a study of abstract spaces—an abstract space being defined as a set of elements, called *points*, all of the same nature which is either unknown or voluntarily ignored. In order to determine the differential of y, it is necessary to know the variation of y when x remains "near" a fixed point x_0 in this space. That is, some idea of "nearness," of neighborhood or limiting point or distance, must be attached to the space in which x lies, before differentiation of y is possible. Thus before introducing the idea of differentiation into general analysis it is necessary to study abstract spaces of the types considered in the latter part of the book. The remaining three divisions of the book are to be considered as merely a preamble to general analysis, which the author hopes to develop in detail in a later volume.

The second main division of the book is Part 1, Section 1, which is devoted to a generalization of the notion of number of dimensions of a space. Much work has been done in recent years on the subject of the number of dimensions of a set of points considered as a space, the most successful treatment probably being that due to Menger and Urysohn. The generalization given in the present volume is due to Fréchet and is based upon his 1910 paper in the Mathematische Annalen. Fréchet's definition has certain advantages and certain disadvantages over that of Menger and Urysohn, the main difference being that Fréchet

considers the number of dimensions as being a property of the space as a whole (see however pp. 111–113), while Menger and Urysohn consider it as a property of a point of the space. For instance, a straight line and a circle are each of dimension 1 according to the Menger-Urysohn definition, but according to Fréchet's definition the line is of dimension 1 and the circle of dimension greater than 1 and less than 2.

In the third main division (Part 1, Section 2) are discussed a number of loosely related topics, the most important being the subject of Fréchet's spaces (D), or metric spaces, as Hausdorff calls them. These are spaces in which there exists a definition of the "distance" between each pair of points of the space, which definition is subject to certain light restrictions. In this division are also given a number of examples of spaces of an infinite number of dimensions, in continuation of the topic discussed in the preceding division of the book.

The fourth main division (Part 2) is devoted to a discussion of the various types of abstract spaces which have been studied by Fréchet. The most general of these is called by Fréchet a topological space. This is a space in which is given merely some rule for finding the set of limit points (points of accumulation) of a given subset of the space. Very little, if anything, can be proved concerning a space of so general a type, because a relation between two subsets of the space does not imply any relation at all between the corresponding sets of limit points. But if we impose the condition that every limit point of a set E is also a limit point of every set which contains E, then the space becomes a space (V), and in a space (V) a remarkably large number of theorems hold true. To quote a few of the more elementary ones: (1) The set of points common to two closed sets is closed; (2) There exists an irreducible closed set containing a given set; (3) The sum of two connected sets having a point in common is connected; (4) If C is a connected set which has a point in common with a set E but is not a subset of E, then C has a point in common with the frontier of E.

Other spaces described in this division of the book are: accessible spaces, or spaces (H), which can be defined by means of a property due to Professor E. R. Hedrick, and are therefore designated by his initial; spaces (L); spaces (E); spaces (S); Hausdorff spaces, which are the same as the spaces which Hausdorff calls topological spaces; regular spaces; normal spaces; completely normal spaces; and finally, spaces (D). The logical relations between these different types of spaces are studied, and for most of them a set of theorems is given which are true in that space but are not true in a space of more general type.

The apparent purpose of this hierarchy of spaces is to enable one to determine the most general space in which a given theorem holds true. Research of this particular type appears to the reviewer to be valueless if it results in merely multiplying the number of spaces to be considered. It can be carried to the absurd extremity of having a countable infinity of distinct spaces in each of which some particular theorem (and any theorem logically equivalent to it) holds true, but in which no other theorem holds true unless it also holds true in some more general space.

The book closes with two short notes by Fréchet reprinted verbatim from the Comptes Rendus, a bibliography, an index, and a table of contents.

The reviewer regrets to state that although the contents of this book are of great mathematical value, the printing and editing of the book are not up to the same high standard. There are numerous misprints, none of them serious to be sure, but all of them annoying. Furthermore, there is a serious disagreement between the text and the table of contents concerning the division of Part 2 into chapters and sections. HARRY MERRILL GEHMAN

A Short Table of Integrals, By B. O. Peirce. Third Revised Edition. Ginn and Company, Boston, 1929. 156 pages, $1.48.

The familiar brown volume which for years has lain within easy reach on every mathematician's desk is now to be replaced by an attractive green pebbled binding.

In this new edition of the indispensable Table of Integrals, Professor Osgood has retained the arrangement and the serial numbering of the integrals practically unaltered, so that no confusion can arise as between the old volume and the new. Two or three formulas have been replaced by more useful ones, and in a number of other cases the integrals have been recast in such a way as to avoid possible ambiguities and misinterpretations; notably, integrals 49, 50, 229, 230, 298, 300, 314, 315, 316, 319, etc. Special attention is given to the correct determination of principal values of multiple-valued functions, and a brief preface defines these principal values for the most common functions.

The tables at the end of the book have been somewhat amplified by the inclusion of more extensive tables of the hyperbolic functions, and of tables of square roots, taken from E. V. Huntington's *Handbook of Mathematics for Engineers*. R. A. J.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3382. *Proposed by J. H. Butchart.*

Given a fixed conic, other than a circle, and the right to draw parallels, construct with the straight edge alone the mean proportional to two segments of a straight line.

3383. *Proposed by S. A. Corey, Des Moines, Iowa.*

A bag contains $(m+n)$ balls, each of which may be either black or white with equal probability, m being the number of balls of one color and n being the number of balls of the other color. A white ball is dropped into the bag, and then a ball is drawn out at random and found to be white. What is now the chance that of the original balls n were white and m black?

3384. *Proposed by Nathan Altshiller-Court. University of Oklahoma.*

A circle (C) touches an equilateral hyperbola (H) in A and passes through the diametric opposite B of A on (H). (1) Prove that the third point common to the two curves is the diametric opposite of A on (C). (2) The lines joining A to the ends of the diameter of (C) perpendicular to AB are parallel to the asymptotes of (H). (3) If a line through A meets the two curves again in P and Q, show that BP, BQ are equally inclined to BA.

SOLUTIONS

3325 [1928, 261]. *Proposed by Paul Capron, U. S. Naval Academy.*

Two circles, S_1 and S_2 with centers O_1 and O_2 intersect at A; O_1A meets S_2 at K_2, O_2A meets S_1 at K_1. Show that the triangle K_1AK_2 is similar to the triangle O_1AO_2.

Solution by E. G. Olds, Carnegie Institute of Technology.

Extend AO_2 to meet S_2 at T_2 and AO_1 to meet S_1 at T_1. Then AK_2T_2 and AK_1T_1 are right triangles and are similar, having an acute angle of the one equal to an acute angle of the other. Therefore,

$$AK_2/AK_1 = AT_2/AT_1 = 2AO_2/2AO_1 = AO_2/AO_1.$$

Also $\angle K_2AK_1 = \angle O_2AO_1$. Therefore the triangles K_2AK_1 and O_2AO_1 are similar.

Also solved by Nathan Altshiller-Court, H. W. Brown, Rufus Crane, R. A. Johnson, L. S. Johnston, J. H. Neelley, A. Pelletier, Paul Wernicke, F. L. Wren, and Margaret M. Young.

3326 [1928, 321]. *Proposed by J. J. Ginsburg, Student, Cooper Union.*

Prove that, if R, r, $2s$ are the circumradius, inradius, and perimeter of a triangle, its altitudes satisfy the cubic

$$2Rx^3 - (s^2 + r^2 + 4Rr)x^2 + 4rs^2x - 4r^2s^2 = 0.$$

Solution by Boyd C. Patterson, Hamilton College.

Let a, b, c be the sides of a triangle and α, β, γ the corresponding altitudes. Then from plane geometry we have:

$$(1) \qquad a\alpha = b\beta = c\gamma = 2rs\,; \qquad (2) \qquad bc/\alpha = ac/\beta = ab/\gamma = 2R\,;$$

$$(3) \quad r^2 s = (s-a)(s-b)(s-c).$$

By combining (1) and (2) we obtain the following results:

$$abc = 2Ra\alpha = 4Rrs\,; \qquad 2R\alpha\beta\gamma = (b\beta)(c\gamma) = 4r^2s^2\,; \qquad 2R\alpha(b\beta) = bc(2rs)$$

and, hence

$$2R(\alpha\beta + \beta\gamma + \gamma\alpha) = 2rs(c + a + b) = 4rs^2\,; \quad 2R(\alpha + \beta + \gamma) = bc + ca + ab.$$

Expanding (3) and inserting the above values where needed, we find

$$r^2s = s^3 - (a + b + c)s^2 + (ab + bc + ca)s - abc$$
$$= -s^3 + 2Rs(\alpha + \beta + \gamma) - 4Rrs.$$

Hence $2R(\alpha+\beta+\gamma) = r^2+s^2+4Rr$, and the desired equation is

$$2Rx^3 - (r^2 + s^2 + 4Rr)x^2 + 4rs^2x - 4r^2s^2 = 0.$$

Note by the Editors. If z is the altitude corresponding to the side x, then $zx = 2rs$ and the required equation easily follows from the equation obtained [1924, 502] in the solution of problem 3059 [1924, 101].

Also solved by C. A. Barnhart, E. H. Clarke, P. S. Dwyer, F. A. Lewis, R. E. Lowney, A. Pelletier, N. Petroff, C. H. Smiley, Guy Stevenson, C. J. Stowell, Paul Wernicke, Roscoe Woods, F. L. Wren, and the Proposer.

3327 [1928, 321]. *Proposed by J. H. Neelley, Carnegie Institute of Technology.*

Bisect a given line segment by means of compasses only.

Solution by Roscoe Woods, University of Iowa.

The solution of this problem is made to depend upon the following problem, i.e., to construct by means of compasses only the inverse A' of a given point A with respect to a given circle $O(B)$. [The notation $O(B)$ means a circle whose center is O and whose radius is OB.] For the case $OA > OB$ we proceed as follows. Describe the circle $A(O)$; it cuts $O(B)$ in the points C and D which are real and distinct. Describe the circles $C(O)$ and $D(O)$ to meet in A'. Then A' is the inverse of A with regard to $O(B)$. For by symmetry $OA'A$ is a straight line. Also the triangles OAC and OCA' are both isosceles by construction, and have a common base angle at O; they are therefore similar and $OA/OC = OC/OA'$, or $OA \cdot OA' = OC^2$, which proves that A and A' are inverse points.

In the problem at hand, let P and Q be the extremities of the given line segment. Describe the circle $Q(P)$ and then in succession the circles $P(Q)$, $R(Q)$, $S(Q)$ to meet $Q(P)$ in the points R, S, P' respectively, so that P, R, S, P' are successive angular points of a regular hexagon inscribed in $Q(P)$. We have now only to construct the inverse X of P' with respect to the circle $P(Q)$. Then X is the midpoint of PQ, for $PX \cdot PP' = PQ^2$ by construction; and since $PP' = 2PQ$, we have immediately that $2PX = PQ$.

For possible constructions with compasses only, consult H. P. Hudson's *Ruler and Compasses*, (Longmans, Green, and Co., 1916). For the original literature on this subject see Mascheroni's *Geometria del Compasso* (Pavia, 1797), Adler's *Zur Theorie der Mascheronischen Constructionen*, (Wiener Sitzungsberichte, 1890), and Hobson's *On geometrical constructions by means of the compasses* (Mathematical Gazette, March, 1913).

Also solved by W. E. Buker, L. C. Mathewson, B. C. Patterson, A. Pelletier, J. Rosenbaum, Guy Stevenson, Paul Wernicke, and the Proposer.

3328 [1928, 321]. *Proposed by Paul Wernicke, Washington, D. C.*

Let A, B, C, D be four points in a plane no three of which are collinear. If AC, BD, produced, meet in F, and AD, BC meet in G, prove that

$$(AF/FC)/(AG/GD) = (BG/GC)/(BF/FD).$$

Solution by the Proposer

Consider the triangle ABC and the point D. The lines joining D to the vertices meet the opposite sides in G, F, E, and it follows from a known theorem that

$$(AE/EB)(BG/GC)(CF/FA) = +1.$$

In a similar manner the triangle ABD and C give

$$(AE/EB)(BF/FD)(DG/GA) = +1.$$

From these two equations it follows at once that the proposed equality is true both as to sign and magnitude.

Also solved by P. S. Dwyer, Nobuichi Kobora, A. Pelletier.

3330 [1928, 321]. *Proposed by J. B. Reynolds, Lehigh University.*

A uniform elastic thread of natural length a, weight W, and coefficient of elasticity e lies within and is attached to one end of a smooth tube. The tube is rigidly attached at this end to a thin vertical rod with which it makes an angle θ. Find the length of the elastic thread in steady motion under a constant angular velocity n about the vertical rod and locate the center of gravity of the thread. θ is measured clockwise from the tube to the vertical.

Solution by the Proposer

Consider a length Δs of the thread at P, a point distant s from the point of attachment, the tension at the ends of Δs being $T+\Delta T$ and T in the sense of increase of s. Let w be the weight per unit length at P. Resolving along the smooth tube, we have

$$T - (T + \Delta T) = [w \cos\theta + wg^{-1}n^2 s \sin^2\theta]\Delta s,$$

from which arises the differential equation,

$$(1) \qquad -dT/ds = w\cos\theta + wg^{-1}n^2 \sin^2\theta s.$$

Suppose that in the unstretched state P moves back to the position P_0 at distance s_0 from the point of attachment. In that state let the density and cross section area be d_0 and c_0 respectively, while in the stretched state they are d and c. Suppose an increment of length Δs_0 stretches into a length Δs; then since the weight does not change, $c_0 d_0 \Delta s_0 = cd\Delta s$. But cd is w, the weight per unit length when the thread is unstretched. Hence, we have $w_0 \Delta s_0 = w\Delta s$ or

(2) $$w = w_0\, ds_0/ds.$$

By Hooke's Law, $\Delta s = (1 + Te^{-1})\Delta s_0$; whence,

(3) $$ds/ds_0 = 1 + Te^{-1},$$

from which

(4) $$dT/ds_0 = ed^2s/ds_0^2.$$

Substituting from (2) and (4) in (1), we derive

(5) $$d^2s/ds_0^2 + w_0 g^{-1}e^{-1}n^2 s \sin^2\theta = -\, w_0 e^{-1}\cos\theta.$$

If we let $L = an \sin\theta(w_0 g^{-1} e^{-1})^{1/2}$, the general solution of (5) is

(6) $$s = A \sin(Ls_0 a^{-1} + B) + C,$$

A, B, C being constants which to meet the condition of the problem and of (5) must satisfy the relations:

(7) $$L^2 C a^{-2} = -\, w_0 e^{-1}\cos\theta.$$

(8) $$0 = A \sin B - w_0 a^2 e^{-1} L^{-2}\cos\theta\,; \qquad \text{since} \qquad s = 0 \text{ for } s_0 = 0.$$

(9) $1 = ALa^{-1}\cos(L + B)$; since $ds/ds_0 = 1$ for $s_0 = a$, as is shown by (3) because $T = 0$ when $s_0 = a$.

From these relations, we derive the following values:

$$A = w_0 a^2 e^{-1} L^{-2}\cos\theta\csc B\,;$$
$$B = \operatorname{arc\,cot}\left[\tan L + eLw_0^{-1}a^{-1}\sec L\sec\theta\right]\,;$$
$$C = -\, w_0 a^2 e^{-1} L^{-2}\cos\theta.$$

Now the length of the stretched thread, s_a, is given by (6) for $s_0 = a$, and is, therefore,

$$s_a = w_0 a^2 e^{-1} L^{-2}\cos\theta\left[\csc B \sin(L + B) - 1\right].$$

Expanding and substituting the value of $\cot B$, we find that this may be put in the form

(10) $$s_a = Wae^{-1}L^{-2}\left[(\sec L - 1)\cos\theta + eLW^{-1}\tan L\right].$$

There are checks which we may apply to (10). If $L \to 0$ because $\theta \to 0$, the thread is vertical and $s_a = Wae^{-1}(\frac{1}{2} + eW^{-1})$, whence the extension $s_a - a = \frac{1}{2}Wae^{-1}$, as it should. If $L \to 0$ because $n \to 0$, the tube is still but inclined at an angle θ

to the vertical, and we have $s_a = Wae^{-1}(\frac{1}{2}\cos\theta + eW^{-1})$ and the extension $s_a - a = \frac{1}{2}Wae^{-1}\cos\theta$. If $\theta = 90°$, in this case $s_a = a$ which is correct. Again if in (10), $\theta = 90°$, the tube rotates in a horizontal plane and $s_a = aL^{-1}\tan L$, as it should.

To determine $\bar{s}$, the distance from the point of attachment to the center of gravity of the stretched thread, we consider the moving mass as concentrated at its center of gravity; then resolving along the smooth tube, we have

$$T_0 = Wg^{-1}\bar{s}n^2\sin^2\theta + W\cos\theta = eL^2a^{-1}\bar{s} + W\cos\theta, \tag{11}$$

in which T_0 is the tension at the point of attachment.

Now $ds/ds_0 = 1 + Te^{-1}$, and $T = T_0$ for $s_0 = 0$. From (6) we find for $s_0 = 0$,

$$ds/ds_0 = We^{-1}L^{-1}\tan L\cos\theta + \sec L,$$

and therefore

$$T_0 = WL^{-1}\tan L\cos\theta + e(\sec L - 1). \tag{12}$$

Equations (11) and (12) give for $\bar{s}$, which locates the center of gravity,

$$\bar{s} = aWe^{-1}L^{-2}(L^{-1}\tan L - 1)\cos\theta + a(\sec L - 1)L^{-2}. \tag{13}$$

In (13), if $L \to 0$ because $\theta \to 0$, $\bar{s} = \frac{1}{2}a + \frac{1}{3}aWe^{-1}$ for the thread when vertical. If $L \to 0$ because $n \to 0$, $\bar{s} = \frac{1}{2}a + \frac{1}{3}aWe^{-1}\cos\theta$. If $\theta \to 90°$ so that the thread is revolving in a horizontal plane, $\bar{s} = a(\sec L - 1)L^{-2}$.

Also solved by William Hoover.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor and Mrs. E. B. Van Vleck will sail from San Francisco on August 15, for a trip around the world. Professor Van Vleck retired from active teaching in June, 1929.

Professor George B. Birkhoff, of Harvard University, has been elected correspondent of the Paris Academy.

Professors A. A. Michelson, of the University of Chicago, and R. A. Millikan, of the California Institute of Technology, were awarded gold medals by the Society of Arts and Sciences at a meeting in New York City, February 22, 1929.

The King of Italy has conferred the decoration of Officer of the Crown of Italy upon Dr. Raymond Pearl, director of the Institute of Biological Research of Johns Hopkins University.

The American Institute of Electrical Engineers has made the following awards: the Edison gold medal for achievement in electrical science to Dr. F. B.

Jewett, for his pioneer research and development work in connection with the theory and practice of voice transmission; the Lamme medal to Mr. Allan Bertram Field, "for the mathematical and experimental investigation of eddy current losses in large slot-wound conductors in electrical machinery."

Princeton University has received from its class of '87 the sum of $200,000 to endow the chair of astronomy held by Professor H. N. Russell, a member of that class. The class also presented a portrait of Professor Russell to the University.

A symposium on theoretical physics is being conducted at the University of Michigan, from June 24 to August 16, 1929. The following courses of lectures will be delivered: by E. A. Milne, of the University of Oxford, "Problems in astrophysics, and vector and tensor methods in statics and dynamics"; by K. P. Herzfeld, of Johns Hopkins University, "Statistical mechanics"; by Leon Brillouin, of the University of Paris, "Quantum statistics"; by Edward Condon, of Princeton University, "Introduction to quantum mechanics"; by P. A. M. Dirac, of the University of Cambridge, "Advanced quantum mechanics"; by D. M. Dennison, of the University of Michigan, "Band spectra."

Assistant Professor H. M. Gehman, of Yale University, has been appointed head of the department of mathematics at the University of Buffalo.

Mr. W. J. Hazard has been promoted to an assistant professorship of mathematics at the University of Colorado.

Miss Marie Litzinger has been promoted to an assistant professorship of mathematics at Mount Holyoke College.

Dr. Edward D. McAllister, who has been assistant professor of mathematics and physics at the University of Oregon during the year 1928–29, will continue the work there next year.

Assistant Professor A. D. Michal, of the Ohio State University, has been appointed associate professor of mathematics at the California Institute of Technology.

Professor W. E. Milne, of the University of Oregon, who has been on leave as professor of mathematics at Stanford University this year, will return to the University of Oregon next fall. During his absence Dr. H. C. Hicks has served as assistant professor of mathematics at the University of Oregon. Dr. Hicks has recently been elected professor of mathematics and aeronautics at Texas Technological College.

Associate Professor L. L. Smail, of Lehigh University, has been promoted to a professorship of mathematics.

Dr. P. A. Smith has been promoted to an assistant professorship of mathematics at Barnard College, Columbia University.

Dr. W. H. Taylor has been appointed head of the department of mathematics at Alabama College.

Dr. W. J. Trjitzinsky has been promoted to an assistant professorship of mathematics at Lehigh University.

Professor James V. Uspensky of the Russian Academy of Sciences has been appointed professorial lecturer at the University of Minnesota for the summer quarter and part of the spring quarter. During the spring quarter he lectured on "Integration in finite form". He will lecture on the theory of numbers in the first session of the summer quarter and on the theory of probability in the second session.

Mr. Charles H. Vehse, of Brown University, has been appointed assistant professor of mathematics at the University of West Virginia.

Dr. Louis Weisner has been promoted to an assistant professorship of mathematics at Hunter College.

Professor G. T. Whyburn, recently appointed at Johns Hopkins University, has been appointed Guggenheim Fellow for next year, and will have leave of absence from Johns Hopkins University.

The following appointments to instructorships in mathematics are announced:

Barnard College, Dr. Lulu Hofmann, Mr. H. W. Raudenbush.

Brown University, Mr. G. N. Carmichael, Mr. K. G. Fuller of Northwestern University.

Professor W. H. Sherk, of the University of Buffalo, died in January, 1929.

Doctorates in 1928

The following forty-nine doctorates with mathematics or mathematical physics as major subject were conferred by American Universities during 1928; the university, month in which the degree was conferred, minor subject (other than mathematics), and title of dissertation are given in each case if available:

A. A. ALBERT, Chicago, August, *Algebras and their radicals and division algebras.*

H. E. ARNOLD, Yale, June, *The rational space quintic curve of the second species and its relation to the rational plane quartic curve.*

MAY M. BEENKEN, Chicago, June, *Surfaces in five-dimensional space.*

T. C. BENTON, Pennsylvania, June, *On continuous curves which are homogeneous except for a finite number of points.*

A. H. BLUE, Iowa, July, *On the structure of sets of points of classes one, two, and three.*

G. B. BRIGGS, Princeton, June, *On types of knotted curves.*

P'EI YUAN CHOU, California Institute of Technology, June, theoretical physics, *The gravitational field of a body with rotational symmetry in Einstein's theory of gravitation.*

L. W. COHEN, Michigan, June, *On subsets of separable metric space homeomorphic with subsets of the linear continuum.*

H. A. DAVIS, Cornell, June, physics, *Involutorial transformations belonging to a linear complex.*

H. A. DOBELL, Cornell, February, industrial organization, *On the geometry of the triangle.*

D. C. DUNCAN, California, May, astronomy, *Rational quintic curves autopolar to a finite number of conics.*

J. M. EARL, Minnesota, July, physics, *Polynomials of best approximation on an infinite interval.*

JOHN J. GERGEN, Rice, June, *Generalized lacunae, On Taylor's series admitting the circle of convergence as a cut, On accessible points on the boundary of a three dimensional region.*

A. O. HICKSON, Chicago, August, *An application of the calculus of variations to boundary-value problems.*

E. L. HILL, Minnesota, June, major, physics, minor, mathematics, *Quantum mechanics of the rotational distortion of spin multiplets in molecular spectra.*

ROSA L. JACKSON, Chicago, August, *The boundary-value problem of the second variation for parametric problems in the calculus of variations.*

R. L. JEFFERY, Cornell, June, physics, *The sequences of functions which define a definite integral containing a parameter.*

MARIE M. JOHNSON, Chicago, August, *Tensors of the calculus of variations.*

B. W. JONES, Chicago, June, *Representation by positive ternary quadratic forms.*

E. G. KELLER, Chicago, August, *On the origin of a planet from a ring system.*

G. H. KEULIGAN, Johns Hopkins, June, *Vibrations of an elongated U-bar.*

M. S. KNEBELMAN, Princeton, June, *Collineations and motions in generalized spaces.*

MARK KORMES, Columbia, January, *On basis sets.*

LINCOLN LAPAZ, Chicago, August, *An inverse problem of the calculus of variations.*

W. T. MACCREADIE, Cornell, February, physics, *On the stability of the motion of a viscous fluid.*

MORRIS, MARDEN, Harvard, June, *On the location of the roots of the jacobian of two binary forms and of the derivative of a rational function.*

W. L. MOORE, Illinois, June, mathematical physics, *On the geometry of the Weddle surface.*

D. C. MORROW, Chicago, June, *The determination of all quaternary quadratic forms which represent every positive integer.*

F. W. PERKINS, Harvard, February, *On the oscillation of harmonic functions.*

J. W. PETERS, Johns Hopkins, June, *Invariants of sets of points under inversion.*

O. J. PETERSON, Michigan, June, *On the rational plane quintic with three cusps.*

C. G. PHIPPS, Minnesota, July, physics, *Problems in approximation by functions of given continuity.*

ALLIE W. RICHESON, Johns Hopkins, June, *Pentagons inscribed in circles.*

W. C. RISSELMAN, Minnesota, July, physics, *Approximation to a given function by means of polynomials in another given function.*

V. B. ROJANSKY, Minnesota, August, major, physics, minor, mathematics, *The Stark effect of hydrogenic atoms in the new quantum mechanics.*

W. E. ROTH, Wisconsin, June, applied mathematics, *A solution of the matrix equation* $P(X)=A$.

C. A. RUPP, Chicago, June, *An extension of Pascal's theorem to space of r dimensions.*

N. E. RUTT, Pennsylvania, June, *Concerning the cut points of a continuous curve when the arc curve contains exactly n independent arcs.*

S. A. SCHELKUNOFF, Columbia, May, *On certain properties of the metrical and generalized metrical groups in linear spaces of n dimensions.*

A. A. SHAGHOIAN, California, May, analytical mechanics, *Solution of homogeneous linear difference equations by means of infinite determinants.*

C. D. SMITH, Iowa, February, *On generalized Tchebycheff inequalities in mathematical statistics.*

F. E. SMITH, Catholic, June, physics and philosophy, *The triangles in- and circum-scribed to the triangular symmetric rational quartic curve.*

DAN SUN, Chicago, August, *Projective differential geometry of quadruples of surfaces with points in correspondence.*

E. L. THOMPSON, Chicago, June, *Systems of two differential equations from the Lie group standpoint.*

MORGAN WARD, California Institute of Technology, June, physics, *The foundation of general arithmetic.*

MARIE J. WEISS, Stanford, June, *Primitive groups which contain substitutions of prime order p and of degree 6p or 7p.*

C. O. WILLIAMSON, Chicago, August, *Stability of an airplane with rotating propeller.*

D. W. WOODWARD, Pennsylvania, June, *Two dimensional analysis situs, with special reference to the Jordan curve-theorem.*

KO-CHUEN YANG, Chicago, August, *Various generalizations of Waring's problem.*

A Congress of Mathematicians of Slavic Countries will be held at Warsaw, Poland, on September 23–27, 1929. The Congress will have sections in the following subjects: (1) Foundations of mathematics, history, didactics; (2) Arithmetic, algebra, analysis; (3) Point set theory, topology and applications; (4) Geometry; (5) Rational mechanics, applied mathematics. Those who desire to take part in the Congress are asked to indicate this to the Secretary of the Congress where a proper registration blank can be obtained. The Congress will be held under the presidency of Professor W. Sierpinski. The address of the Secretary is Politechnika, Gabinet Matematyczny, p. 72, Warszawa (Pologne), ul. Polna 3.

The Chauvenet Prize

In the year 1925, the ASSOCIATION established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a liberal cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

The retiring President of the Association, Professor W. B. Ford, has given an additional endowment for this prize whereby it will hereafter be awarded every three years. The next award, however, will be in December, 1929, for the period 1925-1928 inclusive.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Thirteenth Summer Meeting of the Association, Boulder, Colorado, August 26-27, 1929.

Fourteenth Annual Meeting, Des Moines, Iowa, December 31, 1929, January 1, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled.

ILLINOIS, Carthage, Ill., May 3-4.
INDIANA, Culver Military Academy, May 3-4.
IOWA, Fairfield, Iowa, April 26-27.
KANSAS, Topeka, Kansas, February 2.
KENTUCKY, Lexington, Ky., April 13.
LOUISIANA-MISSISSIPPI, Lafayette, La., April 12-13.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, May 4.
MICHIGAN, Ann Arbor, Mich., March 16.
MINNESOTA, St. Paul, Minn., May 11.
MISSOURI, Kansas City, Mo., November 16.
NEBRASKA.
OHIO, Columbus, Ohio, April 4.
PHILADELPHIA, University of Pennsylvania, November 30.
ROCKY MOUNTAIN, Greeley, Colo., April 12-13.
SOUTHEASTERN, Macon, Ga., April 19-20.
SOUTHERN CALIFORNIA, University of Redlands, March 9.
TEXAS, Houston, Texas, Jan. 26.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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COLLEGE ENTRANCE REQUIREMENTS IN GEOMETRY

A proposal has been made to the College Entrance Examination Board that it should modify its requirements so as to bring about the more extensive introduction of courses including an appreciable amount of solid geometry in the first year of geometry, in place of a part of the plane geometry ordinarily taught. In response to a request from the Board, a committee has been appointed by the Mathematical Association of America and the National Council of Teachers of Mathematics to discuss the feasibility of the proposal. The membership of the committee is as follows:

Miss Gertrude E. Allen, University High School, Oakland, Cal.
C. M. Austin, High School, Oak Park, Ill.
Ralph Beatley, Graduate School of Education, Harvard University, Cambridge, Mass.
Walter F. Downey, English High School, Boston, Mass.
Mrs. Elizabeth L. Hall, East High School, Rochester, N. Y.
J. O. Hassler, University of Oklahoma, Norman, Okla.
Dunham Jackson (Chairman), University of Minnesota, Minneapolis, Minn.
C. N. Moore, University of Cincinnati, Cincinnati, Ohio.
W. D. Reeve, Teachers College, Columbia University, New York, N. Y.
Edwin W. Schreiber, University High School, Ann Arbor, Mich.

The primary function of this committee is not to draw up detailed recommendations for the proposed new requirement, but to discover if there is sufficient interest in the project on the part of colleges and schools to justify the Board in proceeding with a careful study of it.

The inclusion of any significant amount of new material in the first-year course clearly implies the elimination of much that has been regarded as of genuine importance. The question is not whether the existing course can absorb the additional material, but whether it can clear a place for it without sacrificing its own essential character.

Those who believe in the existing courses in geometry at all will agree that the pupil ought to carry away with him:

An adequately comprehensive knowledge of geometric ideas, facts, and processes;

An intimate acquaintance with the nature of deductive reasoning, as applied not only to detached items of argument, but also to the sustained building up of an extensive and coherent logical structure;

Familiarity with the independent use of deductive reasoning through the study of substantial "originals";

Some facility in the application of geometrical knowledge in the world of experience.

It is suggested that it may be found possible to preserve these essentials, with considerably more liberal recognition than has been customary hitherto of the principle that an elementary course need not aim at the final articulation of *all* the facts that it embraces into a single logical framework. It may be possible to arrive at a readjustment of emphasis which will admit some of the important ideas of three-dimensional geometry in the first year, and at the

same time bring geometry closer to the rest of mathematics and to the other sciences.

While the present committee does not aim at a detailed working out of the project, it invites discussion by individuals and groups of teachers who may be interested. It especially desires opinions from teachers who have experimented with a one-year combined course in plane and solid geometry. Communications may be addressed to the chairman or to any member of the committee.

THE NOVEMBER MEETING OF THE MISSOURI SECTION

The twelfth annual meeting of the Missouri Section of the Mathematical Association of America was held at the Junior College of Kansas City, Kansas City, Missouri, on Saturday, November 17, 1928, in connection with the meeting of the Missouri State Teachers' Association. The session was held at 10:00 A.M. and was presided over by Professor W. A. Luby, Chairman of the Section.

The attendance was about twenty-one, including the following seven members of the Association: B. F. Finkel, W. A. Luby, Arria Murto, A. D. Pierson, H. L. Rietz, R. A. Wells, and Jessica M. Young.

The 1929 meeting will be held in St. Louis, Mo., at the time of the meeting of the Missouri State Teachers Association on November 16, 1929.

The following three papers were presented at the meeting:

1. "On the chi-square test of the closeness of agreement of theoretical and observed frequencies," by Professor H. L. RIETZ, University of Iowa.

2. "The mathematics teacher and the history of mathematics," by Professor U. G. MITCHELL, University of Kansas.

3. "Numerical differentiation and mechanical quadrature as astronomical tools," by Professor J. M. YOUNG, Washington University.

Due to the fact that Professor Wahlin was marooned at Bloomington by the flood, the following paper was read by title:

4. "Quadratic number fields," by Professor G. E. WAHLIN, University of Missouri.

JESSICA M. YOUNG, *Acting Secretary*

THE SIXTH ANNUAL MEETING OF THE MICHIGAN SECTION

The sixth annual meeting of the Mathematical Association of America was held at Ann Arbor, Michigan, on Saturday, March 16, 1929, in conjunction with the Michigan Academy of Science, Arts, and Letters. Professor R. W. Clack called the meeting to order and presided until Professor L. C. Plant arrived. Fifty-seven persons registered, but there were others in attendance. Thirty-five members of the Association who attended the meeting are as follows: N. H. Anning, J. W. Baldwin, W. D. Baten, Wm. M. Borgman, J. W. Bradshaw, J. B. Brandeberry, R. W. Clack, C. C. Craig, S. E. Crowe, W. M. Davis, W. W.

Denton, L. C. Emmons, C. M. Erikson, J. W. Glover, V. G. Grove, T. H. Hildebrandt, L. A. Hopkins, L. C. Karpinski, W. S. Kimball, Theodore Lindquist, C. E. Love, R. E. Lowney, G. A. Miller, C. N. Mills, H. L. Olson, L. C. Plant, G. Y. Rainich, L. J. Rouse, T. R. Running, R. H. Schoonover, E. W. Schreiber, R. C. Shellenbarger, E. R. Sleight, G. G. Speeker, and C. C. Wagner.

The following papers were presented during the morning meeting:

1. "Report on placement tests at Michigan State College," by Professor S.E. Crowe, Michigan State College.

2. "Cissoids of certain curves," by Mr. W. M. Davis, Albion College (introduced by Professor E. R. Sleight).

3. "On certain inequalities in the theory of definite integrals," by H. J. Bushey, University of Michigan (introduced by Professor J. A. Shohat).

4. "Enrollment in mathematics classes in junior colleges," by Professor R. C. Shellenbarger, Bay City Junior College.

5. "Are our requirements for certification of high school teachers adequate?" by Professor L. C. Emmons, Michigan State College. Discussion by Professor R. W. Clack, Alma College and Professor G. Masselink, Ferris Institute.

6. "A generalization of Cauchy-Riemann equations," by H. C. Chang, University of Michigan (introduced by Professor G. Y. Rainich).

7. "Application of the theory of probability to the evaluation of certain definite integrals," by W. D. Baten, University of Michigan.

At the luncheon at noon in the Ladies' dining room of the Michigan Union, Professor L. C. Plant, Chairman, made a few remarks on the requirements now demanded of teachers of mathematics in the high schools in the State of Michigan. There was a general expression of opinion from those present that a task well worth the efforts of the Association for the next year would be a study of the requirements for teachers of mathematics in the state with recommendations for higher standards.

The following officers were elected for the ensuing year: Professor R. C. Shellenbarger, Chairman; Professor L. A. Hopkins, Secretary-treasurer; Professor C. C. Richtmeyer, Member of the Executive Committee.

The afternoon session of the Association convened at two o'clock when by special invitation Professor G. A. Miller of the University of Illinois delivered an address on the subject, "Profitable use of errors in the history of mathematics." This paper was discussed by Professor L. C. Karpinski of the University of Michigan. An abstract of Professor Miller's follows: He compared the correction of errors in the mathematical literature with the destruction of weeds on a farm and emphasized the fact that such work is constructive as well as destructive. He said that both the student of mathematics and the mathematical investigator have to correct many errors in their views as their insight is enlarged, and hence the training in the correction of errors is valuable to both. Most errors to be corrected arise from misconceptions on the part of the student and not from mistakes in the literature; but a training in the detection of the latter naturally tends towards a clearer view as regards the former. He emphasized the fact

that many of the statements found in historical accounts require modification as our knowledge of the subjects concerned is extended, and he said that just as the fourth figure of a four-place logarithm table is not expected to be always in formal agreement with the fourth figure in a larger table, so a brief historical statement should not always be expected to be in formal agreement throughout with a longer account.

Among the special questions considered by him were the following: Is it true that rational fractions were first used as numbers by the later Greeks and that the quadratic equation was completely solved by them? He pointed out that in the work of Ahmes rational fractions were given as solutions of linear equations and that the number system of the Greeks was not sufficiently extensive to solve the general quadratic equation in the modern sense of this term. He also called in question the term "Hindu-Arabic numerals" and pointed out that some of the latest investigations seem to tend towards a European origin of these numerals. He emphasized the usefulness of the history of mathematics and pointed out that as our knowledge grows we naturally desire more complete historical information, so that the room for work along this line seems to be unlimited. In particular, he noted that we know little in regard to the history of the graphical solution of the quadratic equation when the roots are imaginary.

LOUIS A. HOPKINS, *Secretary*

FOURTEENTH ANNUAL MEETING OF THE OHIO SECTION

The fourteenth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, April 4, 1929, in connection with the meetings of the Ohio College Association. Chairman E. H. Clarke presided over the afternoon session, and Dean T. M. Focke over the evening session.

Fifty-nine persons registered attendance, among whom were the following forty-nine members of the Association: J. P. Albert, R. B. Allen, W. E. Anderson, Grace M. Bareis, I. A. Barnett, H. M. Beatty, H. A. Bender, L. T. Black, H. Blumberg, R. L. Borger, R. S. Burington, W. D. Cairns, E. H. Clarke, R. Crane, P. S. Dwyer, T. M. Focke, C. A. Garabedian, B. C. Glover, H. Hancock, E. J. Hirschler, F. C. Jonah, Margaret E. Jones, E. M. Justin, L. C. Knight, H. W. Kuhn, A. C. Ladner, Anna D. Lewis, C. C. MacDuffee, A. D. Michal, C. C. Morris, J. J. Nassau, Anna B. Peckham, S. E. Rasor, B. H. Redditt, F. W. Reed, C. E. Rhodes, Hortense Rickard, S. A. Rowland, R. A. Sheets, Mary E. Sinclair, G. W. Spenceley, T. S. Peterson, M. O. Tripp, J. H. Weaver, R. B. Wildermuth, F. B. Wiley, C. O. Williamson, J. B. Winslow, C. H. Yeaton,

The following officers were elected for the coming year: Chairman, S. A. Rowland; Secretary-treasurer, Rufus Crane; Member of executive committee. S. E. Rasor; Member of program committee, M. O. Tripp. It is expected that

before beginning work and to look for short cuts, also to check each operation as it is performed. Time should be allowed for checking but not for dawdling. No credit should be allowed unless answer is correct.

9. The college teacher of mathematics should do his part in the professional preparation of those candidates for high school positions who expect to teach mathematics. The teacher's course should be a review of subject matter as well as a course in the pedagogical aspects of the subject. The prerequisites for this pedagogical training should include at least one year of collegiate instruction in mathematics, including analytic geometry.

RUFUS CRANE, *Secretary*

THE EIGHTEENTH MEETING OF THE IOWA SECTION

The eighteenth annual meeting of the Iowa Section of the Mathematical Association of America was held with The Iowa Academy of Science at Parsons College, Fairfield, Iowa, on April 26 and 27, 1929.

The attendance was about forty, including the following eighteen members of the Association: E. W. Chittenden, L. M. Coffin, Julia T. Colpitts, N. B. Conkwright, Marian E. Daniels, Annie W. Fleming, Dora E. Kearney, F. M. McGaw, J. F. Reilly, H. L. Rietz, B. D. Roberts, Fred Reusser, E. R. Smith, J. S. Turner, L. E. Ward, C. W. Wester, Roscoe Woods, C. C. Wylie. Sessions were held Friday afternoon and Saturday morning, the Section chairman, Professor Wester, presiding. Dinner was enjoyed together Friday evening, at which time Professor Colpitts gave a very interesting account of "The Bologna Congress." Officers were elected for 1929–1930 as follows: Chairman, E. W. Chittenden, University of Iowa; Vice-chairman, L. M. Coffin, Coe College; Secretary-treasurer, J. F. Reilly, University of Iowa. The program consisted of seventeen papers, as follows:

1. "On Stieltjes integrals and their geometric interpretation," by E. W. Chittenden, University of Iowa.

2. "Fitting an n-dimensional surface to an equation of the form: $X_1=F(X_2, X_3, X_4, \cdots, X_n)$," by A. E. Brandt, Iowa State College, by invitation.

3. "A note on Leibnitz's theorem for the differentiation of the product of two functions," by J. F. Reilly, University of Iowa.

4. "A torsion problem in curvilinear coordinates," by E. W. Anderson, Iowa State College, by invitation.

5. "Approximate solution of the problem of Mr. Anderson," by D. L. Holl, Iowa State College, by invitation.

6. "Remark on the existence theorem for the differential equation $dy/dx=f(x, y)$," by N B. Conkwright, University of Iowa.

7. "An examination of the remainder after n terms of Fourier series from the point of view of contour integration," by N. B. Conkwright.

8. "Each diagonal of a parallelopiped passes through the centroids of two diagonal triangles," by J. S. Turner, Iowa State College.

9. "A property of coaxial circles," by J. S. Turner.

10. "The Bologna Congress," by Miss Julia Colpitts, Iowa State College.

11. "Mathematics and common sense," by C. W. Wester, Iowa State Teachers College.

12. "Some properties of a class of curves," by Roscoe Woods, University of Iowa.

13. "Some mistakes of the amateur mathematicians," by E. R. Smith, Iowa State College.

14. "A comparison of the distribution curves of variance and of standard deviation," by H. L. Rietz, University of Iowa.

15. "An evaluation of periodicals from a mathematician's point of view," by E. S. Allen, Iowa State College.

16. "The diurnal path of the end of a shadow," by C. C. Wylie, University of Iowa.

17. "The interest rate on a loan repaid by installments," by C. C. Wylie.

Abstracts of these papers follow:

1. Professor Chittenden's paper contained an elementary discussion of the properties of Stieltjes integrals in terms of a three dimensional representation.

2. In his paper, Professor Brandt set forth a method of fitting a surface to an n-dimensional equation, using least squares and assuming all relations linear. The true nature of the functional relation is approximated by graphic methods and the coefficients determined by least squares.

3. In his paper, Professor Reilly developed a formula for expressing the n-th integral of the product of two functions in terms of the integrals of the functions, analogous to Leibnitz's formula for differentiation.

4. Professor Anderson showed how the torsion problem for a right prism whose boundary consists of arcs of orthogonal parabolas could be effected by means of conjugate cylindrical coordinates, characterized by the relation $2Z = W^2$. Methods were indicated for computing the torque and the stresses.

5. Using Raleigh's approximate method, which consists in assuming a suitable stress function, containing an arbitrary parameter and satisfying all the static boundary conditions on the contour, Professor Holl exhibited a solution of the above torsion problem. He determined the arbitrary parameter by the condition that the energy of deformation is equal to the work of the torquing couple.

6. In his first paper, Professor Conkwright made an elementary application of the method of successive approximations to establish the uniqueness of the solution of the differential equation on a certain interval.

7. In his second paper, Professor Conkwright indicated a method for expressing the remainder in certain cases as the remainder after n terms of the Fourier series for $f(x) \equiv x$, plus a contour integral of relatively simple form.

8. In his first paper, Professor Turner proved that in a tetrahedron $OABC$, if planes through A, B, C, parallel to the opposite faces meet in O', then OO' will pass through the centroid of the triangle ABC.

9. Professor Turner, in his second paper, showed that if E is a limiting point of two circles, and if any chord AB of one circle cuts the other in C, then EC meets the circle AEB in a point which lies on a fixed circle.

10. Professor Colpitts, in her paper, gave first a few points on the history of mathematics at Bologna. A book on that subject, which was given to each member at the Congress, was shown. In it is a photograph of a page from a manuscript in the University of Bologna in which Scipione dal Ferro, a professor at Bologna, gives correctly the rule for the solution of the cubic. This was some years before Cardan published the "Ars Magna."

At the opening meeting of the Congress, Professor Birkhoff as representative of the foreign mathematicians gave the reply to the address of welcome. General meetings were held in mornings and sectional meetings in afternoons. The papers were in either Italian, French, German, or English.

About eight hundred active and three hundred associate members were at the Congress, making the attendance considerably larger than at any other.

The social life was very enjoyable. Members were entertained at three receptions, a very fine concert, and a luncheon. One day was given up to excursions. The one to Ravenna was particularly enjoyable.

At the last meeting which was held at Florence, Professor Birkhoff spoke on some mathematical elements of art. The next Congress will be in Switzerland in 1932.

12. A class of curves was suggested to Professor Woods by the parametric equations of a curve generated when one curve is rolled upon another. They may be defined as the common intersection of a system of concentric circles and a system of lines passing through their common center. The author represents these curves parametrically and derives the conditions for some of their singularities.

13. Professor Smith's departmental mail bag gives a very good glimpse into the activities of a large number of amateur mathematicians. This paper, based on a number of unsolicited communications, shows the attitude of many persons towards a group of famous problems in mathematics and the errors which are generally made by such persons in attempting to obtain solutions.

14. In this paper by Professor Rietz, a discussion is given by both graphical and numerical methods of the distribution curves for variances and standard deviations obtained from small samples. It is obvious that the two sets of curves are very closely related because the standard deviation is the square root of the variance, but the differences in general appearance of the curves is such that for small samples the practical statistician would almost surely feel that the distribution of standard deviations tends more towards symmetry and normality than the distribution of variances. Moreover, this feeling is justified by a comparison of the measures of skewness of the curves.

15. This paper by Professor Allen was a report on the investigation of the relative importance of magazines for mathematicians, based upon a complete study of the citations made in the 1928 volumes of nine of the leading mathematical journals of the world.

16. In his first paper, Professor Wylie stated that neglecting parallax, refraction, and the sun's motion in declination, the paths, shown by a model of an equatorial telescope, are all conic sections. If the sun goes below the horizon during the 24 hours, the path is a hyperbola including the straight line at the equinoxes as a limiting case. This means that in the torrid and temperate zones the path is always hyperbolic.

In polar regions, the path is a hyperbola if the sun sets, a parabola if the midnight sun is just on the horizon, and an ellipse if the sun is above the horizon throughout the 24 hours. At the pole itself the paths are circular, the circle becoming infinite in radius at the equinoxes.

17. Assuming that each interest payment should be figured on the money the borrower has in use, and that each installment should include payment of the interest to date, and a balance to be applied to reduction of the principal, it follows that the interest rate on an installment loan is that given by annuity tables.

Professor Wylie discussed a few plausible and rather common methods of figuring interest rates on such loans, giving rates from about half the correct up to double the correct. A new approximate interest formula was presented, by which the actual interest rate can be figured almost as quickly as the simple interest when no tables are at hand. The formula is given in a paper which is now ready for publication.

J. F. Reilly, *Secretary*

THE TENTH ANNUAL MEETING OF THE ILLINOIS SECTION

The tenth annual meeting of the Illinois Section of the Mathematical Association of America was held May 3 and 4, 1929, at Carthage College, Carthage, Illinois under the chairmanship of Professor A. E. Gault of Bradley Polytechnic Institute, Peoria, Illinois.

The attendance was thirty-one including the following twenty-four members of the Association: Edith I. Atkin, S. F. Bibb, O. E. Brown, C. E. Comstock, A. E. Gault, R. M. Ginnings, Lois Griffiths, Mabel Heren, Byron Ingold, Nellie M. Johnston, B. W. Jones, E. C. Kiefer, W. C. Krathwohl, Martha P. McGavock, C. I. Palmer, H. P. Rogers, Theresa Renner, C. H. Smiley, G. T. Sellew, H. E. Slaught, C. J. Stowell, Mildred Taylor, C. A. Van Velzer, Alice Winbigler.

The time outside of the meetings was fully employed by the entertainment which was provided through the courtesy and cooperation of Professor Van Velzer and Carthage College. An automobile trip to the thirty million dollar dam and power plant at Keokuk, Iowa, was of unusual interest. At the dinner Friday evening Dr. Wickey, president of Carthage College, was an honored guest and gave a message of welcome. During the meeting the women members of the Association were guests of the College at Denhart Hall, the women's

dormitory, and the college invited the entire group to a luncheon after the Saturday morning meeting.

In the absence of the secretary-treasurer, Professor C. N. Mills, who is on leave of absence at the University of Michigan, Miss Edith I. Atkin served as secretary-treasurer pro tem. The following officers were elected for 1929–1930: Chairman, C. A. Van Velzer, Carthage College; Vice-chairman, H. B. Curtis, Lake Forest College; Secretary-treasurer, C. N. Mills, Illinois State Normal University. The meeting of 1930 will be held at Lake Forest College, May 2 and 3.

The following program was presented:

1. "Four years of unified mathematics," by Professor C. J. Stowell, McKendree College.

2. "The determination of cometary orbits," by Dr. C. H. Smiley, University of Illinois.

3. "High school mathematics as an index to college ability," by Miss Theresa M. Renner, Registrar, Blackburn College.

4. "The representation of integers by forms in number theory," by Dr. B. W. Jones, University of Chicago.

5. "The mathematics major for the bachelor's degree," by Professor Byron Ingold, Culver-Stockton College, Canton, Missouri.

6. "A theorem on polygonal numbers," by Dr. Lois W. Griffiths, Northwestern University

7. "Organization of freshmen mathematics at Armour Institute of Technology," by Dr. C. W. Krathwohl, Armour Institute.

8. "Sets of numbers with equal sums of like powers," by Mr. O. E. Brown, Graduate Student, University of Chicago.

9. "Euclid and the boy," by Professor Martha P. McGavock, Rockford College.

Abstracts of these papers follow:

1. Professor Stowell presented a history of the reorganization of mathematics courses at McKendree College, whereby in four years the traditional freshman and sophomore courses were replaced by the unified plan, with the Griffin series for texts. The old and the new plans were compared, with the following principal conclusions:

(a) The unified course, as exemplified by the Griffin texts, succeeds best with students well grounded in entrance requirements; for other students more algebra review is desirable.

(b) The unified course presents certain definite advantages along the line of concreteness and correlation, as well as some routine advantages.

(c) The unified course is to be preferred to the traditional courses for students whose principal interest will be in the applications of mathematics.

(d) The traditional courses are to be preferred for students majoring in mathematics for the sake of teaching and research in this subject.

(e) On account of a growing demand from those intending to be teachers

the traditional separate courses are to be resumed for a time at McKendree College.

2. Dr. Smiley pointed out that the field of orbit methods is one which has interested many of the greatest mathematicians. Two of the three methods extensively used today are modifications of those introduced by Gauss and Laplace. There are both mathematical and physical difficulties inherent in the problem of orbit determination. This is especially true in the case of cometary orbits. However, because of the importance of comets in the question of cosmogony, it is very desirable that the improvement and development of orbit methods continue.

3. Miss Renner's paper was a discussion of the general qualities required for doing satisfactory work during the first year of college and a study of the high school courses, by correlating students' grades in different courses with ratings at the end of the first semester in college, to determine what courses are indicative of success in college. Is the correlation between high school mathematics and college ratings higher than that of other courses or of the general average of the entire high school course? Study to cover a period of five years and has not been completed.

4. Dr. Jones gave a brief survey of some of the more important results on the representation of integers by forms. Classical theorems concerning the integers represented by the sum of two, three, and four squares were given as well as the number of ways in which a given integer may be represented in one of these forms. He gave some of the outstanding results of Professor L. E. Dickson and his students in investigating the integers represented by ternary quadratic forms and forms of higher degree. Mention was made of some methods of finding the number of representations by certain ternary forms.

5. In this paper Professor Ingold stressed the need of uniformity among schools in regard to general, departmental, and group requirements for all majors, and the impossibility of constructing a uniform "mathematics major" under the present conditions. He also pointed out that there should be some flexibility in the major itself to allow selection of courses suited to the subsequent work of the individual student. Prospective teachers in college and high school, engineers, and those who aspire to university professorships and research work were all considered in the proposed "mathematics major."

6. If m is a positive integer, the polygonal numbers of order $m+2$ are 0 and certain positive integers, the values of $p(x)=x+\frac{1}{2}m(x^2-x)$ for $x=0, 1, 2, \cdots$. The Fermat theorem for polygonal numbers of order $m+2$ states that if A is a positive integer then there are $m+2$ polygonal numbers of order $m+2$, namely $p_1, \cdots, p_{m+2}$, such that $A=p_1+\cdots+p_{m+2}$; by the theorem of this paper it is not necessary to use $m+2$ different polygonal numbers of order $m+2$. For example, if $m=3$ the polygonal numbers of order 5 are the pentagonal numbers $0, 1, 5, 12, 22, 35, \cdots$. The Fermat theorem states that if A is a positive integer there are five pentagonal numbers $p_1, \cdots, p_5$ such that $A=p_1+\cdots+p_5$; by the theorem of this paper, if A is a positive integer there are four pentagonal

numbers $p_1, \cdots, p_4$ such that $A = p_1+p_2+p_3+2p_4$. Generally, let $n, a_1, \cdots, a_n$ be positive integers, and $p_1, \cdots, p_n$ be polygonal numbers of order $m+2$. The function $f = a_1p_1 + \cdots + a_np_n$ is said to represent A when $p_1, \cdots, p_n$ can be chosen so that $A = a_1p_1 + \cdots + a_np_n$. In this paper $m \geqq 3, a_1 + \cdots + a_n \leqq m+2$, and there are found all values of $a_1, \cdots, a_n$ such that f represents every positive integer A. This theorem is more powerful than the Fermat theorem, since it implies the latter and also reduces the problem to one in fewer than $m+2$ variables.

7. This paper describes a plan, used by the Armour Institute of Technology for a number of years, which automatically separates the freshmen who need a further review in mathematics from those better fitted to carry on college work, and which tends to remedy an inadequate preparation.

8. Some 63 published papers on this subject are treated by Professor Dickson in his history. This paper is an expository treatment of those results. The notation $x_i \overset{m}{=} y_i (i=1, \cdots, n)$ is used to represent the conditions

$$\sum_{i=1}^{n} x_i^j = \sum_{i=1}^{n} y_i^j (j = 1, \cdots, m).$$

We may roughly divide the paper into three parts: I. Development of the theory, a collection of theorems, most of which show how to set up solutions or how to pass from one to another; II. Exhibitions of the more elegant formulas; III. Exhibition of formulas giving all solutions of $x_1, x_2, x_3 \overset{2}{=} y_1, y_2, y_3$ in integers and all rational solutions of $x_1, x_2, x_3, x_4 \overset{3}{=} y_1, y_2, y_3, y_4$.

9. Professor McGavock suggests that the answer given by Euclid to the boy asking the use of mathematics was not a wise one, and she raises the question whether the reply that Hipparchus would have given would not be a better guide to the modern teacher.

EDITH I. ATKIN, *Secretary pro tem.*

REMARKS ON LINEAR EQUATIONS[1]

By AUBREY J. KEMPNER, University of Colorado

The theory of a system of m linear equations with n unknowns $(m \lesseqgtr n)$ has been completely developed. The following points, which have come up in courses on higher Algebra, refer only to details.

I. Consider n equations with n unknowns, non-homogeneous. According to the theory, if the rank r of the matrix = rank R of the augmented matrix, the system has solutions. If $r \neq R$, then necessarily $r = R-1$, and the system has no solutions.

For example, for $n=3$, only the following familiar cases can arise:
$r=3$, $R=3$: three planes intersecting in one point; one solution.

[1] Presented at the Meeting of the Rocky Mountain Section, Greeley, Colorado, April 12–13, 1929.

$r=2$, $R=3$: the point of intersection has receded to infinity, the planes form the faces of a three sided prism; no finite solution.

$r=2$, $R=2$: the cross section of the prism reduces to a point, the three planes pass through a line; every point on this line represents a solution, we say that there are ∞^1 solutions.

$r=1$, $R=2$: the common line of intersection recedes to infinity, the planes are parallel but not coinciding; no solution.

$r=1$, $R=1$: the parallel planes all coincide, every point in the plane represents a solution; we say that we have ∞^2 solutions.

$r=0$, $R=1$: the triple plane recedes to infinity; no solution.

$r=0$, $R=0$: trivial, satisfied by all points of space; we have ∞^3 solutions.

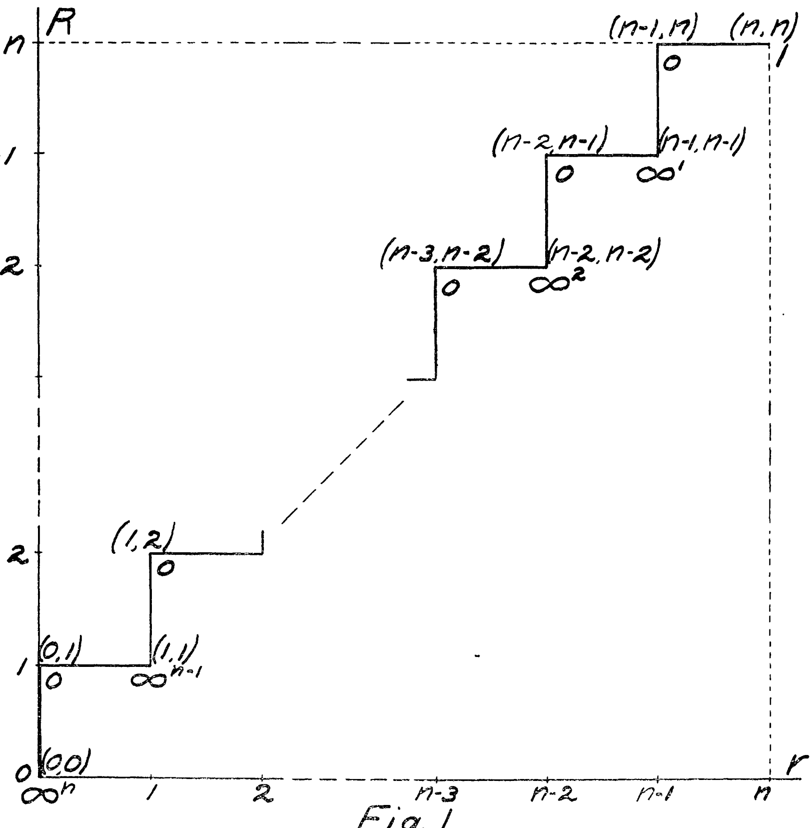

Fig. 1

We see that with increasing degree of degeneracy of the problem we have 1, 0, ∞^1, 0, ∞^2, 0, ∞^3 solutions. This is characteristic of the behavior in the general case of a non-homogeneous system of n equations with n unknowns. This well known state of affairs may be graphically represented by Fig. 1

II. A peculiarly satisfying aspect of the general theory is the uniformity of

procedure in all cases ($m \lesseqgtr n$, homogeneous or non-homogeneous). However, at the very end, this uniformity seems to be given up and the discussion is allowed to follow slightly differing lines in the homogeneous and the non-homogeneous case. The following extremely simple modification of the usual treatment may perhaps not be considered too artificial.

The customary discussion is followed up to the point of reduction of the problem to a non-homogeneous system (A) or a homogeneous system (B), each of n equations with n unknowns. In case A assume the system consistent, i.e., there are solutions. In both cases assume $r(=R)<n$. A: Let first $r=n-1$. There exists more than one solution, say $(x_1', x_2', \cdots, x_n')=(x_i')$, as we write for brevity,[2] and (x_i''). Then for any given k_1, k_2,

$$(1)\qquad \left(\frac{k_1x_i' + k_2x_i''}{k_1 + k_2}\right) = \left(\frac{k_1x_1' + k_2x_1''}{k_1 + k_2}, \cdots, \frac{k_1x_n' + k_2x_n''}{k_1 + k_2}\right), \quad k_1 + k_2 \neq 0,$$

is also a solution. If $r=n-1$, there are no further solutions. The solutions are the coordinates of all points of a line in n-dimensional space through which all the ($n-1$ dimensional) planes pass. However, since the line of intersection extends to infinity, there will also be infinite solutions. These correspond to $k_1+k_2=0$. We will carry them along, although it is understood that the finite solutions correspond to $k_1+k_2\neq 0$.

Let next $r=n-2$. Now, besides (1), there are still other solutions. Let (x_i''') be such an one. Then

$$\left(\frac{k_1x_i' + k_2x_i'' + k_3x_i'''}{k_1 + k_2 + k_3}\right),$$

k_1, k_2, k_3 arbitrary, but $k_1+k_2+k_3\neq 0$ represents the totality of finite solutions, while $k_1+k_2+k_3=0$ corresponds to infinite solutions.

This continues in an obvious fashion:

For $r=n-\mu$,

$$(2)\qquad \left(\frac{k_1x_i' + k_2x_i'' + \cdots + k_{\mu+1}x_i^{(\mu+1)}}{k_1 + k_2 + \cdots + k_{\mu+1}}\right),$$

$k_1, \cdots, k_{\mu+1}$ arbitrary, represents all finite solutions for $\sum k\neq 0$, and the infinite solutions for $\sum k=0$.

B. Let first $r=n-1$: There exists a solution (x_i') besides the trivial solution $(0, 0, \cdots, 0)$, which does not permit a geometrical interpretation in homogeneous coordinates. Then, (k_1x_i') is also a solution for all k_1. There are no other solutions. For $k_1=0$ we have the trivial solution which is usually rejected.

Next let $r=n-2$: Now there is at least one solution besides (k_1x_i'). Let such a solution be (x_i''). Then $(k_1x_i' + k_2x_i'')$ is also a solution, and all solutions are contained in this form. In homogeneous coordinates, $(k_1x_i' + k_2x_i'')$ represents the same point as does (1), provided $k_1+k_2\neq 0$. For $k_1+k_2=0$, we still

[2] Similarly, throughout.

assume that in homogeneous coordinates $(k_1x_i'+k_2x_i'')$ and (1) represent the same point. For $k_1=k_2=0$, we have the trivial solution.

This continues in an obvious fashion:

For $r=n-\mu$, the general solution is given by $(k_1x_i'+\cdots+k_\mu x_i^{(\mu)})$, which is, in homogeneous coordinates, the same point[3] as

$$\left(\frac{k_1x_i'+\cdots+k_\mu x_i^{(\mu)}}{k_1+\cdots+k_\mu}\right),$$

provided $\sum k\neq 0$. For $\sum k=0$, we still assume it to represent the same point. If each k is 0, we assume it to represent the trivial solution.

The general solutions for A and B, respectively, $r=n-\mu$ in both cases, are now given by

$$A:\qquad \left(\frac{k_1x_i'+\cdots+k_{\mu+1}x_i^{(\mu+1)}}{k_1+\cdots+k_{\mu+1}}\right),$$

$$B:\qquad \left(\frac{k_1x_i'+\cdots+k_\mu\cdot x_i^{(\mu)}}{k_1+\cdots\dotplus k_\mu}\right).$$

The only discrepancy left is the lag in the number of terms in each coordinate in (B) as compared with (A). This can be remedied by saying:

Theorem: *The general solution for the non-homogeneous case, $r=n-\mu$, is given by*

$$\left(\frac{k_1x_i'+\cdots+k_{\mu+1}x_i^{(\mu+1)}}{k_1+\cdots+k_{\mu+1}}\right).$$

Here, $k_1+\cdots+k_{\mu+1}\neq 0$ corresponds to the totality of finite solutions, while $\sum k=0$ takes care of the infinite solutions.

The general solution for the homogeneous case, $r=n-\mu$, is given by the same expression, with $k_{\mu+1}=0$, $x_i^{(\mu+1)}$ arbitrary. When $k_1+\cdots+k_{\mu+1}=0$, we ignore the denominator. When all $k=0$, we have the trivial solution.

The assumptions made concerning the interpretation of the fractions with vanishing denominators appear natural under the assumptions of the problem.

III. One may enquire what corresponds in a system of linear equations to the theorem on an equation in one unknown, $a_0x^n+\cdots+a_n=0$: each root is a function of $a_0,\cdots,a_n$, continuous in $a_1,\cdots,a_n$, and continuous for all $a_0\neq 0$.

Consider a system $a_{i1}x_1+\cdots+a_{in}x_n=b_i$, $i=1, 2,\cdots,n$, not all $b_i=0$, and of non-vanishing determinant Δ. One finds immediately:

Theorem: *Each x_i is a linear function of $b_1,\cdots,b_n$.*

Each x_i is a fractional linear function of each a_{ij}.

A simple geometrical interpretation of these results is the following:

Let b_1 be the only one of the quantities a_{ij}, b_i which is permitted to vary.

[3] It is quite customary to consider two solutions of homogeneous equations, $(x_1,\cdots,x_n)$ and $(cx_1,\cdots,cx_n)$, equivalent, on account of the identical geometrical representation of the two when interpreted as homogeneous coordinates in $n-1$ dimensional space.

Plot, in the same system of cordinates, all of the x_i against a b_1-axis. Fig. 2 then explains itself:

Letting next b_1 and b_2 vary independently, and using a three-dimensional system of coordinates, plotting each x_i as a function of b_1, b_2, one finds of course that each x_i moves in a plane. Et cetera.

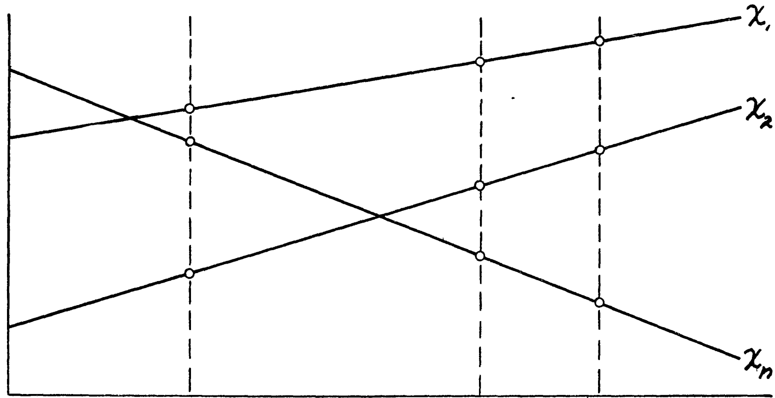

Fig. 2

If we let a_{11} be the only one of the quantities a_{ij}, b_i which is allowed to vary, and plot all x_i in the same system of coordinates as functions of a_{11}, we obtain a system of hyperbolae with horizontal asymptotes and a common vertical asymptote. A very simple transformation (of type $u=(Ca_{11}+D)^{-1}$, C, D constant) transforms all of the hyperbolae into straight lines.

The first theorem may be employed in the solution of a non-homogeneous system (n equations n unknowns, $\Delta\neq 0$). There must be at least one non-vanishing first minor; assume $\Delta_{1n}\neq 0$. Select arbitrarily two values x_n', x_n'' for x_n, and solve the two systems of $n-1$ equations in $n-1$ unknowns:

$$\sum a_{mi}x_i = b_m - a_{mn}x_n', \qquad m = 2, \cdots, n,$$
$$\sum a_{mi}x_i = b_m - a_{mn}x_n'', \qquad m = 2, \cdots, n.$$

Since these two systems differ only in the constant terms, the solution of either of them by determinants (expanding in first minors according to the column containing the terms on the right side) gives the solution of the other system with very little additional work. Let $(x_1', \cdots, x_n')$, $(x_1'', \cdots, x_n'')$, respectively, be solutions of these two systems; let $b_1'=a_{11}x_1'+\cdots+a_{1n}x_n'$, $b_1''=a_{11}x_1''+\cdots+a_{1n}x_n''$, and let k_1, k_2 be two numbers such that $b_1=(k_1b_1'+k_2b_2')/(k_1+k_2)$. Then

$$\left(\frac{k_1x_1'+k_2x_1''}{k_1+k_2}, \quad \frac{k_1x_n'+k_2x_n''}{k_1+k_2}\right)$$

is the solution of the given system.

The solution of the original problem is thus reduced to the solution of two (very closely related) systems of $n-1$ equations in $n-1$ unknowns. This method may be extended by letting more than one of the b_i vary simultaneously (instead of only b_1, as above), but the result seems of doubtful practical value.

A FORERUNNER OF MASCHERONI

By FLORIAN CAJORI, University of California

The Italian Lorenzo Mascheroni who published in 1797 a well known work on the *Geometry of the Compasses*,[1] in which all constructions are effected without a ruler and by the use only of compassses, was anticipated by 125 years, as is now first shown, by a Danish writer, Georg Mohr[2] whose book, *Euclides Danicus* of 1672, the Royal Danish Scientific Society at Kopenhagen has just published in *facsimile* and also in translation into German. The book was overlooked by mathematicians, nowithstanding the fact that there appeared two editions in 1672, one in Danish, the other in Dutch. There is nothing to indicate that Mascheroni had any knowledge of Mohr's book. The two worked independently. The *Euclides Danicus* in Dutch covers 36 pages and is a much smaller book than that of Mascheroni. It consists of two parts, the first part containing 54 constructions in Euclid's *Elements*, effected by the use of only the compasses. The last few problems call for the construction of a figure similar to a given figure and equal in area to another. An easy problem is the following: Given the line BA, to find the end point of a line twice as long. Draw an arc with A as center and AB as radius. Starting at B, apply to this arc, using the compasses, BA three times successively as a chord; the final intersection on the arc is the required point E of the straight line BAE. The second part of Mohr's book gives 24 constructions of various selected problems, ending with a rather involved problem on the erection of a sun dial.

Mohr's book is mentioned by some bibliographers but without a hint as to the nature of its content. From its title one might surmise that it was an edition of Euclid's *Elements*. Leibniz refers to him in a letter to Oldenburg (May 12, 1676) as "Georgius Mohr Danus, in geometria et analysi versatissimus." More is known of him than is indicated by the editors of the 1928 edition of his book. Cantor[3] refers to Mohr's trip to England and thence to France where, about 1676, he met Leibniz and informed him that Collins was in possession of infinite series for arc sin x and sin x. Before this, Oldenburg had mentioned Mohr, in a letter to Leibniz of September 30, 1675, as one well versed in algebra,

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[1] Lorenzo Mascheroni, *Geometria del compasso* (Pavia 1797, Palermo, 1901.)

[2] Georg Mohr, *Euclides Danicus*, Amsterdam, 1672. Mit einem Vorwort von Johannes Hjelmslev und einer deutschen Uebersetzung von Julius Pál. Udgivet af det Danske Videnskabernes Selskab, København, Hovedkommissionaer: Andr. Fr. Høst & Søn, Kgl. Hof-Boghandel 1928.

[3] Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. III, 2nd ed., p. 179.

and mechanics, who left with John Collins a manuscript on roots of $A+\sqrt{B}$ written in Dutch. This tract was forwarded to Leibniz. Born in 1640 in Copenhagen, Mohr went to Holland as a young man and did not return to Denmark except for a short time in 1682.[4]

It may be proper to introduce here some historical remarks. Ever since the dawn of abstract geometry, mathematicians have taken delight in limiting the kind and number of instruments to be used in effecting geometric constructions. Apparently urged by the ideal of simplicity and economy in instrumental equipment, the Greeks ordained that in the science of geometry only an ungraduated straight edge and compasses shall be used. This limitation has given rise to some of the most interesting and famous discussions in geometry and analysis—on the squaring of the circle, trisection of an angle, duplication of a cube, and the inscription of regular polygons in a given circle. From time to time, further instrumental restrictions have been made in the interest of speculative geometry and mathematical recreation. Inspired perhaps by a remark of Pappus, the Arabic scholar Abu'l Wefa[5] of the tenth century, in constructing the corners of the regular polyhedrons on a circumscribed sphere, set himself the condition that all constructions be effected with a ruler and a *single* opening of the compasses. The German painter Albrecht Dürer and the Italian mathematicians of the sixteenth century, including Benedetti and Tartaglia,[6] effected many constructions under these limitations. J. V. Poncelet[7] in 1822 and Jakob Steiner[8] in 1833 went a step further and showed that all constructions possible with a straight edge and compasses can be effected also by the use of a straight edge, and a circle fixed in position and drawn once for all. In 1890 A. Adler[9], of Vienna, went still further and demonstrated that all these constructions can be made by the use of only an ordinary ruler with two parallel straight edges, or only a ruler in the form of a right angle, or only a ruler in the form of a fixed acute angle. If we take cognizance also of the fact that all constructions possible by the straight edge and compasses can be effected by the compasses alone, as was shown by Mohr, Mascheroni, and later writers, then the remarkable result stares us in the face that all Euclidean constructions can be made with *any one* of the four ordinary instruments of geometric construction taken singly; viz., the compasses, or the ruler with parallel straight edges, or the ruler with a right angle, or the ruler with an acute angle.

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[4] See G. Eneström, Bibliotheca Mathemateca, (3) vol. 10 (1909–1910), pp. 71, 72; vol. 12 (1911–1912), p. 77.

[5] See Woepcke, Journal Asiatique, (5) vol. 5 (1855), pp. 241, 352–358.

[6] See Moritz Cantor, *op. cit.*, vol. II, 2nd ed., 1913, p. 566.

[7] J. V. Poncelet, *Traité des propriétés projectives des figures*, Paris, 1822, p. 187–190.

[8] Jakob Steiner, *Ueber die geometrischen Constructionen ausgeführt mittels der geraden Linie und eines festen Kreises*, Berlin, 1833.

[9] A. Adler, *Ueber die zur Ausführung geometrischer Constructionsaufgaben zweiten Grades notwendigen Hülfsmittel*, Wiener Sitzungsberichte d. Akademie d. Wiss., Math.-Naturw. Classe, vol. 99 (1890), Abth. IIa, pp. 846–859.

TRANSFORMATIONS ON CUBIC EQUATIONS

By RAYMOND GARVER, University of California at Los Angeles

The general Tschirnhaus transformation on the roots of a cubic equation (which we may take in the reduced form $x^3+a_2x+a_3=0$) may be written

$$y=m\,(x^2+k_2\,x+k_3)\ ; \tag{1}$$

the general linear fractional or homographic transformation is

$$y=\frac{ax+b}{cx+d}\qquad (ad-bc\neq 0). \tag{2}$$

It is well-known that any transformation of form (2) is reducible to one of form (1); in fact, any rational algebraic transformation on the roots of an equation of the nth degree is reducible to an integral or Tschirnhaus transformation. Conversely, it is sometimes important to determine whether or not a Tschirnhaus transformation is equivalent to a rational transformation of some particular form, for example, whether or not transformation (1) on the roots of a cubic is equivalent to transformation (2).

It seems to be generally stated that this last equivalence does exist. Burnside and Panton say, in their *Theory of Equations* (4th edition, vol. 2, page 167): "Whence it appears that the most general [Tschirnhaus] transformation of a root of a cubic may be reduced to a homographic transformation." In Niewenglowski's *Cours d'Algèbre* (5th edition, vol. 2, page 314) is found the statement: "Il résulte de là que la transformation rationnelle la plus générale, relativement à l'équation du 3e degré, est la transformation homographique." And other similar references might be given.

These statements are not entirely correct, however, since there are certain exceptional Tschirnhaus transformations on cubic equations which are not equivalent to homographic transformations. The purpose of this paper is to study these transformations, which do not seem to be mentioned in the literature.

That such exceptional transformations do exist, it is not difficult to show. Assume that transformation (1) is given, and that we wish to determine a, b, c, d (if possible) so that (2) will be equivalent to it. Equate the right-hand members of (1) and (2), clear of fractions, and reduce by using $x^3+a_2x+a_3=0$. This gives:

$$ax+b=m(ck_2+d)x^2+m(ck_3+dk_2-ca_2)x+m(dk_3-ca_3). \tag{3}$$

If (3) is to be an identity in x, corresponding coefficients on the two sides must be equal and there are three equations to satisfy. The first gives the condition $d=-ck_2$, and the others then reduce to

$$a=mc(k_3-k_2^2-a_2),\qquad b=mc(-k_2k_3-a_3). \tag{4}$$

In general, c may be chosen at will ($\neq 0$), and a, b, d are then determined.

Furthermore, the steps in the process can be reversed to show that the two transformations are actually equivalent.

However, *in case* k_2 *is a root of the cubic* $x^3+a_2x+a_3=0$, a valid linear fractional transformation is not obtained. For the denominator $c(x-k_2)$ will vanish when x takes on the value k_2. Also, as the reader can easily verify with the aid of (4), $ad-bc=0$ for this case. Hence we may state:

THEOREM 1. *The Tschirnhaus transformation (1) on the roots of the cubic* $x^3+a_2x+a_3=0$ *is not reducible to a linear fractional transformation in case* k_2 *is a root of the cubic.*

The question next arises as to what is the effect of applying one of these exceptional transformations. This is answered in

THEOREM 2. *A cubic equation can be transformed by a Tschirnhaus transformation into an equation with a double root.*[1]

For if we apply[2] $y=x^2+x_3x+k_3$ we shall have

$$
\begin{aligned}
y_1 &= x_1^2 + x_3x_1 + k_3 = -x_1x_2 + k_3, \\
y_2 &= x_2^2 + x_3x_2 + k_3 = -x_1x_2 + k_3, \\
y_3 &= x_3^2 + x_3x_3 + k_3 = 2x_3^2 + k_3.
\end{aligned} \tag{5}
$$

A similar result obviously follows if k_2 is taken equal to x_1 or x_2.

Let us now compute the transformed equation in y explicitly. I have considered this question in a previous paper[3] and shall not reproduce the details of the work here. Briefly, if m in (1) is taken equal to 1, and if k_3 is chosen as $2a_2/3$, the transformed equation will be $y^3+A_2y+A_3=0$, where

$$
\begin{aligned}
-2A_2 &= \sum y^2 = \frac{2}{3}a_2^2 - 2k_2^2a_2 - 6k_2a_3, \\
-3A_3 &= \sum y^3 = 3a_3^2 + \tfrac{2}{9}a_2^3 + 3k_2a_2a_3 + 2k_2^2a_2^2 - 3k_2^3a_3.
\end{aligned} \tag{6}
$$

For the present case, where k_2 satisfies the equation $k_2^3+a_2k_2+a_3=0$, the second equation of (6) becomes

$$
-3A_3 = 6a_3^2 + \tfrac{2}{9}a_2^3 + 6k_2a_2a_3 + 2k_2^2a_2^2. \tag{7}
$$

But we also know that the transformed equation has a double root, say r. And from its form the third root is $-2r$. Hence the equation is

$$
y^3 - 3r^2y + 2r^3 = 0. \tag{8}
$$

We may obtain an expression for r which does not involve k_2 by eliminating the latter from the equations $A_2=-3r^2$, $A_3=2r^3$, where A_2 and A_3 are replaced by

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[1] This result is clearly not obtainable with a linear fractional transformation.

[2] We hereafter take $m=1$. The roots of the given cubic are x_1, x_2, x_3; those of the transformed cubic, y_1, y_2, y_3.

[3] This Monthly, vol. 34 (1927), pp. 521–525.

their values as given in (6). The result of the elimination is a cubic equation in r, which can be written as

$$r^3 - a_2 r^2 - (D/27) = 0 \quad (D = -4a_2^3 - 27a_3^2), \tag{9}$$

or as

$$(r - \tfrac{2}{3}a_2)^2(r + \tfrac{1}{3}a_2) = -a_3^2. \tag{10}$$

We should expect r to satisfy a cubic, its three values corresponding to the three possible values of k_2.

Equations (9) and (10) are of interest because they may be used in the derivation of a number of well-known properties of the cubic equation. First, let r represent the double root $y_1 = y_2$ of (5); we then have, replacing y_3 by $-2r$, which is correct if $k_3 = \frac{2}{3}a_2$, and dividing the third equation of (5) by -2,

$$r = -x_3^2 - \tfrac{1}{3}a_2. \tag{11}$$

The other two values of r are similar, with x_3 replaced by x_1 and x_2 in turn. Hence we may say that the transformation $r = -x^2 - \frac{1}{3}a_2$ leads from the given equation $x^3 + a_2 x + a_3 = 0$ to the transformed equation (10). If we put $s = -(r + \frac{1}{3}a_2) = x^2$, (10) reduces at once to $s(s + a_2)^2 = a_3^2$; and we have found by an indirect method the equation whose roots are the squares of those of $x^3 + a_2 x + a_3 = 0$.

Next, consider the case in which the original cubic in x already has a double root. Then two of the three possible values of k_2 will lead to a transformed equation (8) with a triple root. But (8) clearly has a triple root only if $r = 0$. Referring to (9), we obtain the well-known condition $D = 0$ on the coefficients of the given cubic. The argument can be modified easily to show that this condition is also sufficient to insure that the given cubic have a double root.

Another known result of a similar nature follows immediately from a consideration of (5) and (11). Assume in (5) that $x_1 = x_3$. Then $y_1 = y_2 = y_3$, and, as above, r must be zero. Then from (11), $x_1^2 = x_3^2 = -\frac{1}{3}a_2$; and we have proved the theorem that if the equation $x^3 + a_2 x + a_3 = 0$ has a double root it is $\pm\sqrt{(-a_2/3)}$.

Finally, with the aid of elementary calculus, we can supplement the result of the paragraph before the last and obtain the usual relations between the value of D and the nature of the roots of the given cubic. If higher powers of x are eliminated between $x^3 + a_2 x + a_3 = 0$ and $r = -x^2 - \frac{1}{3}a_2$, the transformation which, as pointed out above, leads to (9) or (10), we have $x = a_3/(r - \frac{2}{3}a_2)$, which can be interpreted as a rational transformation leading from (9) or (10) back to the original equation in x. But the main point is that x is determined *rationally* in terms[4] of r; hence we can investigate the nature of the roots of $x^3 + a_2 x + a_3 = 0$ by examination of (9).

To this end, we consider the cubic polynomial[5]

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[4] At least in case $r \neq \frac{2}{3}a_2$. And this condition is satisfied if $a_3 \neq 0$.

[5] The y in (12) has no connection with the y used before in this paper. We are simply considering the graph of (12) in an (r, y) axis-system. The student is supposed to be familiar with the essential features of the graph of a cubic polynomial.

(12) $$y = r^3 - a_2r^2 - (D/27).$$

Three cases must be distinguished: (1) a_2+, $D-$; (2) a_2-, $D-$; (3) a_2-, $D+$. In all cases (12) has its maximum and minimum values at $P_1(0, -D/27)$ and $P_2(\frac{2}{3}a_2, a_3^2)$. In case (1) both of these points are above the r-axis, and the graph of (12) will cut the r-axis in one and only one point, which will of course be to the left of the y-axis. The same thing is true in case (2); the only reason for distinguishing it from (1) is that now P_2 is to the left of P_1, and is the maximum point, whereas in (1) it is to the right of P_1 and is the minimum point. Combining these two results, we may say that when D is negative equation (9) and hence the equation $x^3+a_2x+a_3=0$ has but one real root. In case (3) the minimum point P_1 is below the r-axis, while the maximum point P_2 is still above it; a construction of the graph shows that it now cuts the r-axis in 3 real points. That is, when D is positive, equation (9) and hence the equation $x^3+a_2x+a_3=0$ has three real roots. The other possibility, $D=0$, has already been treated above by a purely algebraic method.

ON NOVEL MAGIC SQUARES

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1. Magic Squares are defined as squares that are divided into n^2 smaller squares in which are inserted numbers in arithmetical progression (usually the numbers from 1 to n^2) so that the sum of the n numbers in any row, column, or diagonal is constant. These squares have been known from the earliest times and in many countries magic properties have been ascribed to them. In recent years there have been many investigations concerning these squares and the methods of their construction, and some interesting and ingenious squares have been constructed by different writers. However, in all these squares with the various combinations and permutations of the numbers within them, their common characteristic remains unaltered: the sum of the n numbers in the rows and columns and the diagonals is always the same constant for the given square.

In my research into these squares I came to notice in a certain combination of the numbers a peculiar feature that led me afterwards to construct a new type of magic squares with different properties than those already known. I present here the new kind of squares and the method of their construction, as I think that they may be of interest and value as a contribution towards the research into the properties of magic squares.

2. *Features of the New Square.* In Fig. 1 is represented a magic square of 7 of the ordinary type where the numbers from 1 to 49 are so arranged that the sum of the seven numbers in any row, column, or diagonal is equal to 175. These numbers can be rearranged in many different ways so that the constant sum will always remain the same, according to the usual definition of magic squares.

In Fig. 2 is represented also a square of 7, but of the new type. There the numbers from 1 to 49 are inserted, and have the following property: the sum of *any* four numbers that lie in a sub-square of four is constant, and in this instance is equal to 93. For example: $4+15+32+42=93$, $42+30+8+13=93$, $42+32+11+8=93$, etc. So also is the sum of the four numbers at the four corners of every even sub-square or rectangle; $46+28+18+1$, $28+43+8+14$, $17+47+22+7$, etc., all have the same constant sum, 93.

35	23	18	13	1	45	40
4	48	36	31	26	21	9
22	17	12	7	44	39	34
47	42	30	25	20	8	3
16	11	6	43	38	33	28
41	29	24	19	14	2	46
10	5	49	37	32	27	15

Fig. 1

26	46	28	44	24	43	27
17	4	15	6	19	7	16
40	32	42	30	38	29	41
10	11	8	13	12	14	9
47	25	49	23	45	22	48
3	18	1	20	5	21	2
33	39	35	37	31	36	34

Fig. 2

32	28	46	14	18	7	39
17	22	3	36	31	43	10
34	26	48	12	20	5	41
16	23	2	37	30	44	9
35	25	49	11	21	4	42
15	24	1	38	29	45	8
33	27	47	13	19	6	40

Fig. 3

This *four-number-constant* is the basic characteristic of this type of squares. In all squares of this kind, of whatever size they may be, only the sum of these four numbers is looked for. But unlike the ordinary magic square, this sum, while it remains constant for a given arrangement of the numbers in a given square, is not constant for the given square. In the same square the numbers can be arranged in a different order so as to give a different constant sum. This is shown in Fig. 3, where the four-number-constant is 99, instead of 93 as in Fig. 2. It will be proved later that in every odd square it is possible to obtain *nine* different constants. These are the two basic features of the new type of squares that distinguishes them from the old ones.

3. *Method of Construction.* In the arithmetical series $1, 2, 3, \cdots, 2m-2, 2m-1$, the sum of any two terms equidistant from the two ends respectively is equal to $1+(2m-1)$, or $2m$, and the sum of any two such pairs is evidently $4m$. When we divide the same series into two groups of m and $m-1$ terms, viz. $1, 2 \cdots m-1, m$; and $m+1, m+2, \cdots, 2m-2, 2m-1$; then the sum of each pair in the first group will be $m+1$, and the sum of any such pair in the second group will be $3m$. The sum of one pair of the first group and one pair of the second group will then be $4m+1$. Again, if we divide the original series into two groups of $m-1$ and m terms respectively, viz. $1, 2, \cdots, m-2, m-1$; and $m, m+1, \cdots, 2m-2, 2m-1$; then the sum of any pair in the first group will be m and that of the second group $3m-1$; and the sum of two such pairs will be $4m-1$. Thus we arrive at three different sums for two pairs of numbers in the same series, viz. $4m+1$, $4m$, and $4m-1$. Since for our purpose it is imposed that the terms in each sub-group shall constitute an arithmetical progression in themselves, it is obvious that no other subdivision of the original series into two such groups is possible.

We shall illustrate the foregoing by an example: In the upper row of Fig. 4 the numbers 1 to 7 are written in any order. In the second row the numbers are written to supplement the upper numbers so as to form pairs as described above, i.e., their sum shall be 8. These two rows are the *basic* rows of the square. In the rows that follow below, the *basic* rows are repeated cyclically. It can be seen at a glance that any two adjacent numbers (or numbers that are separated by an *even* number of rows) in any column form a pair and hence the sum of any four numbers that lie at the four corners of a square will be constant, because they are composed of two pairs with a constant sum.

3	7	6	4	5	2	1
5	1	2	4	3	6	7
3	7	6	4	5	2	1
5	1	2	4	3	6	7
3	7	6	4	5	2	1
5	1	2	4	3	6	7
3	7	6	4	5	2	1

Fig. 4

2	7	1	5	3	6	4
3	5	4	7	2	6	1
2	7	1	5	3	6	4
3	5	4	7	2	6	1
2	7	1	5	3	6	4
3	5	4	7	2	6	1
2	7	1	5	3	6	4

Fig. 5

5	3	4	2	6	1	7
6	1	7	2	5	3	4
5	3	4	2	6	1	7
6	1	7	2	5	3	4
5	3	4	2	6	1	7
6	1	7	2	5	3	4
5	3	4	2	6	1	7

Fig. 6

To construct the square shown in Fig. 5, we divide the numbers 1 to 7 into two groups of 4 and 3 numbers respectively, i.e. into the two series 1, 2, 3, 4 and 5, 6, 7; and we write the numbers of the first group in any order in the *odd* places in the first row, and the numbers of the second group in the *even* places in the same row, also in any order. In the second row the numbers of each group are written to supplement the corresponding numbers of the same group. The rows are then repeated cyclically as in the preceding example. Here, too, every four-number-square is composed of two pairs with a constant sum, but different from that in Fig. 4. The square in Fig. 6 is constructed in the same manner as that of Fig. 5, but with the sub-groups 1, 2, 3 and 4, 5, 6, 7 instead, which results in a different four-number-constant.

These are the three fundamental constructions of this type of squares. Evidently the numbers may be written in columns instead of in rows; and we shall denote them as vertical and horizontal basic squares respectively.

In order to form a resultant square similar to that in Fig. 2, two of the basic squares are combined together, one horizontal and one vertical, each having one of the three basic constructions. This gives rise to *nine* different constant sums, as was pointed out before-hand.

In Fig. 7 is represented such a composite square. To obtain Fig. 8 from Fig. 7, the left hand digit in each cell of Fig. 7 is diminished by unity and the resulting number in the cell is thought of as a number in the septenary scale of notation. Fig. 8 then is derived by translating from the septenary to the decimal

scale. For example: the upper left hand corner cell of Fig. 7 contains the digits 25; diminishing the left hand digit by unity gives 15; 15 thought of as a number in the septenary scale is equal to $1\times 7+5=12$ in the decimal scale; so 12 is the entry in the upper left hand cell of Fig. 8. Similarly 33 of Fig. 7 is translated into 17 of Fig. 8; 24 is translated into 11; etc. The same composite square (Fig. 7) can be used to construct another resultant square by changing the right hand digit instead of the left hand one in each cell of Fig. 7 before translating to the decimal scale. This second square is shown in Fig. 9.

25	33	24	32	26	31	27
76	51	77	52	75	53	74
15	43	14	42	16	41	17
56	71	57	72	55	73	54
35	23	34	22	36	21	37
66	61	67	62	65	63	64
45	13	44	12	46	11	47

Fig. 7

12	17	11	16	13	15	14
48	29	49	30	47	31	46
5	24	4	23	6	22	7
34	43	35	44	33	45	32
19	10	18	9	20	8	21
41	36	42	37	40	38	39
26	3	25	2	27	1	28

Fig. 8

30	17	23	10	37	3	44
42	5	49	12	35	19	28
29	18	22	11	36	4	43
40	7	47	14	33	21	26
31	16	24	9	38	2	45
41	6	48	13	34	20	27
32	15	25	8	39	1	46

Fig. 9

From our foregoing investigations about the sums of the different pairs in the arithmetical series $1, 2, 3 \cdots 2m-2, 2m-1$, it can be easily shown that in any square of n numbers in a side, where n is equal to $2m-1$, the nine four-number-constants will be $2(n^2+1)$, $2(n^2+1)\pm 1$, $2(n^2+1)\pm n$, and $2(n^2+1)\pm n\pm 1$. It is also evident from the method of construction that each square is subject to a great number of permutations without altering its constant. Thus the mode of construction of the square in Fig. 4 gives rise to $(2m-1)!$ permutations. And if the composite square is a combination of two such squares, the resulting square may then be formed in $[(2m-1)!]^2$ different ways. The number of permutations in squares like those in Figs. 5 and 6 is much less, as there are only $m!(m-1)!$ permutations. In practice, however, some of these combinations will prove impossible.

Squares of this type may also be constructed so as to have added features besides those already described. We shall give only two examples: In Fig. 10, the sum of any two pairs that lie in a broken horizontal line equals 52, the constant of the square. For instance: $18+9+24+1=52$, $18+9+14+11=52$, $9+17+1+25=52$, etc. The same will hold also when the "break" in the line is filled with an even number of rows; thus: $13+12+7+20=52$, $17+10+3+22=52$, etc.

13	12	14	11	15
18	9	17	10	16
23	2	24	1	25
8	19	7	20	6
3	22	4	21	5

Fig. 10

9	4	3
2	5	8
7	6	1

Fig. 11

The square of 3, shown in Fig. 11, has the peculiar feature that, besides every square of four numbers having the constant sum of 20, every line or diagonal can be made to have this same sum by taking double its middle number. Thus: $9+1+(2\times 5)=20$, $9+7+(2\times 2)=20$, etc.

4. *The Even Square.* Those who are acquainted with magic squares know that even squares present certain difficulties and that more caution is necessary in their construction. While it is comparatively easy to construct odd squares, it is not so with even squares. The same difficulty I found in attempting to construct even squares of the new type. The difficulty consists in this, that in what ever way we divide into two sub-groups, of m terms each, the series $1, 2, 3 \cdots 2m-1, 2m$, the sum of two pairs from these groups will always be $4m+2$. This renders the impossibility of construction of even squares with different constants, which is the most interesting feature of this new kind of squares. Every even square has thus only one four-number-constant, $2(n^2+1)$. It may be added in passing, that this mode of construction in some instances makes the square "magic" in the usual sense also, the rows and columns, excepting the diagonals, having a constant sum.

In Fig. 12 the four-number-constant is 130, and the sum of the numbers in each row or column is equal to 260.

1	63	3	61	8	58	6	60
56	10	54	12	49	15	51	13
17	47	19	45	24	42	22	44
40	26	38	28	33	31	35	29
57	7	59	5	64	2	62	4
16	50	14	52	9	55	11	53
41	23	43	21	48	18	46	20
32	34	30	36	25	39	27	37

Fig. 12

The lack of variety mentioned above is overcome when we consider the square as composed of several sections. This at once opens the road to many possibilities of combinations. The square is then divided into sections, each having its own constant. These constants may be different from each other, or

all alike. The square of 6 in Fig. 13 is divided into 4 sub-squares with different constants viz. 68, 70, 80, and 82; whereas all four sub-squares in Fig. 14 have their constants alike, viz. 75.

33	5	31	6	34	2
13	17	15	14	16	18
3	35	1	36	4	32
7	29	9	26	10	30
21	23	19	24	22	20
25	11	27	8	28	12

Fig. 13

2	29	3	25	4	30
21	23	20	24	22	19
26	5	27	1	28	6
9	35	8	36	10	31
14	17	15	13	16	18
33	11	32	12	34	7

Fig. 14

It is the selection of the proper numbers for the sub-division that will prove one's ability in the construction of these squares.

THE FUNDAMENTAL THEOREM IN RIGID KINEMATICS

By LOUIS BRAND, University of Cincinnati

In this note a simple proof is given for the existence of the angular velocity vector $\boldsymbol{\omega}$ in the most general motion of a rigid body by the use of the Plücker coordinates of a line. If O and P are two fixed points of a rigid body in motion, we then prove the fundamental theorem,

$$v_P = v_O + \boldsymbol{\omega} \times \boldsymbol{r} \quad (\boldsymbol{r} = \overrightarrow{OP}), \tag{1}$$

giving the velocity distribution in the body.

1. *Plücker Coordinates.* Let α be a line in space and P any one of its points of position vector $\boldsymbol{r}$ referred to a fixed origin O_1. Then if $\boldsymbol{a}$ is a vector along α and $\boldsymbol{a}'=\boldsymbol{r}\times\boldsymbol{a}$, the vectors $\boldsymbol{a}$, $\boldsymbol{a}'$ (taken in this order) are called the Plücker coordinates of the line. Obviously $\boldsymbol{a}'$ is the same for all choices of P. The line itself is denoted by $(\boldsymbol{a},\boldsymbol{a}')$; its equation is $\boldsymbol{r}\times\boldsymbol{a}=\boldsymbol{a}'$.

In order that $(\boldsymbol{a},\boldsymbol{a}')$ represent a line it is necessary that

$$\boldsymbol{a}\cdot\boldsymbol{a}' = \boldsymbol{a}\cdot\boldsymbol{r}\times\boldsymbol{a} = 0. \tag{2}$$

Conversely if $\boldsymbol{a}\cdot\boldsymbol{a}'=0$, the line is determined by choosing

$$\boldsymbol{r} = \frac{\boldsymbol{a}\times\boldsymbol{a}'}{\boldsymbol{a}^2}; \quad \text{then} \quad \boldsymbol{a}' = \boldsymbol{r}\times\boldsymbol{a}.$$

A necessary and sufficient condition that the lines $(\boldsymbol{a},\boldsymbol{a}')$ and $(\boldsymbol{b},\boldsymbol{b}')$ be coplanar is that

$$\boldsymbol{a}\cdot\boldsymbol{b}' + \boldsymbol{b}\cdot\boldsymbol{a}' = 0. \tag{3}$$

If $\mathbf{a}' = \mathbf{r} \times \mathbf{a}$, $\mathbf{b}' = \mathbf{s} \times \mathbf{b}$, this follows at once from the identity

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{r} - \mathbf{s}) = \mathbf{a} \cdot \mathbf{b}' + \mathbf{b} \cdot \mathbf{a}'.$$

2. Let O be a point fixed in a rigid body moving in any manner, and $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ non-coplanar position vectors to three of its points. Since the body is rigid $\mathbf{r}_i^2$ and $\mathbf{r}_i \cdot \mathbf{r}_j$ do not vary with the time. Hence on differentiating with respect to the time,

$$\mathbf{r}_i \cdot \dot{\mathbf{r}}_i = 0 \qquad (i = 1,2,3),$$

$$\mathbf{r}_i \cdot \dot{\mathbf{r}}_j + \mathbf{r}_j \cdot \dot{\mathbf{r}}_i = 0 \qquad (i,j = 1,2,3).$$

From (2) and (3) above we see that $(\mathbf{r}_1, \dot{\mathbf{r}}_1)$, $(\mathbf{r}_2, \dot{\mathbf{r}}_2)$, $(\mathbf{r}_3, \dot{\mathbf{r}}_3)$ are the Plücker coordinates of three lines which are coplanar in pairs. Since $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ were chosen as three non-coplanar vectors, these lines cannot be coplanar; at every instant they must therefore be concurrent in some point J. Then if $\boldsymbol{\omega} = \overrightarrow{OJ}$, the second Plücker coordinates of the lines are given by

$$\dot{\mathbf{r}}_i = \boldsymbol{\omega} \times \mathbf{r}_i \qquad (i = 1,2,3). \tag{4}$$

Now the position vector $\overrightarrow{OP}$ to any point of the body may be written

$$\mathbf{r} = l\mathbf{r}_1 + m\mathbf{r}_2 + n\mathbf{r}_3$$

where l, m, n, are constant scalars. Hence

$$\dot{\mathbf{r}} = l\dot{\mathbf{r}}_1 + m\dot{\mathbf{r}}_2 + n\dot{\mathbf{r}}_3,$$

or in view of (4)

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}. \tag{5}$$

The vector $\boldsymbol{\omega}$ defined as above is called the *angular velocity* of the body. At any instant it is independent of the choice of O. For if we write

$$\mathbf{s} = \overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = \mathbf{r}_P - \mathbf{r}_Q,$$

$$\dot{\mathbf{s}} = \dot{\mathbf{r}}_P - \dot{\ }_Q = \boldsymbol{\omega} \times \mathbf{r}_P - \boldsymbol{\omega} \times \mathbf{r}_Q = \boldsymbol{\omega} \times \mathbf{s}.$$

3. We are now in position to prove the fundamental theroem (1). If O_1, O are points fixed in space and in the body respectively,

$$\overrightarrow{O_1P} = \overrightarrow{O_1O} + \overrightarrow{OP}.$$

The time derivatives of these three vectors are precisely the corresponding terms of (1).

4. If $\boldsymbol{\omega}$ were the time derivative of a vector function, this function might be appropriately defined as the "angle." However, in general, no such vector function exists, so that the term "angular velocity" for $\boldsymbol{\omega}$ is to some extent a misnomer.

If $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and $\mathbf{i}_1$, $\mathbf{j}_1$, $\mathbf{k}_1$ are two sets of orthogonal unit vectors fixed in the

body and in space respectively, we may regard the rotation dyadic (tensor of the second order)

$$R = \boldsymbol{i}\boldsymbol{i}_1 + \boldsymbol{j}\boldsymbol{j}_1 + \boldsymbol{k}\boldsymbol{k}_1$$

as a species of generalized angle which defines the orientation of the body at any instant. In view of (5) its "velocity" is

$$\frac{dR}{dt} = \omega \times R.$$

ON COMPLEX VALUES OF A REAL PARAMETER

By BOYD C. PATTERSON, Hamilton College

1. *Introduction.* In a recent paper in the American Journal of Mathematics,[1] the writer made use of the fact that if given a curve parametrically expressed by a complex function of a real parameter, $x = x(\mu)$, we may allow μ to take complex as well as real values. For such values of μ the parametric equation gives not points on the curve but image points with respect to the curve. The equation of the curve is obtained by eliminating the real parameter from $x = x(\mu)$ and its conjugate equation $\bar{x} = \bar{x}(\mu)$. The result will be a self-conjugate equation[2] in x and $\bar{x}$:

$$f(x, \bar{x}) = 0.$$

It is the purpose of this paper to illustrate by a more elementary application than that in the above mentioned paper the effectiveness of such a parametric representation in certain instances and at the same time to further elucidate the general principle under consideration.

2. *Parametric equation of a circle.* It is known that the transformation $x = e^w$ maps the complex w-plane upon the complex x-plane in such a way as to send straight lines parallel to the real axis of the w-plane into half-rays proceeding from the origin of the x-plane; and straight lines parallel to the axis of imaginaries of the former into circles with centers at the origin of the latter.[3]

Putting $w = u + iv$ and giving u the constant value u_0 we have a parametric representation of a circle with radius $\rho_1 = \exp u_0$ and center at the origin:

$$x = \rho_1 e^{iv}.$$

Writing $x - c_1$ for x in this equation we obtain

$$x = \rho_1 e^{iv} + c_1 \tag{2,1}$$

a circle of radius ρ_1 and center c_1.

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[1] Vol. 50 (1928), pp. 553–568.

[2] The condition that the equation be self-conjugate is equivalent to the condition that the coefficients be real in a cartesian equation. Professor Morley has made extensive use of these coordinates with much success. See also Winger's *Projective Geometry*, D. C. Heath & Co., 1923, p. 324.

[3] For a complete discussion see E. J. Townsend's *Functions of a Complex Variable*, p. 126.

Elimination of the parameter v from (2,1) and its conjugate

$$\bar{x} = \rho_1 e^{-iv} + \bar{c}_1 \tag{2,2}$$

gives us the equation of the circle:

$$x\bar{x} - \bar{c}_1 x - c_1 \bar{x} + c_1\bar{c}_1 - \rho_1^2 = 0. \tag{2,3}$$

Upon replacing $\bar{x}$ by $\bar{y}$ in (2, 3), we get

$$x\bar{y} - \bar{c}_1 x - c_1 \bar{y} + c_1\bar{c}_1 - \rho_1^2 = 0; \tag{2,3$'$}$$

and the equation is no longer self-conjugate and does not represent a curve. It is rather a relation between two points, x and y, of the x-plane. For a given x, y is determined by (2,3′) and conversely. Each of these points is termed the image of the other with respect to the circle (2,3) and for that reason we call (2,3′) the image equation of the circle. Such an equation is of much broader significance than the usual cartesian equation for while the latter simply tells what points lie on the curve the former, that is the image equation, is a relation on all points of the plane. In the case at hand it is evident if we write (2,3′) in the form,

$$(x - c_1)(\bar{y} - \bar{c}_1) = \rho_1^2,$$

that x and y are inverse points with respect to a circle of radius ρ_1 and center c_1—that is, the circle (2,1) or (2,3).

In the foregoing, v has been restricted to real values. If we extend its range to complex values, writing for that purpose $v = \alpha + i\beta$, equation (2,1) takes the form

$$x = \rho_1 e^{i\alpha}\lambda^{-1} + c_1, \tag{2,1$'$}$$

where $\lambda = e^{\beta}$ is real, while (2,2) becomes

$$\bar{y} = \rho_1 \lambda e^{-i\alpha} + \bar{c}_1. \tag{2,2$'$}$$

Here we have put $\bar{y}$ instead of $\bar{x}$ as in (2,2) for equation (2,2′) is not the conjugate of (2,1′). It is easily verified that the x and y of these two equations are inverse points of the circle (2,1).

3. *Application.* Given two circles, C_1 and C_2, to find their common inverse points. Let the circles have respectively the radii ρ_1 and ρ_2 and the centers c_1 and c_2. The image equation of the circle C_2 is

$$x\bar{y} - \bar{c}_2 x - c_2 \bar{y} + c_2\bar{c}_2 - \rho_2^2 = 0. \tag{3,1}$$

Any pair of points inverse with respect to the circle C_1 is given by the two equations

$$x = \rho_1 e^{iv} + c_1, \quad \bar{y} = \rho_1 e^{-iv} + \bar{c}_1, \tag{3,2}$$

where the parameter v is complex. (Put $v = \alpha + i\beta$ and equations (2,1′) and (2,2′)

are recovered.) Substituting the values of x and y as given by (3,2) in equation (3,1) we have a function of exp iv, namely,

$$\rho_1(\bar{c}_1 - \bar{c}_2)e^{iv} + [(c_1 - c_2)(\bar{c}_1 - \bar{c}_2) + \rho_1^2 - \rho_2^2] + \rho_1(c_1 - c_2)e^{-iv}.$$

The zeros of this function are the values of v which give with the aid of (3,2) the common inverse points. The distance between the centers of C_1 and C_2 is $(c_2 - c_1)$, a complex number which we shall write $\delta \exp i\phi_0$. With this simplification the above reduces to the solution of

$$[\exp i(v - \phi_0)]^2 - 2A[\exp i(v - \phi_0)] + 1 = 0,$$

where

$$A = (\delta^2 + \rho_1^2 - \rho_2^2)/2\delta\rho_1.$$

Therefore

$$\exp i(v - \phi_0) = A \pm (A^2 - 1)^{1/2}$$

and

$$v = \phi_0 - i \log [A \pm (A^2 - 1)^{1/2}].$$

Since ϕ_0 is the direction angle of the line of centers and is constant, the above values of v obviously depend on A. We consider the two cases:

(i) When $A^2 > 1$. The quantity $A + (A^2 - 1)^{1/2}$ is a real number, say λ_0. Then $A - (A^2 - 1)^{1/2} = 1/\lambda_0$ and $v = \phi_0 \pm i \log \lambda_0$. Hence $\alpha = \phi_0$; $\beta = \pm \log \lambda_0$, giving the two common inverse points[4]

$$x_0 = \rho_1\lambda_0^{-1} \exp i\phi_0 + c_1; \quad y_0 = \rho_1\lambda_0 \exp i\phi_0 + c_1. \tag{3,3}$$

(ii) When $A^2 \leqq 1$. If $A^2 < 1$ we may put

$$A \pm (A^2 - 1)^{1/2} = \cos\theta_0 \pm i \sin\theta_0 = \exp(\pm i\,\theta_0),$$

for the left member is a complex number of absolute value 1. Here $\theta_0 = \cos^{-1} A$ is the angle included between the line of centers and that radius of C_1 drawn from its center to the point of intersection of the two circles. For v in this case has the values $v = \phi_0 \pm \theta_0$, which are real and are hence parameters of points on the circle. If $A^2 = 1$ we find $\theta_0 = 0$ or π and the circles are tangent.

The results of (i) are of course the most important. A little calculation will show that $(x_0 - c_1)$ and $(y_0 - c_1)$ have the same direction angle as the line of centers, namely ϕ_0. Hence x_0 and y_0 lie on the line of centers. Furthermore the circle C_0 on x_0y_0 as diameter has a radius equal to $\rho_1(A^2 - 1)^{1/2}$ and center $c_0 = \rho_1 e^{i\phi_0} A + c_1$. The absolute value of the segment $(c_0 - c_1)$ is $\rho_1 A$. With this information we apply the trigonometrical law of cosines and find C_0 to be orthogonal to C_1 and to C_2.

4. *Conclusion.* The construction of these points from the viewpoint of

[4] (3.3) gives the values of x_0, y_0 for $\beta = +\log \lambda_0$. If the negative sign is taken, x_0 and y_0 are interchanged.

synthetic geometry probably has been recalled already by the foregoing analysis. The easily proved theorem that "every line through the center of a given circle cuts all orthogonal circles in two points inverse with respect to the given circle" and its converse that "all circles through inverse points with respect to a circle cut that circle orthogonally" enable us to restate the problem as follows: Given two circles, to construct a third orthogonal to both. The intersections of this orthogonal circle with the line of centers of the given circles are the points sought.

For a further application of this principle the reader is refered to the paper first cited. The analysis therein is particularly effective and as is frequently the case gives results not possible by synthetic geometry.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

DISCUSSIONS

I. INTERPRETATION OF SOLUTIONS FOR THE TIME OBTAINED BY INTERPOLATION IN THE MATHEMATICS OF INVESTMENT

BY W. L. HART, UNIVERSITY OF MINNESOTA

1. *Introduction.* In an elementary application of the mathematics of investment leading to an exponential equation, the solution can usually be obtained either by logarithmic methods or by interpolation in standard tables. It frequently happens that the interpolation method leads to results which have definite and useful interpretations, while the so-called exact logarithmic methods give only approximate results, as judged by current business practice. The purpose of this note is to emphasize the useful interpretations which can be given to the solutions obtained by interpolation in certain types of problems. The most important result considered is in connection with the amortization of debts; a convenient method[1] is formulated for determining the final partial payment in case a debt is discharged by equal periodic payments of specified size.

2. *Fundamental equations for linear interpolation.* Suppose that a table is available giving the values of a real-valued function $f(n)$ for integral values of n. Consider solving for the unknown n in an equation

$$f(n) = c, \tag{1}$$

where c is a given constant. If $c=f(k)$, where k is some integer, the solution of

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[1] This method is implicity described in *Politische Arithmetik*, by Emil Foerster (Walter de Gruyter and Company, Berlin), a book of small circulation in this country. The method deserves wider appreciation than it has been given heretofore.

equation 1 can be found by mere inspection of the given table of values of $f(n)$. If c lies between two table entries, $f(k)$ and $f(k+1)$, and if we solve for n by linear interpolation between these values, we conclude that $n=k+h$, where

$$h = \frac{[c - f(k)]}{[f(k+1) - f(k)]}\,. \tag{2}$$

On placing $c=f(n)=f(k+h)$ in equation (2), we see that our solution by interpolation assumes that, if n has the form $n=k+h$ where k is an integer and $0<h<1$, then

$$f(k+h) = f(k) + h[f(k+1) - f(k)]. \tag{3}$$

That is, by interpolation we obtain the exact solution of equation (1) under the assumption that equation (3) defines $f(n)$ when n is not an integer.

Since the right member of equation (3) is linear in h, the geometric assumption equivalent to equation (3) is that, for values $n=k+h$ between $n=k$ and $n=k+1$, the graph of $f(n)$ is a straight line joining the points $[k, f(k)]$ and $[(k+1), f(k+1)]$. Under condition (3), the whole graph of $f(n)$ would be a broken line with its corners at the points where n has integral values.

3. *An application to compound interest.* Consider the equation

$$A = P(1+i)^n, \tag{4}$$

where A is the compound amount at the end of n conversion periods, if a principal P is invested at the rate i per conversion period. Suppose that A, P, and i are given and that we solve for the unknown time n by linear interpolation in a table of values of $(1+i)^k$ for integral values of k. By use of equation (3) with $f(n)=P(1+i)^n$, we find that our interpolated value of n is the exact solution of equation (4) under the assumption that the amount A at the end of $(k+h)$ periods is given by

$$A = P(1+i)^k + hiP(1+i)^k.$$

$$A = P(1+i)^k(1+hi). \tag{5}$$

We can translate equation (5) into the following directions for finding the amount A at the end of a given time, $(k+h)$ periods, which is not a whole number of periods: *First, find the compound amount at the end of the last whole period contained in the given time; then, accumulate the resulting amount for the remaining time at simple interest at the given nominal rate.*

The value of A given by equation (5) is sometimes called the practical compound amount, and is usually employed as the working approximation to the compound amount in current business whenever the time of investment is not an integral number of conversion periods. Hence, we have arrived at the following well known result: *If the equation of compound interest is solved for the time by linear interpolation, the result is precisely the time at the end of which the practical compound amount on the given principal P equals the given amount A.*

4. *Determination of a final partial payment.* In the equation

$$A = Ra_{\overline{n}|i}, \tag{6}$$

i is the interest rate per conversion period, and A is the present value of an annuity of R payable at the end of each period for n periods. This description of equation (6) has meaning only when n is an integer, although the algebraic expression for $a_{\overline{n}|}$ is defined for all values of n. The question arises as to the interpretation which can be given to a non-integral solution for n in equation (6) if A, R, and i are given. No attempt will be made to interpret a value of n obtained by solving this exponential equation by use of logarithms. We shall suppose that n is obtained by use of linear interpolation in a table of values of $a_{\overline{k}|}$ for integral values of k. By use of equation (3) with $f(n) = Ra_{\overline{n}|i}$, we find that such a solution of equation (6) assumes that, when $n = k + h$, A is given by

$$A = Ra_{\overline{k}|i} + hR(a_{\overline{k+1}|i} - a_{\overline{k}|i}). \tag{7}$$

By use of the geometric progressions for $a_{\overline{k}|}$ and $a_{\overline{k+1}|}$, from equation (7) we obtain

$$A = Ra_{\overline{k}|i} + hR(1 + i)^{-(k+1)}. \tag{8}$$

Thus, when we solve equation (6) for n by interpolation, we are actually solving equation (8) for two unknowns k and h, where k is an integer and $0 < h < 1$. We note that the right member of equation (8) represents the present value of an annuity of R per interest period for k periods, plus the present value of a smaller payment hR due at the end of $(k+1)$ periods.

Now, consider a debt whose present value is A, which is to be discharged, principal and interest at the rate i per conversion period included, by payments of R at the end of each period. For given A, R, and i, it would be unusual if an integral number of payments R exactly sufficed to extinguish the debt A. Hence, usually, a final smaller payment $M < R$ will be necessary one interest period after the last full payment R. If A, R, and i are known, we desire to find the number, k, of regular payments R and the concluding payment M due at the end of $(k+1)$ interest periods. If we let $(M \div R) = h$, or $M = hR$, it is verified that the equation which the unknowns h and k must satisfy is identical with equation (8). Thus, in view of the discussion leading to equation (8) we can state the following conclusion:

To find the number of payments of R per interest period and the final partial payment M which will discharge a specified debt A, principal and interest included, first solve $A = Ra_{\overline{n}|i}$ for n by linear interpolation, obtaining $n = k + h$, where k is an integer and $0 < h < 1$. Then, the debt requires k payments of R each, and a final payment $M = hR$ due at the end of $(k+1)$ interest periods.

Illustration: Suppose a debt of \$8800, with interest at the rate 5%, compounded semi-annually, is to be discharged by payments of \$1200 at the end of each 6 months. We write

$$8800 = 1200a_{\overline{n}|.025},$$

whose solution, obtained by interpolation, is $n = 8.20381$. Hence, the debt requires eight payments of \$1200 each, and a final payment

$$M = (.20381)(1200) = \$244.57$$

at the end of $4\frac{1}{2}$ years. The reader can easily verify that this result is accurate to the nearest cent, by determining M by use of an equation of value.[1]

5. *A problem relating to the amount of an annuity.* Consider the equation

$$S = Rs_{\overline{n}|i}, \tag{9}$$

when S, R and i are given. Suppose that we obtain $n = k + h$ on solving for n by linear interpolation in a table of values of $s_{\overline{k}|}$ for integral values of k. By use of equation (3) with $f(n) = Rs_{\overline{n}|i}$, we find that the interpolated value of n is the exact solution of equation (9) under the assumption that, when n is of the form $n = k + h$, S is given by

$$S = Rs_{\overline{k}|i} + hR(1+i)^k. \tag{10}$$

The right member of equation (10) represents the amount of an annuity of R per interest period for k periods, plus the compound amount at the end of the term of the annuity if the additional payment hR is invested at the beginning of the term. If we solve equation (9) for n by interpolation, we are actually solving equation (10) for the unknowns (h, k), where k is an integer and $0 < h < 1$, and our result $n = k + h$ can be interpreted by reference to the preceding sentence.

Illustration: If the equation $10{,}000 = 500\, s_{\overline{u}|.025}$ is solved for n by linear interpolation, the result is $n = 14.20272$. We find that $(.20272)(500) = \$101.36$. Hence, if interest is at the rate 5% per conversion period, an initial deposit of \$101.36, plus payments of \$500 at the end of each interest period, will accumulate exactly \$10,000 by the end of 14 periods.

II. On the Limit of the Ratio of an Arc to its Chord

By H. A. Simmons, Northwestern University

1. *Introduction.* Let $y = f(x)$ be a function of one of the following types[2]: (A) one-valued, with a continuous derivative at each point of a closed interval from $x = 0$ to $x = k$ (where k is a finite constant conveniently assumed to be greater than 0) except at the point for which $x = k$, at which point $f(x)$ becomes infinite, e.g., $\lim_{x\to k} f(x) = \pm\infty$); (B) one-valued, with a continuous derivative for every $x \geqq 0$, and such that $\lim_{x\to\infty} f(x) = k_1$, where k_1 is any constant; (C)

[1] See W. L. Hart, *Mathematics of Investment*, D. C. Heath and Co., p. 83.

[2] The following examples may help one to see what are the types of functions (A), (B), (C) which we consider: the curve $y = \sec x$ comes under type (A); $y = 1/(x+1)$ comes under type (B); $y = e^x$ comes under type (C); $y = \sin x$, which neither approaches a constant nor becomes infinite as $x \to \infty$, is excluded from consideration; $r = a^\theta$, which when expressed in the form $y = f(x)$ is multiple-valued, is also excluded.

(III). Divide numerator and denominator of the second quotient of (2) by x and use the fact that lim $[(f-a)/x]=\lim f'=k$; the result follows immedietely.

(IV). The hypotheses that x approaches ∞ and $\lim f=k_1$ imply that $\lim f'=0$ as can be shown by an application of the mean value formula stated in the footnote No. 2. Hence division in the second quotient of (2) by x leads to the result.

4. *Consequences.* The functions $y=f(x)$ for which lim (s/c) as $c\to\infty$ is 1 are bounded and continuous on a closed interval from $x=0$, to $x=k$, or from $x=0$ to $x=\infty$, the meaning of ∞ as a right hand end-point being clear from§ 1. These functions therefore attain a maximum value (and the minimum value 1) on their intervals.

This result is surprising in the cases of numerous functions. For example, one would scarcely believe at first thought that s/c attains a maximum in the case of the parabolic arc $y=2(ax)^{1/2}$.

III. A Note on a Simple Theorem in Relative Velocity

By A. F. Stevenson, University of Toronto

If a closed circuit of any shape is marked out in a river flowing with uniform velocity, the circuit being fixed with respect to the bank, then a boat, whose speed in still water is constant, takes the same time to journey round the circuit, in whichever sense the circulation is completed.

This result would be suspected in advance and is easily proved; but inasmuch as no mention of it appears to be made in the text-books, it may perhaps be worth while to reproduce a proof. We can prove the result, somewhat more generally, when the flow of the river is not uniform, provided only that it is steady, and that the velocity of flow is the same (in sign and magnitude) at all points of the circuit which are at the same distance downstream.

Let V, v denote, respectively, the constant velocity of the boat in still water, and the velocity of flow at any point of the circuit. (We must, of course, suppose $V>|v|$ everywhere.) Also let ds denote an element of the circuit at a point whose downstream distance from some fixed origin is x, and let ds make an angle θ with the x-axis.

Then for the velocities of the boat relative to the circuit when traversing ds in the two directions, we easily find

$$v_1=\sqrt{(V^2-v^2\sin^2\theta)}-v\cos\theta;\; v_2=\sqrt{(V^2-v^2\sin^2\theta)}+v\cos\theta).$$

Hence the *difference* in the times taken to traverse ds in the two directions is

$$ds\left(\frac{1}{v_1}-\frac{1}{v_2}\right)=\frac{2vds\cos\theta}{V^2-v^2}=\frac{2v}{V^2-v^2}dx,$$

where dx is the projection of ds on the x-axis. The difference in times for the whole circuit is therefore $\int 2v(V^2-v^2)^{-1}dx$, taken round the circuit. But from the

hypotheses made as to v, it follows that the integrand is a single-valued function of x only; the value of the integral is therefore zero, and the theorem is proved.

We can further suppose that the direction of flow is also variable; but in this case the direction of flow and the circuit must both satisfy restrictive conditions of symmetry. In the particular case where the direction and velocity of flow are the same everywhere, we see from the above that the difference in times for an *unclosed* circuit is proportional to the downstream distance between the end points; and, whether v is constant or not, depends only on this distance, the shape of the circuit being immaterial.

IV. Recurrence Formulae Involving Bessel Functions of the First Kind

By W. O. PENNELL, St. Louis, Mo.

The object of this note is to call attention to the following formulae involving Bessel functions:

$$(1)\ (2n+1)\int_0^\theta x^n\epsilon^{\alpha x}J_n dx = \left[x^{n+1}\epsilon^{\alpha x}(J_n - \alpha J_{n+1})\right]_0^\theta + (\alpha^2+1)\int_0^\theta x^{n+1}\epsilon^{\alpha x}J_{n+1}dx,$$

$$(2)\ (2n+1)\int_0^x x^n\epsilon^{\pm ix}J_n dx = x^{n+1}\epsilon^{\pm ix}(J_n \mp iJ_{n+1}),$$

$$(3)\ (2n+1)\int_0^x x^n \cos x\, J_n dx = x^{n+1}\cos x\, J_n + x^{n+1}\sin x\, J_{n+1},$$

$$(4)\ (2n+1)\int_0^x x^n \sin x\, J_n dx = x^{n+1}\sin x\, J_n - x^{n+1}\cos x\, J_{n+1},$$

$$(5)^*(2n+1)\int_0^x x^n\epsilon^{\mp x}I_n dx = x^{n+1}\epsilon^{\mp x}(I_n \pm I_{n+1}),$$

$$(6)\ (2n+1)\int_0^x x^n \cosh xI_n dx = x^{n+1}\cosh xI_n - x^{n+1}\sinh xI_{n+1},$$

$$(7)\ (2n+1)\int_0^x x^n \sinh xI_n dx = x^{n+1}\sinh xI_n - x^{n+1}\cosh xI_{n+1}.$$

Formulae (2) to (7) are special cases of (1). In all the formulae $n > -1/2$.

Some search has been made through the standard works on Bessel functions without finding any mention of these relations and they are published with the hope that they may be of some service.

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* The notation $I_n(x)$ is used to denote $i^{-n}J_n(ix)$.

The proof of (1) is as follows: Two integrations by parts give:

$$(n+1)\int x^n e^{\alpha x} J_n(x)dx = x^{n+1}e^{\alpha x}J_n - \alpha\int e^{\alpha x}x^{n+1}J_n dx - \int e^{\alpha x}x^{n+1}J_n' dx$$

$$= x^{n+1}e^{\alpha x}[J_n - \alpha J_{n+1}] + \int e^{\alpha x}x^{n+1}[\alpha^2 J_{n+1} - J_n']dx.$$

Upon replacing J_n' by $J_n' = nx^{-1}J_n - J_{n+1}$, and transferring the resulting term in J_n to the left member of the equation, we get the recursion formula (1). (2) is obtained from (1) by giving α the values $\pm i$. (3) and (4) are obtained by taking respectively half the sum and half the difference of the two values of (2). (5) is obtained from (1) by giving α the values $\pm i$ and making θ imaginary. (6) and (7) are obtained by taking respectively half the sum and half the difference of the two values of (5).

When $2n$ is not an integer the origin is a singular point of $x^n\epsilon^{\alpha x}J_n(x)$ and in this case it would have to be understood that the path of integration is either a radial line from 0 to θ or its equivalent.

RECENT PUBLICATIONS

Edited by Roger A. Johnson, Hunter College of the City of New York.

All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Currier, C. H. and *Watson, E.* A Course in General Mathematics. New York, Macmillan, 1929. x+414 pages.

Prasad, Ganesh. Six Lectures on Recent Researches in the Theory of Fourier Series. Published by the University of Calcutta, 1928. xiv+140 pages.

" · · · in as compact and simple a form as possible, the present state of the convergence problem in the theory of Fourier series."

Hessenberg, Gerhard. Vorlesungen über Darstellende Geometrie, herausgegeben von E. Salkowski. Leipzig, Akademische Verlagsgesellschaft M. B. H. xiv+274 pages.

Schwam, W. Elementare Geometrie. Erstes Band, Die Ebene. Leipzig, Akademische Verlagsgesellschaft M. B. H. xviii+402 pages.

Two new volumes in the Hilb series "Mathematik und ihre Anwendungen in Monographien und Lehrbüchen."

Springer, Julius. Quellen und Studien zur Geschichte der Mathematik. Herausgegeben von O. Mengebauer, J. Stenzel, O. Toplitz. Abteilung B: Studien, Band 1, Heft 1, Berlin, 27 März, 1929.

Contents: Das Verhaltius von Mathematik und Ideenlehre bei Plato, von O. Toeplitz. Zur Theorie des Logos bei Aristoteles. Zur Geschichte der Babylonishen Mathematik, von O. Neugebauer. Über die Geometrie des Kreises in Babylonien von O. Neugebauer und W. Struve. Platos Einfluss auf die Bildung der Mathematischen Methode, von J. Johnsen.

Oglesby, E. J. and *Cooley, H. R.* Plane Trigonometry. New York, Prentice Hall, 1929. x+154+76 pages, with tables. $1.60.

Nevalinna, R. Le Theoreme de Picard-Borel et la theorie des fonctions meromorphes. Paris, Gauthier-Villars, 1929. viii+176 pages.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Gino Loria, Professore nell'università di Genova, *Storia delle Matematiche, Vol. I, Antichità—Medio Evo—Rinascimento. Con numerose figure nel testo.* Torino, Società Tipografico-Editrice Nazionale, 1929, 497 pages.

The appearance of a new history of mathematics by a writer as widely and favorably known as Gino Loria is to be hailed with satisfaction. Loria's aim is not to present a severely technical account of mathematical development, but to make the story readable by a popular and attractive style of exposition. Combining readability and the requirements of scientific accuracy is an undertaking in which it would be difficult to surpass the distinguished historian from Genoa.

In fifteen chapters of this, the first volume, the story is told, beginning with the number systems, the intuitive geometry and astronomy of Babylonia and Egypt, and ending with the syncopated algebras of Widman and Pacioli. Somewhat unusual chapter headings are "The Chinese Enigma," referring to the difficulty of historical research in Chinese mathematics and the variety of judgments on the achievements of Cathay, and "Geometry in the service of painting" as seen in Leonardo da Vinci and Albrecht Dürer. Of questionable propriety is "S. P. Q. R.," signifying *Senatus Populusque Romanus*, and used as the heading of the chapter on Roman mathematics.

The account on Babylonian astronomy unfortunately omits reference to the greatest achievement of observational astronomy of antiquity, namely, the discovery of the precession of the equinoxes. In histories of astronomy this great discovery has been ascribed to the Greek astronomer Hipparchus. Loria so ascribes it. But it is now proved conclusively that a few centuries before Hipparchus, the Babylonian astronomer Cidenas[1] on the banks of the Euphrates had reached this result. As regards the Egyptians, recent research is revealing more and more clearly that the early dwellers of the Nile had proceeded much further in mathematics than is indicated by Greek authors. Of this fact the Rhind papyrus was the first distinct indication. Loria refers to an Egyptian papyrus now being studied in Russia, but not yet published in full. It contains a calculation of the surface-area of a hemisphere, made 1500 years before Archimedes! It is worthy of remark that Loria overlooked an equally startling result in the Moscow Egyptian papyrus[2] containing the computation of the

[1] See Paul Schnabel in *Zeitschrift für Assyriologie*, N. S., vol. 3 (1926), pp. 1–60.

[2] *Ancient Egypt*, 1917, p. 100.

volume of the frustum of a square pyramid, at a period of 1500 or 2000 B.C.

Loria's treatment of Greek mathematics is in general of high quality. However, we were disappointed in his discussion of Zeno's arguments against motion. In the first place he fails to indicate the perennial interest in this topic as revealed even by recent writers, for instance Bertrand Russell and Henri Bergson. Nor is there a reference to Paul Tannery's brilliant interpretation of Zeno's arguments. In the second place, Loria does not make plain how profoundly Zeno of Elea influenced Greek science by indirectly inducing philosophers to advance the atomic theory, the "atom" (indivisible) being created to afford what seemed to be a safe escape from the subtle grip of Zeno's logic. It seems true also that Zeno's arguments were partly responsible for the banishment from Euclidean geometry of the notion of the fixed infinitesimal and the infinite. It is pleasing to find in Loria a heading "Arithmetical recreations of the Greeks." In fact, mathematical recreations are almost as old as mathematics itself. Witness in the Rhind papyrus Ahmes' problem, leading to a geometrical progression, and interpreted to signify: Seven persons have each seven cats, each cat eats seven mice, each mouse eats seven ears of barley, from each ear seven measures of corn may grow. How many persons, cats, mice, ears and measures altogether?

Loria's book possesses qualities which are certain to stimulate general interest in the history of mathematics.

FLORIAN CAJORI

Freshman Algebra. By James Byrnie Shaw. Thomas Y. Crowell Co., New York, 1929. 137+xi pages.

I. Shaw's *Freshman Algebra* is a text to be used primarily by women's divisions in college algebra.

The book is not intended to be used in the usual courses in algebra where the subject is applied in as many ways as time permits. It is the author's purpose to present algebra devoid of applications, stressing the beautiful and the aesthetic. Poetry, rather than utility, furnishes the background for the course.

Many topics are treated in comparatively few pages, each in a necessarily superficial manner. The subjects include integers, equations, determinants, irrationals, identities, invariants, and hypernumbers. It is to be remembered, however, that the book is not a mathematics text in the ordinary sense; it is rather a mathematical companion for a course of study in the purpose, beauty, and mission of mathematics, or, more specifically, algebra. After such a course it seems reasonable that the students might have clearer conceptions of what are the real value and place of algebra than most students acquire from the present courses.

"The book is not intended to contain everything that is said or done in the class. The author assumes that the class is in charge of a teacher who is interested enough to bring in any extra material. . . . ." The contents are chiefly explanatory, describing briefly the various topics, the fundamental operations

with each, and the type of work that can be done with each. Solution of the usual types of problems is minimized, and there are very few problems for illustration and solution by students. Although the course of study is intended to be based on literature and fine arts there seem to be very few references to these phases; this material being left to the instructor to furnish. It seems as if more specific references to poems and other literature would prove a helpful addition in guiding an instructor toward making the best use of the text and conducting a better course.

The book is novel and will be received with welcome by many who have courses where such a text can be used. The explanations are especially clear, are presented in a conversational manner, and should prove appealing to students.

E. C. Warren

II. In these days of statistical investigation into theories of education, with resultant revolutions in our college and secondary school curricula, mathematics almost more than any other single subject has received a great deal of attention. Old theories of transference of values are being discarded despite the repeated warnings of certain groups of educators steeped in the theory of discipline in mathematics and the classics. No one of course attempts to deny the utilitarian place of mathematics in the education and training of the engineer and scientist, for in all studies of such curricula, investigators feel the need for more time and more thorough training in subject matter and methods of mathematical development. But it has come to be a different story for the so called "liberal arts" group, to whom the mathematics requirement is just one more hurdle to leap in the path of credit gathering for a degree. The student in this group will probably never have occasion to use any of the actual subject matter of his freshman year, aside from some incidental work in arithmetic, of which, in most cases, he is woefully ignorant. The solution in an increasing number of institutions is to strike mathematics entirely out of the curriculum, or allow as an elective some other course.

Mathematics, however, has too extensive a background in the development of education to be so lightly discarded simply because the student does not like the usual doses he receives of it, or feels he has had enough "discipline" to enable him to tackle something more attractive. The mathematician may feel that his standing and the standing of his field in the intellectual world is endangered if we allow the ancient subject to be replaced in our curricula. But perhaps the teacher of mathematics, himself, is somewhat to blame for the whole trend of development in the liberal arts courses. Professor Shaw's book seems to the reviewer to be a big step, if not too radical, in the direction towards a solution of the difficulties sketched above. It may be dangerous, for in the hands of the inexperienced, unsympathetic, mathematical drill master it would be more a farce than some of our ordinary courses. In the hands of a genuine master of his field with sympathetic guidance, extensive mathematical training, keen interest in arts, literature, and philosophy, knowledge of the history of

developments in modern mathematics, and above all an unbounded enthusiasm for the subject himself, it will form the basis for a course of utmost value and interest for the student. I will not maintain that he will solve equations better or expand binomials more accurately, or manipulate logarithms more quickly than the student who has had the ordinary drill and routine of college algebra, although it is certainly possible. He will, however, see something of the beauty of mathematics, have some knowledge of its contribution to the advance of civilization, know something of its aesthetic qualities, which any person who aspires to be a cultured individual should feel when viewing such a sublime creation of the intellect, and the conviction that mathematics is a living, growing, vital part of the intellectual life of the world.

The author is convinced that mathematics would be more properly classed under the "fine arts." Its relation to poetry is constantly stressed with numerous examples and references. Its symmetry, design, rhythm, harmony, unity, creativeness are some of the aesthetic properties analogous to the same properties of other fields of art that are frequently pointed out and developed. Nor is the book entirely a discourse *about* mathematics, for a wealth of subject matter is presented, some of which is given in a manner wholly out of the ordinary, showing the author's individuality in methods. The approach is generally from an intuitive standpoint, encouraging the student to use his imagination to arrive at the general conclusions before they are actually stated by the author. The reviewer feels that unless the teacher in planning the course is careful to make his own background as broad as possible, the student will get just as naïve notions about mathematics as he would in an ordinary course and more likely extremely fantastic conceptions. Experience and personal limitations ought to guide the instructor as to how far he should go with the more unusual topics introduced.

The text is not long and might otherwise pass as a well written set of notes on lectures delivered by the author. The style is somewhat informal and discursive making the contact between the author and the student more potent. The first four pages give a concise account of the place of algebra in history with several references for further reading. This closes with a list of the divisions of the work, namely: arithmetic, equations, identities, invariants, operators and hypernumbers. Chapter II gives a number of theorems on divisibility of integers as examples of number theory, with the binomial coefficients as the goal and end of the chapter.

Chapter III is on elimination, bringing in naturally determinants and their properties and, as an unusual topic, Sylvester's dialytic method. Chapter IV is a discussion of the rational domain and some remarks on the connection between mathematics and other fields of creative thought. Chapter V, which is the most extensive, deals with equations in one unknown, with presentations in an individualistic manner of such topics as rational roots, Horner's method, use of continued fractions in approximating roots, symmetric functions, Galois resolvent equations, and isolation of roots by Sturm's functions. Chapter VI is

on irrational numbers and laws of exponents with logarithms introduced incidentally. Chapter VII is on identities, sums of finite series, method of differences, partial fractions (though not so named), arrangements and probabilities. Chapters VIII and IX give short introductions to the notions of groups, invariants, and quaternions, closing with a brief mention of the direction of continuation of the various topics considered.

A half dozen or so misprints were noticed. Most of them are quite obvious and can be corrected by the student. Large type was used throughout in the make-up of the book, making it mechanically easily readable. Some minor departures from usual printing were used, scarcely detracting or adding to the book's worth.

The personality of the instructor, mastei of his subject, a refined, cultured individual acquainted with philosophy and art and filled with a conviction of the place of mathematics in the make-up of a well educated person, must dominate such a course for which Professor Shaw's book would form a basis. It is perhaps most of all a sketch of some of the author's methods of treatment and an indication of how the individual instructor may vary the work to the tastes and make-up of the class. The reviewer believes that it is at least a very potent answer to the problem of the liberal arts student although inadequate for use by the inexperienced instructor with insufficient mathematical training. It should arouse perplexing questions in the mind of the student demanding some kind of intelligent exposition.

A competent, open minded teacher seeking a modus vivendi for the liberal arts freshman in mathematics might do well to conduct an experiment of value to himself if not to others. Let him take two classes of equal ability and previous training so far as can be readily determined and let him give to one class as a control a course based upon the conventional text book with the conventional material and to the other a course based on work such as Professor Shaw has sketched in his *Freshman Algebra*. Then let him draw his own conclusions from the results as he sees them, not measured by some examination of manipulative skill. Let him expect different results from the essentially different aims just as we expect different results from a student in a course in music appreciation and a student of piano technique.

CLYDE M. HUBER

First Course in the Differential and Integral Calculus. By. W. B. Ford. Henry Holt and Co., New York, 1928. 372 pages. $3.00.

This introduction to the calculus by Professor Ford of the University of Michigan follows, to quote from the preface, "in large measure the lines of development already customary in American texts on the calculus. . . . There are 2005 examples and exercises. Of these, 179 are worked in full and serve as illustrative material . . . . Of the remainder, 89 are accompanied by both the answer and a hint as to method of solution, 1132 are accompanied by the answer

only.... As to the proofs, it is well known that it is impossible in an ordinary first course in the calculus to furnish strictly vigorous proofs at all points.... The best one can do ... is to give what may be termed a 'first approximation' to a real proof. This, at least, is all that has been attempted in the present text, the student being told plainly, however, that such is the case .... Special attention has been given to the applications of the calculus to physics.... A brief treatment of the method of least squares has been given in the Appendix.... Tables at the end of the book include not only a considerable number of the more important standard integrals, but also trigonometric formulas and tables showing the sine, cosine, and tangent, correct to four decimal places, of all angles, expressed in either degrees or radians, at intervals of 10′ from 0° to 90°.... A table has been added also showing the squares, cubes, square roots, cube roots, and reciprocals of all integers from 1 to 100."

The scope and plan of the book appear from these quotations. The purpose of the author has been in our opinion fairly well carried out. Certainly any student who worked through the text would have a good knowledge of the elements of the calculus. The appearance of the book is good, with a large, clear page.

The book contains too much for a first course, if by a first course is meant a one-year course. There is a great lack of problems calling for numerical work. The principles and methods of the calculus are best brought home to beginners by continual emphasis on numerical examples. In this connection we note that the book contains no table of natural logarithms, probably a much more useful table in a calculus than a table of common logarithms; the trigonometric tables would be more useful if the functions were given for every hundredth or thousandth of a radian rather than for every ten minutes. The fundamentally important subject of formal integration seems to us to be treated rather lightly: the methods and devices over which the beginner stumbles and struggles appear largely as hints for the solution of problems and do not receive the systematic explanation and discussion which seem to us their due.

The following misprints were noted: on page 30, example 1; the answer should have a factor 2 in the denominator. On page 83, the graph for example 2 is inverted so as to represent $y=x/(x^2-1)$ instead of $y=x/(1-x^2)$. On page 87, exercise 4 is incorrect; the locus, incidentally, is composite. On page 149, fifth line from bottom, for x^1 read x_1. On page 161, exercise 3c; the answer should be half the given value. On page 180, exercise 3a; the value of e is given as 2.7182 ... instead of 2.7183.

J. K. WHITTEMORE

Plane Trigonometry. By Carl A. Garabedian and Jean Winston. McGraw-Hill Book Co., 1929. xviii+306 pages.

In looking back over this text after spending a good deal of time reading its eight chapters, I am struck with three things: the very unique style of presen-

tation, the completeness of the explanations, and something more vague which might be called the "atmosphere" of the book.

The style, called by the authors "the syllabus mode of presentation," is like a very complete set of notes; and if a student should master this style, it would certainly improve his ability to take intelligible notes in a mathematics lecture. It catches the attention and gives the impression that everything said is important, and one can easily see just what is said in any particular article. Here is a sample:

"36. *Application of trigonometry to the solution of triangles.*—Before attempting problems in plane triangles, we first *note that*

(a) Certain important facts concerning triangles should be recalled from plane geometry (Art. 13).

(b) To *solve* a triangle is to find numerical values for *all parts not given.* There are two possible solutions:

(1) the *graphical solution*, whose importance should not be underestimated; and

(2) the *trigonometric solution*, which we shall discuss fully below.

(c) A triangle can be solved provided we have *given three parts*, *at least one of which is a side* [Art. 13, (7)]. (Is the solution always unique?)

(d) In labelling the parts," etc.

The explanations and illustrative examples are unusually complete, the computations are put up in attractive form, and the graphs are beautiful with orange rulings and black curves. This careful work is commendable, and has a real influence on the attitude of the student and his working habits. I think, however, that the authors have overdone it in some places; for instance they devote sixteen pages to "Accuracy of measurements, tables, and computed results"; and the explanations are almost confusingly minute as to how to round off a number when it has too many digits, and the abbreviated methods of multiplication and division. Again, the treatment of "Graphs of the functions" begins with an article on graphing in general and plots a few conic sections, and ends up with compound curves obtained by adding several sine or cosine curves. This takes up about forty pages. It seems to me that if some of these things could be shortened and a set of tables put in their place the book would be more teachable. The authors meet this objection in the preface by saying that since we are so often called upon to give a short course the student needs a complete text all the more, especially if he is going to keep his text and use it for reference later.

In this connection, I should like to voice a thought that has been in my mind for some time. Correspondence courses are becoming increasingly popular, and here the student must indeed depend upon the text. Should we develop a special type of text for these courses? The text we are reviewing would make a very good one for a correspondence course.

The "atmosphere" of the text shows clearly that the authors love their subject and feel that no apology is needed for expecting the students to find an

interest in it. I hardly know how this idea is conveyed throughout the text unless it be by the questions that are constantly raised for discussion. The reader will notice one tucked in under a sub-topic in the extract quoted above. They never pass quietly by a difficult point like the tangent of ninety degrees, but always raise the question even though many questions are raised only to be postponed. In the introduction, after saying that trigonometry is a necessary stepping stone to higher mathematics, the authors say that it "possesses a certain beauty and significance characteristic of an abstract mathematical theory—a beauty comparable in its purity and simplicity to that of a fugue of Bach or a symphony of Beethoven." In a short conclusion, (an unusual feature), they advise the student to review the course for the sake of perspective, and tell him that he is only at the beginning of a large and interesting subject. The text is in all respects consonant with an observable tendency to let the pendulum swing back from the utilitarian toward the cultural.

DAVID F. BARROW

Short Course in Spherical Trigonometry. By Pauline Sperry. Johnson Publishing Company, vi+57 pages.

Short as is this spherical trigonometry, containing only fifty pages, we find in these comparatively few pages a geometrical setting, development of the theoretical spherical trigonometry, applications, and a brief historical sketch. These are set forth in a remarkably complete presentation of the subject, showing careful work at every step. The technique of the composition is good. One is immediately impressed with the attractive page which has good arrangement, clear print, and unusually fine drawings.

The section referring to spherical geometry is none too explicit for review on the part of the student who has studied that subject, yet is sufficiently detailed and logically presented to guide carefully through that ground the student who meets this material for the first time. Such a student would acquire, from the chapters on the sphere, a vivid understanding of the ideas prerequisite to work on the trigonometry of the spherical triangle. The clear-cut definitions and the arrangement of the theory would produce an enquiring state of mind. In this portion of the text the author has presented especially well chosen sets of thought-provoking exercises.

Condensation of theory by means of the principle of reciprocity, complete treatment of the ambiguous case, and stimulating exercises are prominent features of this small book. In the computational work good sets of exercises are given, completely covering the theory presented. The problems for application of this part of the work are both varied and refreshingly modern. The emphasis laid on checking results is very cheering.

In calling attention to many connections between plane and spherical trigonometry, the author enriches the work for the student of lesser initiative. In this fact and in other ways the book brings out what we so often optimistically

hope for, and so seldom find—a treatment of a portion of mathematics in its relation to other connecting material, not as a detached subject to be pigeonholed and separated from its close relatives.

MARY E. WELLS

Studies in the History of Statistical Method—With Special Reference to Certain Education Problems. By Helen M. Walker. Baltimore, Williams and Wilkins Company, 1929. viii+229 pages.

The study of mathematical statistics is just now attracting many students with a foundation of sound mathematical training, but without experience in any field of statistics. In even greater numbers it is gaining recruits among biologists, psychologists, economists, anthropologists and others who hope to find in mathematical statistics the solutions of some of their own special problems. To a person who is approaching this study the first great difficulty is to define the field. This is undoubtedly true of any subject but it is especially true of statistics. This results in part from its topical nature, the methods and purposes of the various topics being quite different. In part, too, it results from the important contributions made by men working in different lines, dealing with different material and often ignorant, as Dr. Walker points out, of results and terminology already in use in other fields. These forces have not only created a field which is difficult to define; they have brought about a great confusion and duplication of formulas, terminology, and notation. It is doubtless among the functions of the mathematicians who are teaching or studying statistics to coördinate the work already done in different portions of the field, to simplify and standardize notation and terminology, and to unify and define the subject. Certainly one of the most satisfactory ways of forming a clear and unified picture of the science as it now stands is by constructing a background of its historical development. For this reason I predict a warm welcome for Dr. Walker's "Studies in the History of Statistical Method."

The title of the book indicates its topical and incomplete character. The two longest chapters are those on the normal curve and correlation. The first of these is supplemented by shorter chapters on moments and percentiles. Written from the point of view of one especially interested in problems of educational psychology, there is a chapter on Spearman's "Theory of Two Factors" and another on the history of the teaching of statistics in American universities. Those who are interested in the more general aspects of statistical study will perhaps share my regret that these two chapters were included at the sacrifice of material on curve fitting, analysis of time series, graduation, least squares, sampling or some of the many other important topics of general interest which were omitted. We may hope that these will be treated in later studies.

The concluding chapter of the book is an alphabetical list of certain technical terms used in statistics together with the name of the person who introduced each term and references to its first publication. This list unfortunately does not

include symbols, and it is most complete in the domain of educational statistics. As the author suggests, this material should be of service to an investigator who proposes a new term or formula by acquainting him with the sources of terms already in use. It would have been still more useful, even to the student of educational statistics, if the terms of other branches of statistics had been included. For duplication of terms results precisely when a writer introduces a new expression for which an equivalent already exists in a part of the field with which he is not familiar. It would be ungrateful, indeed, however, not to appreciate this chapter as the nucleus of a historical dictionary of statistical terms.

Similar to this dictionary are the special bibliographies. The chapter on the normal curve ends with a bibliography of memoirs on the probable errors of frequency constants arranged according to the year of publication. There is another list of memoirs relating to measures of correlation other than the Pearson product-moment, likewise arranged by years, and with annotations amounting to abstracts of the papers. Another annotated bibliography is given for the theory of two factors. Unquestionably these special bibliographies will do more than anything else to make this book valuable as a work of reference. As is true of so many features of the book they suggest a method which, being topical itself, can be extended to topics not considered here. This goes far to answer any criticism that may be made on the score of incompleteness.

The topical method of the bibliographies is largely carried out in the text. Thus in the chapter on the normal curve, after discussions of the origin in the theory of probability and the establishment of the normal curve, we find histories of the application of the law of error to social phenomena, of the use of different units for the abscissa of the curve, of the development of tables of the error function, and of the search for probable errors of frequency constants. This method is sound, I think, and the author has done well to cover portions of the field intensively rather than to attempt a less thorough survey of a wider territory. The general bibliography should be useful, but includes only works actually consulted by the author. This gives impressive evidence of the author's background of reading, but causes some curious omissions.

The author has a keen perception of what is important. It is natural that some of her decisions should surprise us, as, just as instances, her entirely unqualified tribute to the influence of Quetelet, her judgment of the work of Jacob Bernoulli and her negative remark concerning the nature of a Type B Gram-Charlier series.

The style is easy and lively. The book is not intended as a text-book in statistical theory, but is addressed mainly to readers who are somewhat familiar with the subject. It is good to read and good to refer to. I wish that I had had it at hand during the past year while teaching a course in the mathematical theory of statistics. From now on it should be an important resource for teachers and investigators.

RAYMOND W. BRINK

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEM FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3385. *Proposed by J. Rosenbaum, Milford, Conn.*

In any pentagon, $A_1A_2 \cdots A_5$, P_1, P_2, $\cdots$, P_5 are the midpoints of the sides A_1A_2, A_2A_3, $\cdots$, A_5A_1. The points M_1, M_2, $\cdots$, M_5 are the midpoints of P_1P_3, P_2P_4, $\cdots$, P_5P_2. Prove that the lines A_5M_1, A_1M_2, $\cdots$, A_4M_5 are concurrent at a point O, such that the vectors OA_1, OA_2, $\cdots$, OA_5 form a closed polygon.

3386. *Proposed by H. E. Trefethen, Colby College.*

If the incircle passes through the centroid of a triangle, find positive integral values for the sides a, b, c.

3387. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

With the same vertex of a given triangle as center, two circles are drawn respectively orthogonal to the nine-point circle and to the conjugate circle of the triangle. Show that the ratio of the squares of the radii of the two circles is 1:2.

3388. *Proposed by S. A. Corey, Des Moines, Iowa.*

Prove that, if the tenth and higher derivatives of f vanish identically, $f(i)+f(-i) = 120f(0)+30f(r)+30f(-r)+640f(s)+640f(-s)-405f(s)-405f(-s)-324f(t) -324(-t)$, where $i=\sqrt{(-1)}$, $r=\sqrt{\frac{1}{2}}$, $s=\sqrt{\frac{1}{3}}$, $t=\sqrt{\frac{2}{3}}$, and where f is analytic. It follows that when the tenth and higher derivatives are negligibly small the indicated relation holds approximately.

SOLUTIONS

3020 [1923, 206]. *Proposed by John Nichols, Portland, Oregon.*

Rationalize $\alpha^{1/2}+\beta^{1/2}+\cdots+\mu^{1/2}+\nu^{1/2}+p=0$.

I. *Solution by C. K. Robbins, Purdue University.*

Consider $a^{1/2}+b^{1/2}+c^{1/2}+p=0$. This can be rationalized by the usual process.

Consider $a^{1/2}+b^{1/2}+c^{1/2}+d^{1/2}+p=0$. This becomes $-u+v=0$ by letting $a^{1/2}+b^{1/2}+c^{1/2}+u=0$ and $d^{1/2}+p=v$. Rationalize these two equations, the rationalized form of the first one being obtained from the previous case by replacing p by u. Then eliminate u and v from the three equations, using Sylvester's method where convenient.

It is obvious how to continue. The algebraic difficulties seem to be inherent in the problem.

II. *Solution by Otto Dunkel, Washington University.*

Set

$$x = \sum p_i^{1/2}, \quad \text{where} \quad = 1, 2, \cdots, n; \tag{1}$$

and let us assume that no one of the radicals $p_i^{1/2}$ can be expressed rationally in terms of the other radicals and the p's including p_i. This excludes the case of any p being zero, or any radical having a rational value. A rational function of any set of quantities shall mean here a polynomial in these quantities with integral coefficients, or the quotient of two such polynomials. Suppose that x satisfies an irreducible equation $f(x)=0$ of degree m where the left side of the equation is a polynomial in x with coefficients which are rational functions of the p's and the coefficient of x^m is unity. Write $x=P_i+p_i^{1/2}$, where P_i is the sum of the remaining radicals, and insert this value of x in $f(x)=0$. Separating the terms of the result containing even powers of $p_i^{1/2}$ and those containing odd powers, we may write this result $M_i+N_ip_i^{1/2}$, where M_i and N_i are rational functions of the remaining radicals and the p's. If now $N_i \neq 0$, then $p_i^{1/2}$ can be expressed rationally in terms of the remaining radicals and the p's contrary to hypothesis. Hence $N_i=0$ and then $M_i=0$. It now follows that $P_i-p_i^{1/2}$ is also a root. Hence the equation $f(x)=0$ must have at least $N=2^n$ roots and therefore $m \geqq N$. It will now be shown that $m=N$ by constructing the equation. Let $r_1, r_2, \cdots, r_N$ be the N different values of (1) when the radicals in (1) are assinged the N different arrangements of $+$ and $-$ signs. It will be shown that

$$f(x) = \prod_{i=1}^{N}(x - r_i) = \sum_{i=1}^{N} b_i x^{N-i}, \tag{2}$$

The coefficients of the polynomial in x to the right in (2) are the elementary symmetric functions of the r's. The interchange of two of the p's in the set of r's merely changes the order of the r's in the set. Hence these coefficients are symmetric functions of the n radicals. Moreover, if we change the sign of any one radical, say$p_i^{1/2}$, wherever it occurs in the set of r's, the set is unchanged except as to order, and the coefficients of (2) are unchanged. Any one coefficient may be written $A_i+B_ip_i^{1/2}$, where A_i contains all the terms of this coefficient with even powers of $p_i^{1/2}$. Now if the sign of the radical $p_i^{1/2}$ alone is changed, the coefficient is unaltered in value and also A_i and B_i remain unchanged. Hence $A_i+B_ip_i^{1/2}=A_i-B_ip_i^{1/2}$, and therefore $B_i=0$, and the coefficient contains

no odd powers of $p_i^{1/2}$. Hence each coefficient is a rational integral function of the p's. Moreover it is symmetric in the p's and it can therefore be expressed as a polynomial in the elementary symmetric functions $a_1, a_2, \cdots, a_n$ where $a_i = \sum p_1 p_2 \cdots p_i$.

Some methods of computation: The computation of the coefficients of (2) by the direct multiplication of the linear factors is tedious even for small values of n, say $n=4$. For $n=2$ this direct method is rather easy. The following equations for $n=3, 4, \cdots$, may then be formed in turn as follows. Having found the equation $f(x)=0$ for n, write $f(x+q)=f_1(x, q^2)+qf_2(x, q^2)$, in which the even and odd powers of q have been separated as indicated. Then the equation for $n+1$ is

$$[f_1(x, p_{n+1})]^2 = p_{n+1}[f_2(x, p_{n+1})]^2.$$

Here additional labor would be required if we wished to reduce the coefficients to the elementary symmetric functions of $p_1, p_2, \cdots, p_{n+1}$.

This additional labor may be avoided in the following method, which appears to possess other advantages. Let

$$\phi(x) = x^n - \alpha_1 x^{n-1} + \alpha_2 x^{n-2} - \cdots = 0$$

be the equation whose roots are the n radicals $p_i^{1/2}$. Write $\phi(x) = \phi_1(x^2) + x\phi_2(x^2)$, where the even and odd powers of x have been separated. Then $[\phi_1(x)]^2 = x[\phi_2(x)]^2$ is the equation whose roots are $p_1, p_2, \cdots, p_n$, and whose coefficients are $a_1, a_2, \cdots, a_n$ with suitable signs. After expansion we find

$$a_{2i} = \alpha_{2i}^2 + 2\sum \alpha_{2k}\alpha_{4i-2k} - 2\sum \alpha_{2k-1}\alpha_{4i-2k+1},$$
$$a_{2i-1} = \alpha_{2i-1}^2 - 2\sum \alpha_{2k}\alpha_{4i-2k-2} + 2\sum \alpha_{2k-1}\alpha_{4i-2k-1}.$$

The case for $n=4$ will suffice to indicate the process. After suitable multiplication of each side and setting $\alpha_1 = x$, the equations for this case may be written

$$2\alpha_2 = x^2 - a_1, \qquad 4a_2 = 4\alpha_2^2 + 8\alpha_4 - 8\alpha_3 x, \qquad 64a_3x^2 = 64(\alpha_3 x)^2 - 128\alpha_2\alpha_4 x^2.$$

By elimination of $\alpha_3 x$ from the last two equations we obtain a single equation involving α_2 and α_4, and we may set the terms containing even powers of α_4 on the left and the odd powers on the right. Square both sides and replace α_4^2 by a_4, and there results

$$\{[(x^2 - a_1)^2 - 4a_2]^2 - 64(a_3x^2 - a_4)\}^2 - 256a_4\{3x^4 - 2a_1x^2 + 4a_2 - a_1^2\}^2 = 0.$$

It is obvious that by setting $a_4 = 0$ and omitting the last exponent 2 we obtain the equation for $n=3$, and so on.

A method will now be given for calculating the coefficients of the equation (2). Each b with an odd subscript is zero, and we may write $b_{2i} = (-1)^i \sum r_1^2 r_2^2 \cdots r_i^2$, where the summation is extended to half of the roots, the other half having the same values but opposite signs. Hence b_{2i} may be expressed as a polynomial in $s_1, s_2, s_3, \cdots$, where $s_i = \sum r_j^{2i}$, $j = 1, 2, \cdots, \sigma = 2^{n-1}$.

Suppose that r_1 is the sum of the radicals each taken with the positive sign. Then r_1^{2i} may be developed by the multinomial theorem in terms of products of the radicals $p_1^{1/2}, p_2^{1/2}, \cdots$, and for the purpose of summation to find s_i we may omit all terms containing an odd power of $p_j^{1/2}$, since such terms cancel out in the summation. Thus s_i may be expressed in terms of symmetric functions of the p's, and these functions may then be expressed in terms of the a's. Inserting these final values of the s's in the expressions for b_{2i} we obtain the desired results. The method will be clear from the results for the first four coefficients.

$$-b_2 = s_1, \quad 2b_4 = s_1^2 - s_2, \quad -6b_6 = s_1^3 - 3s_1s_2 + 2s_3,$$

$$24b_8 = s_1^4 + 3s_2^2 - 6s_1^2s_2 + 8s_1s_3 - 6s_4, \quad s_i = \sum r_j^{2i}, \; j = 1, 2, \cdots, \sigma = 2^{n-1}.$$

Then by summation we find

$$s_1 = \sigma a_1, \quad s_2 = \sigma(a_1^2 + 4a_2), \quad s_3 = \sigma(a_1^3 + 12a_1a_2 + 48a_3),$$
$$s_4 = \sigma(a_1^4 + 16a_2^2 + 24a_1^2a_2 + 256a_1a_3 + 1088a_4).$$

After substituting these values in the expressions above, we have

$$b_2 = -\sigma a_1, \quad b_4 = \sigma C_2 a_1^2 - 2\sigma a_2,$$
$$b_6 = -{}_\sigma C_3 a_1^3 + 2\sigma(\sigma - 2)a_1a_2 - 16\sigma a_3,$$
$$b_8 = {}_\sigma C_4 a_1^4 - \sigma(\sigma-2)(\sigma-3)a_1^2a_2 + 2\sigma(\sigma-2)a_2^2 + 16\sigma(\sigma-4)a_1a_3 - 272\sigma a_4,$$

where σC_i is the coefficient of x^i in the expansion of $(1+x)^\sigma$.

3332 [1928, 377]. *Proposed by R. E. Gaines, University of Richmond.*

In a given ellipse the two conjugate diameters are drawn which are equally inclined to the major axis, and a similar variable ellipse touches these two diameters and has its major axis on the same line as that of the fixed ellipse. Find the distance between their centers when the area common to the two ellipses is a maximum.

Solution by the Proposer.

The equations of the two ellipses may be written

$$(1) \qquad b^2x^2 + a^2y^2 = a^2b^2, \quad (x - 2^{1/2}az)^2b^2 + a^2y^2 = a^2b^2z^2,$$

where the distance between the centers is $2^{1/2}az$, and z is the parameter. Let u and v be the two segments into which the distance between the centers is divided by the common chord, so that $u+v=2^{1/2}az$. If $2A$ is the area in common, we shall compute A. For this purpose, we shall consider the two ellipses as having their centers at the origin, and we may write

$$(2) \qquad ay_1 = b(a^2 - x^2)^{1/2}, \quad ay_2 = b(a^2z^2 - x^2)^{1/2},$$

where $y_1(u) = y_2(v)$ is one half the common chord in their original position. Then

$$A = \int_u^a y_1 dx + \int_v^{az} y_2 dx,$$

$$(3) \qquad \frac{dA}{dz} = -\, y_1(u)\frac{du}{dz} - y_2(v)\frac{dv}{dz} + \int_v^{az} \frac{dy_2}{dz} dx,$$

$$= -\, 2^{1/2} a y_2(v) + \int_v^{az} abz(a^2z^2 - x^2)^{-1/2} dx,$$

where reductions are made by use of the two relations above. Now set $x = az\cos\theta$, $v = az\cos\psi$, then $y_2(v) = bz\sin\psi$, and we have

$$dA/dz = abz(\psi - 2^{1/2}\sin\psi).$$

For actual intersection we must have $2^{1/2}-1 \leqq z \leqq 2^{1/2}+1$. Since $2^{3/2}zu = a(1+z^2)$, we find that $2^{3/2}\cos\psi = 3 - z^{-2}$. As z increases from its lower limit to its upper limit, ψ passes from π to 0, and therefore dA/dz is at first positive, then vanishes for ψ_o, and finally remains negative until it is zero again when $\psi = 0$. Hence there is a single maximum for A given by $\psi_o = 2^{1/2}\sin\psi_o$. It is found that $\psi_o = 79°43'49''$ approximately, $z = .63299$, and $2^{1/2}az = .89519a$.

3335 [1928, 377]. *Proposed by Paul Wernicke, Washington, D. C.*

Assume, on the sides of a triangle ABC, the points L on BC, M on CA, N on AB. Find the condition for the existence of a real straight line dividing the three joins AL, BM, CN in the same ratio, and, in particular, bisecting them.

Solution by the Proposer.

In homogeneous coordinates ($A \equiv 1, 0, 0$; $B \equiv 0, 1, 0$; $C \equiv 0, 0, 1$), the coordinates of L, M, N, dividing BC, CA, AB in the respective ratios l, m, n are $(0, 1, l)$, $(m, 0, 1)$, $(1, n, 0)$. Then AL, BM, CN are divided in the ratio k in the points $(1+l, k, kl)$, $(km, 1+m, k)$, $(k, kn, 1+n)$. If these last three points are collinear, then

$$\begin{vmatrix} 1+l & k & kl \\ km & 1+m & k \\ k & kn & 1+n \end{vmatrix} = 0.$$

This equation reduces to $k+1=0$, giving the line at infinity, and to

$$(1 + lmn)k^2 - (1+l)(1+m)(1+n)(k-1) = 0.$$

There will thus be two real distinct lines of the required kind if the discriminant

$$(1+l)(1+m)(1+n)[(1+l)(1+m)(1+n) - 4(1+lmn)].$$

is positive, and only one if it vanishes. This refers to the finite lines. The line at infinity is always of the required type.

If $k=1$, the case of bisection, $lmn = -1$, and it then follows from the theorem

of Menelaus that L, M, N are collinear. Conversely, if L, M, N are collinear, $k=1$ is a solution. The others are $k=\infty$ for the straight line LMN, and $k=-1$ for the line at infinity.

Note by the Editors. The derivation of the coordinates of the points above is simplified by recalling that in this system of coordinates, if x, y, z are the coordinates of P, then P is the center of mass of a system of three particles at A, B, C of masses x, y and z.

3336 [1928, 378]. *Proposed by Otto J. Ramler, The Catholic University of America.*

Construct a triangle having given $a-b$, h_b+h_c, and the angle A, where a and b are sides, and h_b, h_c are the altitudes upon the sides b, c, respectively. (Altshiller-Court's College Geometry, p. 28, no. 7).

Solution by the Proposer.

The data are equivalent to having given $a-b$, $b+c$, $a+c$, and angle A, since angle A, h_b+h_c, and $b+c$ form a datum (Cf. "College Geometry" p. 28). We observe $a+c\equiv(b+c)+(a-b)$.

Construct a triangle $A''CE$, with $CE=a-b$, $A''C=a+c$, $A''CE=180°-A$. Produce EC to B' making $EB'=a+c$. Then $CB'=b+c$. Complete the parallelogram $A''EB'D$ so that D is opposite to E. Then $A''D=a+c$, and DB' is parallel to $A''E$. With C as center, and $a+c$ as radius strike an arc meeting DB' produced in A'. Let A be the intersection of $A''A'$, EB'. Through A draw a line parallel to $A''C$ meeting CA' at B. Then triangle CAB is the required triangle.

For the angle $CAB=A''CA=A$, because AB is parallel to $A''C$. Since triangle $A''CA'$ is isosceles, so also is triangle ABA'. Whence, $AB=BA'$. The angle $DA''C=B'AB$ (sides are parallel). And $AB:A''C=AA':A''A'$ $=AB':A''D$. But $A''D=A''C$. Whence, $AB'=AB$. Therefore, $AB'=AB$ $=BA'$, and $CB'=CA+AB=b+c$, $CA'=CB+AB=a+c$, and angle $CAB=A$.

Therefore, triangle ABC satisfies all conditions imposed. The problem has one and only one solution if we consider the parts of the triangle to be positive only.

Also solved by Paul Wernicke.

3337 [1928, 378]. *Proposed by E. B. Escott, Oak Park, Illinois.*

Sum the series

$$\frac{6^3}{2\cdot3\cdot4\cdot5}x^5-\frac{8^3}{4\cdot5\cdot6\cdot7}x^7+\frac{10^3}{6\cdot7\cdot8\cdot9}x^9-\cdots.$$

I. *Solution by Robert E. Moritz, University of Washington.*

The series is obviously absolutely convergent for all values of x for which $|x|<1$, conditionally convergent for $x=1$, and divergent when $|x|>1$. Let us first restrict ourselves to values of x such that $|x|<1$.

The general term of the given series is

$$u_{n-2} = (-1)^{n-1}(2n)^3x^{2n-1}/(2n-4)(2n-3)(2n-2)(2n-1)$$
$$= \overset{(1)}{u}_{n-2} + \overset{(2)}{u}_{n-2} + \overset{(3)}{u}_{n-2} + \overset{(4)}{u}_{n-2},$$

where

$$\overset{(1)}{u}_{n-2} = (-1)^{n-1}(-1/6)x^{2n-1}/(2n-1), \quad ,\overset{(2)}{u}_{n-2} = (-1)^{n-1}(4)x^{2n-1}/(2n-2),$$

$$\overset{(3)}{u}_{n-2} = (-1)^{n-1}(-27/2)x^{2n-1}/(2n-3), \quad \overset{(4)}{u}_{n-2} = (-1)^{n-1}(32/3)x^{2n-1}/(2n-4).$$

Now the original series being absolutely convergent for the values of x under consideration, its sum may be found by adding the sums of the series whose terms are $\overset{(1)}{u}_{n-2}$, $\overset{(2)}{u}_{n-2}$, $\overset{(3)}{u}_{n-2}$, and $\overset{(4)}{u}_{n-2}$ respectively, the initial value of n in each case being 3. Now it is easily seen that when $|x|<1$,

$$\sum_3^\infty \overset{(1)}{u}_{n-2} = -\tfrac{1}{6}\left[\tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \tfrac{1}{9}x^9 - \cdots\right] = -\tfrac{1}{6}\left[\tan^{-1}x - x + \tfrac{1}{3}x^3\right],$$

$$\sum_3^\infty \overset{(2)}{u}_{n-2} = 4x\left[\tfrac{1}{4}x^4 - \tfrac{1}{6}x^6 + \tfrac{1}{8}x^8 - \cdots\right] = +2x\left[x^2 - \log(1+x^2)\right],$$

$$\sum_3^\infty \overset{(3)}{u}_{n-2} = (-27x^2/2)\left[\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 - \cdots\right] = (-27x^2/2)\left[x - \tan^{-1}x\right];$$

$$\sum_3^\infty \overset{(4)}{u}_{n-2} = (32x^3/3)\left[\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4 + \tfrac{1}{6}x^6 - \cdots\right] = (16x^3/3)\log(1+x^2).$$

Adding, we find the sum of the given series, viz.,

$$(A) \quad \sum_3^\infty u_{n-2} = \tfrac{1}{6}x - (104x^3/9) + \tfrac{1}{6}(81x^2-1)\tan^{-1}x + \tfrac{2}{3}x(8x^2-3)\log(1+x^2).$$

Finally, by a theorem of Abel (Crelle's Journal, vol. 1, 1826), since the series is known to be convergent when $x=1$, its sum as x approaches 1 will also be given by (A); in short, whenever the given series has a sum its value will be given by (A).

II. *Solution by Mildred Barr, Iowa State College.*

Let $f(x)$ denote the given series. We shall proceed formally and differentiate it four times with respect to x, and we find that

$$f^{iv}(x) = 8\sum(-1)^{n+1}(n+2)^3x^{2n-1}.$$

We multiply both sides of the above equation by $(1+x^2)^4$, and find that the coefficient of x^{2n-1} for $n>4$ is

$$(-1)^{n+1}8\left[(n+2)^3 - 4(n+1)^3 + 6n^3 - 4(n-1)^3 + (n-2)^3\right] = 0.$$

Therefore

$$f^{iv}(x) = 8(27x + 44x^3 + 31x^5 + 8x^7)/(1+x^2)^4.$$

We integrate this expression for $f^{iv}(x)$ four times, determine the constants of integration by letting $x=0$ in the successive integrals, and we obtain

$$f(x) = \frac{2x(8x^2-3)}{3}\log(1+x^2) + \frac{81x^2-1}{6}\tan^{-1}x - \frac{104x^3}{9} + \frac{x}{6}\cdot$$

It is obvious that the foregoing processes are valid when $-1<x<1$.

Also solved by E. H. Clarke, M. S. Knebelman, J. F. Reilly, and the Proposer.

3338 [1928, 378]. *Proposed by N. A. Court, University of Oklahoma.*

The two pairs of extremities of two harmonic segments determine an involution in which the mid-points of the segments are two conjugate points.

Solution by Paul Wernicke, Washington, D. C.

In Cartesian coordinates, let the abscissae of the midpoints be x, u. Then the abscissae of the pairs may be written $x \pm y$ and $u \pm v$. Let the double points of their involution have the abscissae a and b, then we have

$$(x+y-a)(x-y-b)+(x+y-b)(x-y-a)=0,$$

or

$$x^2-y^2-(a+b)x+ab=0.$$

Also,

$$u^2-v^2-(a+b)u+ab=0.$$

The condition that the pair of midpoints belong to the same involution is

$$(x-a)(u-b)+(x-b)(u-a)=0,$$

or

$$2xu-(a+b)(x+u)+2ab=0,$$

which will be a consequence (namely the sum) of the preceding two equations if we can show that $2xu=x^2-y^2+u^2-v^2$.

This last equation is merely a way of writing the condition,

$$(x+y-u-v)(x-y-u+v)+(x+y-u+v)(x-y-u-v)=0,$$

that the given pairs be harmonic.

Note by the Editors. This theorem is obvious from geometrical considerations. If A, A'; B, B' are four points on a straight line such that the first pair of points is harmonically separated by the second pair, the two circles with diameters AA' and BB' cut orthogonally in a point O. If C and C' are the centers of the two circles, AOA', BOB', COC' are right angles and A, A'; B, B'; C, C' are pairs of points in the involution thus determined by O. The converse is easily proved in the same manner.

3339 [1928, 378]. *Proposed by Paul Capron, U. S. Naval Academy.*

Discuss the nodes of the parabolic spiral $(\rho-a)^2=ca\theta$.

Solution by the Proposer.

The two branches of the parabolic spiral are given by the pair of equations $\rho=a[1\pm(c\theta/a)^{1/2}]$, and there is a node $(\rho_{1,2},\ \theta_{1,2})$, given by their intersection, when and only when

$$\theta_2 = (2k+1)\pi + \theta_1 \ (k \text{ integral}), \tag{1}$$

and

$$-\rho_2 = \rho_1. \tag{2}$$

Hence $(c\theta_2/a)^{1/2}-(c\theta_1/a)^{1/2}=2$, or $m-n=2$, if we set $m^2=c\theta_2/a$, $n^2=c\theta_1/a$. From (1) we have

$$m^2 - n^2 = (2k+1)\pi c/a = (2k+1)l, \quad \text{where} \quad l = \pi c/a. \tag{3}$$

Solving the above pair of equations in m and n, we obtain as the conditions for a node $4(n+1)=4(m-1)=(2k+1)l$. Hence nodes occur at the points given by

$$\theta_1 = n^2\pi/l, \qquad \rho_1 = (n+1)a, \qquad \theta_2 = m^2\pi/l, \qquad \rho_2 = (1-m)a$$

provided, k being any integer, that

$$n = (2k+1)l/4 - 1, \qquad m = (2k+1)l/4 + 1, \qquad l = \pi c/a.$$

The increment of θ between two successive nodes (ρ, θ) and (ρ', θ'), given by $n=\frac{1}{4}(2k+1)l-1$, $n'=\frac{1}{4}(2k+3)l-1$, is

$$\theta' - \theta = (n'^2 - n^2)\pi/l = (n'-n)(n'+n)\pi/l = [(k+1)l - 2]\pi/2.$$

3341 [1928, 445]. *Proposed by Paul Wernicke, Washington, D. C.*

Given a triangle ABC and a point P, find the conditions for the construction of a triangle having its sides parallel and proportional to the joins PA, PB and PC.

Solution by J. Rosenbaum, Milford, Conn.

It is easily seen that there is no loss in generality if the proportionality factor is taken as unity. We accordingly take a point P in the plane ABC, and assuming that for this position of P the construction is possible, we draw PA and then draw AD parallel and equal to PB, and then draw DP.

Since the triangle PAD is supposed to fulfill the requirements, CPD is a straight line with point P lying between C and D. Also since AD is parallel and equal to PB, the figure $ADBP$ is a parallelogram, and hence PD bisects AB.

It thus follows that for a construction to be possible, it is necessary that P lie on the median to AB, and since AB is any side of ABC, the point P has to lie on all the three medians. In other words P has to be the centroid of the triangle ABC.

It is easily proved that this condition is also sufficient.

It is interesting to note that the analogous point in a quadrilateral is the intersection of the lines which join the midpoints of the two pairs of opposite sides (which point is not necessarily the center of area).

Also solved by Rufus Crane, Laurence Hampton, R. A. Johnson, Harry Langman, Enrique Linares, A. Pelletier, and the Proposer.

Note by the Editors. This solution considers a direction requirement which may be stated thus: The position of P is to be determined so that the forces defined in direction and magnitude by PA, PB, and PC shall be in equilibrium. If the direction requirement is dropped, or altered, there are three other solutions given by the vertices of the triangle formed by the three straight lines through the vertices of the triangle parallel to the opposite sides.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor G. D. Birkhoff, of Harvard University, has been elected foreign associate of the Royal Academy dei Lincei.

Professors Marston Morse and J. L. Walsh, of Harvard University, have been elected fellows of the American Academy of Arts and Sciences.

Honorary doctorates have been conferred on Professor A. H. Compton, of the University of Chicago, by Yale University and by The Ohio State University.

The University of Michigan has conferred the honorary degree of doctor of laws on Professor R. A. Millikan, of the California Institute of Technology.

Harvard University has conferred an honorary doctorate on Professor H. N. Russell, of Princeton University.

Swarthmore College has conferred an honorary doctorate on Dr. W. F. G. Swann, director of the Bartol Foundation of the Franklin Institute.

Marietta College has conferred the honorary degree of doctor of science on Dr. Frank C. Jordan, professor of astronomy in the University of Pittsburgh.

The board of trustees of Monmouth College has determined that the most fitting tribute which can be given to Miss Alice Winbigler for the great service she has rendered to Monmouth College during a period of fifty years' continuous teaching is the establishment of a permanent endowment for an Alice Winbigler chair of mathematics in the college. It is proposed to raise the sum of $60,000. A retiring allowance of $1500 each year, which will be approximately one half the income from this endowment, is to go to Miss Winbigler during her lifetime, after which the entire income will be used for the permanent support of the chair of mathematics.

Dr. Charles S. Howe retired from the presidency of the Case School of Applied Science in August.

The corner stone of the new Eckhart Hall on the main quadrangle of the University of Chicago was laid on July 12. This Hall will be occupied by the departments of physics, mathematics, and astronomy.

Associate Professor B. A. Bernstein, of the University of California, has been promoted to a professorship of mathematics.

Professor Florian Cajori, of the University of California, has retired, as professor emeritus of the history of mathematics.

Associate Professor C. C. Camp, of the University of Nebraska, has been promoted to a professorship.

Assistant Professor E. F. A. Carey, of the University of Montana, has been promoted to an associate professorship.

Associate Professor J. C. Fitterer, of the Colorado School of Mines, has been promoted to a professorship.

Dr. William Gafafer has been appointed research associate at Johns Hopkins University.

Professor W. V. N. Garretson, of Ouchita College, Arkadelphia, has been appointed associate professor of mathematics at the Oklahoma Agricultural and Mechanical College.

Assistant Professor J. W. Hurst, of Montana State College, has been promoted to an associate professorship.

Dr. C. G. Jaeger, of the University of Missouri, has been appointed to an assistant professorship at Tulane University.

Miss Elizabeth Knight, of the Milwaukee State Normal School, has been promoted to an assistant professorship.

Mr. T. R. Long has been promoted to an assistant professorship at the University of Rochester.

Associate Professor C. E. Love has been promoted to a professorship at the University of Michigan.

Mr. C. I. Lubin, of the University of Cincinnati, has been promoted to an assistant professorship.

Mr. C. T. Male, of Union College, has been promoted to an assistant professorship.

Dr. Earl S. Mickelson, of the University of Minnesota, has been appointed head of the department of Mathematics and physics at the New Mexico State Teachers' College, Silver City, N. M.

Dr. G. W. Myers, professor of the teaching of mathematics and astronomy in the School of Education at the University of Chicago, has retired.

Mr. G. A. Parkinson, of the University of Wisconsin, has been promoted to an assistant professorship.

Mr. C. S. Porter, of Amherst College, has been promoted to an associate professorship.

Dr. Tibor Rado has been appointed lecturer in mathematics at Harvard University for the first term of 1929–30. He will give courses on minimal surfaces and the problem of Plateau, and on the elementary theory of differential equations.

Assistant Professor H. P. Robertson, of the California Institute of Technology, has been appointed assistant professor of mathematical physics at Princeton University.

Dr. William S. Slauch, chairman of the department of mathematics of the High School of Commerce of New York City, has been appointed assistant professor of mathematics at New York University.

Dr. I. M. Sheffer, National Research Fellow, has been appointed assistant professor of mathematics at Pennsylvania State College.

Assistant Professor J. H. Taylor, of the University of Wisconsin, has been appointed professor of mathematics at George Washington University.

Mr. V. C. D'Unger, formerly of Little Rock College, is now auditor and general office manager of the Pyramid Life Insurance Company of Little Rock.

Assistant Professor R. L. Wilder, of the University of Michigan, has been promoted to an associate professorship.

Mr. F. G. Williams, of Cornell University, has been appointed professor of mathematics at Susquehanna University.

The following appointments to instructorships are announced:

Columbia University, Dr. A. A. Albert.

College of the City of Detroit, Dr. K. W. Folley, Dr. D. C. Morrow.

Harvard University, Dr. T. F. Cope, full time; Mr. A. E. Anderson, Mr. A. E. Currier, Mr. G. A. Hedlund, Mr. S. H. Kimball, Mr. C. B. Morrey, Mr. C. S. B. Myers, Mr. G. Saute, Mr. G. B. Van Schaack, Mr. C. Wexler, part time.

University of Iowa, Dr. J. M. Earl.

Long Island University, Mr. E. A. Knobelauch.

Purdue University, Mr. E. L. Klinger, Mr. J. M. Thompson.

Rice Institute, Dr. L. M. Blumenthal.

Worcester Polytechnic Institute, Mr. S. A. Lepeshkin.

The Chauvenet Prize

In the year 1925, the ASSOCIATION established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a liberal cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

The retiring President of the Association, Professor W. B. Ford, has given an additional endowment for this prize whereby it will hereafter be awarded every three years. The next award, however, will be in December, 1929, for the period 1925-1928 inclusive.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of America the sum of .... Dollars, to be known as the .... Fund, and to be used

for[2] { Endowment—the income only of which may be expended.
Special Projects—for which both principal and income may be expended.

Witness: .... Signature ....

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.
[2] Indicate which one of the two purposes is desired, and omit the other.

The Association needs funds for scientific publications and for the promotion of scientific activities.

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EDITORIAL CORRESPONDENCE should be addressed to the Editor-in-Chief, W. H. Bussey, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. Johnson, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the Secretary-Treasurer of the Association, W. D. Cairns, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Thirteenth Summer Meeting of the Association, Boulder, Colorado, August 26-27, 1929.

Fourteenth Annual Meeting, Des Moines, Iowa, December 31, 1929, January 1, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1929.

Illinois, Carthage, Ill., May 3-4.

Indiana, Culver Military Academy, May 3-4.

Iowa, Fairfield, Iowa, April 26-27.

Kansas, Topeka, Kansas, February 2.

Kentucky, Lexington, Ky., April 13.

Louisiana-Mississippi, Lafayette, La., April 12-13.

Maryland-District of Columbia-Virginia, George Washington University, May 4.

Michigan, Ann Arbor, Mich., March 16.

Minnesota, St. Paul, Minn., May 11.

Missouri, Kansas City, Mo., November 16.

Nebraska.

Ohio, Columbus, Ohio, April 4.

Philadelphia, University of Pennsylvania, November 30.

Rocky Mountain, Greeley, Colo., April 12-13.

Southeastern, Macon, Ga., April 19-20.

Southern California, University of Redlands, March 9.

Texas, Houston, Texas, Jan. 26.

Affiliated Organizations: The New England Association of Teachers of Mathematics. The National Council of Teachers of Mathematics.

ERRATA

CONTENTS FOR AUGUST-SEPTEMBER

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MISSOURI, Kansas City, Mo., November 16.

NEBRASKA.

OHIO, Columbus, Ohio, April 4.

PHILADELPHIA, University of Pennsylvania, November 30.

ROCKY MOUNTAIN, Greeley, Colo., April 12-13.

SOUTHEASTERN, Macon, Ga., April 19-20.

SOUTHERN CALIFORNIA, University of Redlands, March 9.

TEXAS, Houston, Texas, Jan. 26.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXVI, 1929

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PUBLISHED TEN TIMES A YEAR

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THE RHIND MATHEMATICAL PAPYRUS

At last we are able to announce with confidence that the second volume of the Rhind Papyrus is actually in process of completion and that the two volumes are expected to be ready for delivery not later than December first, 1929. The long delay, while unavoidable, has nevertheless contributed to the scientific perfection of the work and has permitted the addition of certain valuable features, especially information concerning the Golenishchev mathematical papyrus in Moscow, and an official account and photograph of the Mathematical Leather Roll, British Museum 10250, a document of the same period as the Rhind Papyrus.

A full description of the Chace publication of the Papyrus was printed in this MONTHLY for August–September, 1926, and ample reasons will there be found to justify the purchase of these volumes for every college library and for many of the larger high schools. The price there stated of fifteen dollars, postpaid, to members of the Association is considerably less than half of the actual cost of preparation and publication, but it was the desire of Doctor Chace that the Association members should be thus favored and he further showed his loyalty to the Association by making an outright gift to it of practically the whole edition with the understanding that the income from the sales should be kept as an endowment fund of the Association.

The Trustees, therefore, feel that they should deposit in that fund the full fifteen dollars for every set sold, and while they will surely fulfill the printed promise to prepay the carrying charges to the members who have already subscribed, nevertheless these will confer a favor and do a service to the Association by returning those charges upon receipt of the volumes. The parcel post charge will be well toward a dollar on an average, according to the distance carried. However, to those members who have not yet subscribed, notice is hereby given that on and after December first, 1929, the price will be fifteen dollars *plus carrying charges* to members who subscribe thereafter, directly through the Secretary of the Association.

This membership price will be extended to all new institutional and individual members whose applications are received by the Secretary on or before December first.

It is understood that the distribution to all non-members will be through the Open Court Publishing Company, 339 East Chicago Avenue, Chicago, Illinois, and that the price will be twenty dollars plus carriage charges.

Those of us who have seen the proof sheets of the plates in the second volume, containing the complete photographic reproduction of the Papyrus, the presentation of the hieratic text in its original colors, the hieroglyphic transcription, the transliteration and literal translation, can only marvel at the extent and character of the work which Doctor Chace and his assistants have accomplished. Likewise, the contents and make-up of the first volume completed two years ago command our admiration. This volume contains 48 pages

of introductory matter, 72 pages of free translation and commentary, and 85 pages of a very elaborate and critical bibliography of Egyptian mathematics contributed by Professor Archibald and including references to the literature of over fifty documents dating from 3500 B.C. to about 1000 A.D. This bibliography is further supplemented in the second volume especially in the light of recent remarkable discoveries in the field of Babylonian mathematics.

THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The thirteenth regular meeting of the Rocky Mountain Section of the Mathematical Association was held at the State Teachers College, Greeley, Colorado on April 12–13, 1929. There were three sessions, at which Prof. G. W. Finley acted as chairman.

The attendance was thirty-five including the following twenty-three members of the association: C. F. Barr, A. S. Clark, J. R. Everett, G. W. Finley, J. C. Fitterer, G. W. Gorrell, C. A. Hutchinson, H. Karnow, A. J. Kempner, Miss Claribel Kendall, A. J. Lewis, W. V. Lovitt, S. L. Macdonald, A. S. McMaster, J. Q. McNatt, R. R. Middlemiss, W. K. Nelson, E. D. Rainville, O. H. Rechard, A. W. Recht, W. J. Risley, L. J. Rote, C. H. Sisam.

The following officers were elected for the coming year: Chairman, G. W. Gorrell, University of Denver; Vice-chairman, A. J. Kempner, University of Colorado; Secretary-treasurer, A. J. Lewis, University of Denver. Following the election of officers a resolution was presented in appreciation of the work of Philip Fitch who passed away last autumn. Mr. Fitch was secretary of the association for many years and rendered a very valuable service to the work of the association in this district.

The following eleven papers were read:

1. "Statistical treatment of the factor of soil heterogeneity in agricultural experimentation," by Professor Andrew G. Clark, Colorado Agriculture College.
2. "Mathematical geography," by Professor Chas. A. Hutchinson, University of Colorado.
3. "The icosahedron," by Mr. A. J. Lewis, University of Denver.
4. "A geometrical approximation of π," by Mr. L. J. Rote, Denver, Colorado.
5. "The hypersurface of bi-secants of a curve in four-way space," by Dr. C. H. Sisam, Colorado College.
6. "A geometrical construction showing the relation between the in-center and circum-center of a triangle," by Mr. J. Q. McNatt, Canyon City, Colorado.
7. "Graphical methods of approximating irrational roots," by Professor James R. Everett, Colorado School of Mines.
8. "Present trend in the organization of subject matter of high school mathematics," by Professor A. E. Mallory, Colorado State Teachers College.
9. "An age sifter," by Professor Walter K. Nelson, University of Colorado.

10. "On the solution of linear equations," by Dr. Aubrey J. Kempner, University of Colorado.

11. "Some problems in elementary research," by Dr. J. L. Gibson, University of Utah, by invitation.

Abstracts of papers follow:

1. Professor Clark showed how an assumption of continuity in the variation of soil fertility would offset the unreliability of results in experimentation plots due to their insufficient number. Using adjoining plots, a correlation is effected which uses a sufficient number of plots to overcome the unreliability of first results.

2. Professor Hutchinson gave a brief outline of the subject of cartography, indicating some of the problems of mathematical interest in that field.

3. This is an outline of the methods used by Felix Klein in developing the icosahedral equation and showing its use in the solution of the quintic equation.

4. Mr. Rote showed a construction for a square which approximated closely the area of a circle.

5. This paper deals with the problem of finding those properties of the hyper-surface of bi-secants to an algebraic curve in four dimensions that can be determined by projecting the given curve from a given line onto a plane.

6. Mr. McNatt developed in a new way the known formula for the distance between the in-center and circum-center of a triangle.

8. Professor Mallory emphasized the fact that the present tendency in the selection of subject matter of high school mathematics was toward the adaptation of material to pupils' ability and toward a more informal treatment of the subject generally.

9. Professor Nelson explained briefly a device consisting of nine cards which may be used to determine the age of a person. When the cards are placed according to instructions the age of the person appears in large type through an opening in the back of the pile of cards.

10. This paper was published in the August-September issue of this "Monthly."

11. The general parametric equations of the space and body centrodes of certain disks and disk-like bodies, and other surfaces and solids supported by and rolling between two intersecting planes are found. The abscissas of the instantaneous axes of rotation give the values of definite integrals whose peculiarities have in these problems specific physical meanings. This makes it possible to use the planes as mechanical analyzers of many integrals, including some elliptic and hyperelliptic integrals. Points rigidly attached to the rolling bodies generate roulettes, the abscissa of each point of which, using the general equations, contains a definite integral. These curves, under certain conditions, degenerate into many well known forms, such as the cycloids. If we assume the equations of the space and body centrodes and study the integrals and equations of curves which follow from them, we find a field in which research has been done by students of limited mathematical attainments. Other problems of

a mechanical nature leading either to new methods or new results were mentioned. It was suggested that more attention be paid to the finding of this type of problem for the purpose of stimulating research earlier in the case of students specializing in mathematics.

The members and friends of the association were guests of the State Teachers College on the evening of April 12 at a banquet. President Finley acted as toastmaster. The address of welcome was given by President Frazier of State Teachers College and the response was given by Professor G. W. Gorrell of Denver University, after which there was a very interesting talk by Dean J. L. Gibson of the University of Utah. Dean Gibson recounted his experiences in visiting Germany and German mathematicians after the war.

A. J. LEWIS, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The regular spring meeting of the Minnesota Section was held at the College of St. Catherine, St. Paul, Minnesota, on Saturday, May 11, 1929. At the request of the chairman, Sister Alice Irene, Professor Dunham Jackson presided at the morning and afternoon sessions.

The attendance was 60 at the luncheon and 80 at the regular session, and included the following 30 members of the Association: W. O. Beal, R. W. Brink, W. E. Brooke, W. H. Bussey, Elizabeth Carlson, H. H. Dalaker, J. M. Earl, Margaret Eide, Gladys Gibbens, C. H. Gingrich, S. Guttman, D. Jackson, C. M. Jensen, W. H. Kirchner, E. L. Mickelson, Marie Ness, M. A. Nordgaard, G. C. Priester, Inez Rundstrom, R. E. Scammon, R. R. Shumway, Sister Alice Irene, Sister Prudentia Morin, F. J. Taylor, Ella Thorp, A. L. Underhill, M. B. White, H. B. Wilcox, G. L. Winkelmann, F. Wood.

The following officers were elected for the coming year: Chairman, Fredrick Wood, Hamline University, St. Paul, Minnesota; Secretary, A. L. Underhill, University of Minnesota; an Executive Committee consisting of the Chairman, the Secretary, Gladys Gibbens, University of Minnesota, F. J. Taylor, College of St. Thomas, St. Paul, C. M. Jensen, Macalaster College, St. Paul.

A motion was passed expressing the appreciation of the Section for the hospitality of the College of St. Catherine.

The following seven papers were read:

1. "An integrating operator," by Mr. Max Scherberg, University of Minnesota.

2. "Insect populations," by Mr. John Stanley, University of Minnesota.

3. "Newton's method of solving equations," by Professor W. O. Beal, University of Minnesota.

4. "Developmental geometry," by Miss Marie Ness, Department of Anatomy, University of Minnesota.

5. "Approximate solutions of problems in the calculus of variations," by Professor C. G. Priester, University of Minnesota.

6. "Teaching higher algebra in large classes," by Professor Elizabeth Carlson, University of Minnesota.

7. "A revision of the college entrance requirement in geometry," by Professor Dunham Jackson, University of Minnesota.

Abstracts of these papers follow:

1. The purpose of this paper was to find an operator which would evaluate the indefinite integral of a product $g_1(x)\ g_2(x)$ by operations upon the separate factors of the product. The author found such an operator and gave two examples of its use. In the first example, $g_1(x)=E^{mx}$, $g_2(x)=\sin nx$; in the second, $g_1(x)=E^{\pm x}$, $g_2(x)=\log x$. He also gave a generalized operator for evaluating the indefinite integral of the product, $g_1(x)\ g_2(x) \cdots g_n(x)$.

2. This paper dealt with mathematical considerations of an insect population in which ancestors accumulate as producing members throughout several generations. The author developed a formula for the total number of eggs laid up to and including the kth day of the rth generation and showed how that formula can be used in actual practise to obtain insect population values.

3. Mr. Beal showed by an example how n simultaneous equations involving n variables may be solved to any desired degree of accuracy provided an approximate solution can be obtained by graphical processes or measurements.

4. Miss Ness's paper gave two striking examples of the manner in which certain phenomena of biological development follow simple geometric forms. The first example had to do with cerebral hemispheres and the second with the frontal-fontanelle area.

5. A method of solving certain problems in engineering is based on the consideration of the potential energy of deformation. When a bar is bent by external forces or moments the equation of its elastic curve is that one in which the potential energy of deformation is the least.

The potential energy of deformation may be expressed by the equation

$$U = K\int (d^2y/dx^2)^2 dx$$

and the work due to the external forces or moments by

$$T = P\int f(x)(dy/dx)^2 dx,$$

where P is the external force or moment.

Equating these two expressions and solving for P,

$$P = \frac{K\int (d^2y/dx^2)^2 dx}{\int f(x)(dy/dx)^2 dx}.$$

This equation gives the value of P when the external work and the energy of

deformation are equal. Since the quantity on the right hand side should be a minimum, the problem is to find the equation of the elastic curve which will make it so. An approximate solution may be found by setting up a series of the form

$$y = a_1 f(x_1) + a_2 f(x_2) + a_3 f(x_3) + \cdots,$$

in which each term satisfies the boundary condition of the problem and where a_1, a_2, a_3, etc. are parameters to be determined so that the curve produces a minimum value for P. This value of P is a critical value and is on the boundary between stable and unstable equilibrium.

6. In this paper Miss Carlson pointed out some of the differences in methods used in teaching classes of about one hundred students as compared with methods used in teaching small classes. Also, she gave figures showing that the grade of work done by the students in the large classes compared favorably with the grade of work done by the students in the small classes.

7. Mr. Jackson spoke briefly about the organization and purposes of a committee on college entrance requirements in geometry, with regard to which a more detailed announcement appeared in the August-September number of this Monthly.

A. L. UNDERHILL, *Secretary*

THE SIXTH ANNUAL MEETING OF THE INDIANA SECTION

The sixth annual meeting of the Indiana Section of the Mathematical Association of America was held May 3–4, 1929 at Culver Military Academy, Culver, Indiana.

There were sixty present at the meeting including the following twenty-nine members of the Association: W. C. Arnold, Gladys Banes, Stanley Bolks, H. T. Davis, S. C. Davisson, C. S. Doan, J. E. Dotterer, P. D. Edwards, E. D. Grant, H. E. H. Greenleaf, G. E. Happell, C. T. Hazard, F. H. Hodge, H. K. Hughes, Juna M. Lutz, William Marshall, T. E. Mason, H. R. Mathias, G. T. Miller, J. A. Reising, C. K. Robbins, L. S. Shively, J. R. K. Stauffer, R. B. Stone, R. O. Virts, C. J. Waits, K. P. Williams, W. A. Zehring, H. A. Zinszer.

On Friday afternoon at 5:30 a reception was given to the visiting members and their guests. At 6:30 a complimentary banquet which was held in the mess hall was attended by approximately ninety guests of the academy. General L. R. Gignilliat, superintendent of the academy, presided at the banquet and made a brief address of welcome. Entertainment was furnished by Major Norman Imrie, head of the public speaking department of the academy, who regaled the guests with stories appropriate to the occasion. Music was furnished during the banquet by the cadet band.

At eight o'clock a public lecture under the auspices of the academy was given in the gymnasium by Professor Warren Weaver of the University of Wisconsin

on the subject, "Science and Imagination." Professor Weaver presented the new view of mathematical and physical science which is emerging from modern speculations. Ancient mathematics, according to the speaker, made use of defined elements and self-evident axioms. Modern mathematics makes use of undefined elements and assumed postulates. The theorems of modern mathematics are thus creations, not discoveries. Since mathematics is now a product of the creative imagination, it deserves consideration as an art. A somewhat similar change has come about in the logical structure of physical theories. The older model theories of physics explaining by analogy are comparable to the older mathematics; while the modern more abstract physical theories are more closely related with modern postulational mathematics. In the development of physical theories, therefore, the imagination now plays a more significant rôle than formerly and the theories have become more artistic in structure.

At 8:30 Saturday morning, a military review was held in honor of the visitors and this was followed by a tour of the academy buildings.

At the session at 10:00 a.m. in the Memorial Building of the Academy presided over by Professor H. E. H. Greenleaf, De Pauw University, chairman, the following officers were elected: Professor H. A. Zinszer, Hanover College, Chairman; Professor E. D. Grant, Earlham College, Vice-chairman; Professor H. T. Davis, Indiana University, Secretary-treasurer.

A chairman's address was made by Professor Greenleaf on the subject, "Mathematics in the Fundamentals of Music." Professor Greenleaf, considering the musical scale and the principal intervals of music as fundamental, discussed the changes in the scale from earliest times to the present. The speaker pointed out the mathematical basis of the Pythagorean, the diatonic, the mean-tone temperament, and the equal temperament scales and intervals and made a comparison of the four in regard to tonality.

The remainder of the program consisted of the following papers:

1. "The sectioning of freshman engineering students in mathematics," by Professor William Marshall, Purdue University.

2. Extracts from a discourse of J. F. Hennert (Utrecht, 1765): "On the necessity of including the study of mathematics in a good education," by Professor T. E. Mason, Purdue University.

3. "Invariance under the symmetric group of order three of a functional equation due to Abel," by Professor P. D. Edwards, Ball State Teacher's College.

4. "A solution of the biquadratic equation," by Professor E. D. Grant, Earlham College.

5. "A certain general type of contact transformation in three dimensions," by Professor C. K. Robbins, Purdue University.

6. "Three methods for finding the shortest distance between two skew lines" by Margaret L. Darragh, Hanover College (Introduced by Professor Zinszer).

7. "Transformations by reciprocal rays," by Mr. James Avas Cooley, Indiana University (Introduced by Professor H. E. Wolfe).

8. "Some properties of the circles that can be connected with the complete quadrilateral," by Mr. Maurice M. Lemme, De Pauw University (Introduced by Professor Greenleaf).

9. "Present status of the theory of the Volterra integral equation of the second kind," by Professor H. T. Davis, Indiana University.

10. "Notes on quantum mechanics," by Professor H. A. Zinszer, Hanover College.

Abstracts of the papers follow:

1. In the fall of 1928 the mathematics department of Purdue University divided the incoming freshmen in engineering mathematics into three groups: a sub-collegiate group, a normal group, and an advanced or honor group. Professor Marshall set forth in some detail the reasons for this sectioning, how it was done, how the various groups were handled, and the results of the experiment in so far as they are apparent at the present time.

2. Professor Mason's paper consisted of a translation of the inaugural address of J. F. Hennert at Utrecht on a subject of perennial interest to mathematicians. The striking feature of this discourse lies in the fact that the criticisms of students made by this professor in 1765 sound very modern. Apparently students have not changed much in the last century and a half. There are some reasons advanced to the people of the commercial town of Utrecht for the study of mathematics and physics which we should put today under the heading of reasons for the study of engineering.

3. In order that $F(x, y)F(y, z)F(z, x)$ be identical with $F(y, x)F(z, y)F(x, z)$ it is obviously sufficient that $F(x, y)$ be composed of factors which are (1) functions of x only, or (2) functions of y only, or (3) symmetric functions of x and y. That the condition is also necessary is not evident. Proof is given that if F is an algebraic function which is rational, or if irrational, one that belongs to a realm in which the unique factorization law holds, then the conditions named are necessary. Extension is made to the invariance of the function $F(x_1, x_2)$ $F(x_2, x_3) \cdots F(x_n, x_1)$ under the symmetric group of order n.

4. The biquadratic equation is first reduced to the form lacking the term in x^3. The roots are assumed to be $a \pm \sqrt{b}$, $-a \pm \sqrt{c}$. If we express the relation between the roots and the coefficients, there are three equations to solve for a, b, and c. The elimination of b and c leads to an equation of the sixth degree in a, containing the terms a^2, a^4, and a^6. This equation may be solved by Cardan's method in any numerical case; b and c may then be obtained, and the four roots written out.

5. If the transformation $x' = f_1(x, y, z, p, q)$, $y' = f_2(x, y, z, p, q)$, $z' = f_3(x, y, z, p, q)$, $p' = f_4(x, y, z, p, q)$, $q' = f_5(x, y, z, p, q)$ transforms a union of plane elements into a union of plane elements, it is a contact transformation. The analytical condition is that the vanishing of $p'dx' + p'dy' - dz'$ is a consequence of the vanishing of $pdx + qdy - dz$. This leads to a set of four partial differential equations, the integration of which can be determined (theoretically) according to the general theory of such equations. The actual application of this condition

seems to lead to insurmountable difficulties, but certain special cases are of interest. In particular if f_4 and f_5 are functions of p and q only, that is if the orientation of the plane in the transformed element depends only on the orientation of the plane in the original element, the system of partial differential equations can be completely solved. If $f_4=p$ and $f_5=q$, and the arbitrary function introduced by integration assumes a certain value, the transformation becomes the well known dilation.

6. The following methods were discussed: I. Through each of the lines any plane is passed perpendicularly. At some position of these planes their line of intersection will intersect each of the skew lines. The distance between these points is the shortest distance between the two lines. II. A plane is constructed perpendicular to one of the lines and through each line a plane is passed perpendicular to it. By expressing the equations for these two planes in their normal form and adding the right members the distance is obtained. III. The last method finds the distance directly by the minimizing process.

7. Laguerre in Nouvelles Annales, 3rd series, volume 1, defined the transformation by reciprocal rays and gave some of its properties. Mr. Cooley introduced a new constant for the modulus of the transformation, defining it as a cross ratio. In terms of the constant, he developed relations between the angle which reciprocal rays make with each other and the axis of transformation. He also gave additional properties and applications of the transformation especially with regard to circles and their tangents.

8. This paper proved by methods of Euclidean geometry alone the following theorem given by Jakob Steiner in the Annales de Gergonne, vol. 18, p. 16: In each of the four triangles formed by the sides of a complete quadrilateral there is one circle inscribed and three circles escribed, making in all sixteen circles. The centers of these sixteen circles arrange themselves in groups such that each of the four circles of one group cuts orthogonally all the circles of the other group. The lines of centers of the two groups of circles are perpendicular to each other. The lines of centers of the two groups of circles intersect at the point of intersection of the circles circumscribed to the four triangles forming the quadrilateral. Various consequences of the theorem were also exhibited by the speaker.

9. The methods used to solve the Volterra integral equation, not only for the case of continuous kernels, but for various types of discontinuities, were discussed. Numerous properties of the equation of the closed cycle, namely the case of the kernel of the form $K(x-t)$, were exhibited.

10. Assuming a closed system consisting of a nucleus and an electron the former being a point charge located at the origin of coordinates, the principle of the conservation of energy expressed in Newtonian notation was imposed. Applying the principle of Maupertuis (least action) and assuming the resulting equation to describe a family of wave-fronts travelling with a speed $E/\sqrt{[2m(E-V)]}$, where E is the total energy and V the potential energy function, a particular form of de Broglie's wave equation finally resulted.

At the afternoon session a resolution was adopted by the members of the section expressing their appreciation of the welcome that had been given them by the academy and of the efforts of General Gignilliat and of Major G. H. Crandall, Captain L. R. Kellam and other members of the department of mathematics who had contributed to the success of the meeting.

H. T. DAVIS, *Secretary*

A MODIFICATION OF A PROOF BY STEINER

By OTTO DUNKEL, Washington University

INTRODUCTION. An elegant and elementary proof was given by Steiner of the theorem that the equilateral triangle has the greatest area of all triangles having the same perimeter.[1] This proof is interesting in that no use is made of either parallels or metrical expressions for the area; it applies therefore whether the sum of the angles of a triangle is supposed to be less than, equal to, or more than 180°, and Steiner showed that his proof applied to spherical triangles without essential change. His proof consists of two parts of which the first part is essentially the proof under Theorem I below, while the second part has been altered to the form of proof under Theorem II. This modified form of proof is applicable to other similar geometrical theorems, and two such theorems are proved in this way without the use of parallels or metrical expressions for the area. The following proofs are worded for spherical triangles since in a few places restrictions are required peculiar to this form of geometry. For the cases where the sum of the angles of the triangle is less than or equal to 180° the proofs are essentially the same but simpler. In conclusion two theorems are given which result from the consideration of a metrical expression for the area. In the discussions below when one side of a triangle is designated as a base the term side will be considered to apply only to the two remaining sides.

THEOREM I. *Two triangles which have equal perimeters and bases of equal lengths have unequal areas if they are neither congruent nor symmetric. The triangle having the smaller area has the smallest base angle, the greatest base angle, the shortest side and the longest side.*

PROOF: Let ABC and $A'B'C'$ be two triangles which are neither congruent nor symmetric, but are such that $AB=A'B'$, $AC+BC=A'C'+B'C'$, $A\leqq B$, $A'\leqq B'$, where A denotes the angle BAC etc. The equality signs in the last relation are assumed to hold for only one triangle, for otherwise the two triangles would be congruent. Let the bases be made to coincide so that A' falls at A and B' at B. If then C and C' fall on opposite sides of the common base, we shall replace one triangle by its symmetric triangle and we shall suppose that the lettering of the vertices of the new triangle is the same as that for the old.

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[1] Steiner, *Sur le maximum et le minimum des figures dans le plan, sur la sphère et dans l'espace en général*, Crelle's Journal, vol. 24 (1842), pp. 96–99.

The new triangle has the same area and parts as the old. With this understanding C and C' lie on the same side of the base, and C' cannot fall within ABC or upon a side, for then we should have $AC'+BC'<AC+BC$; for the same reason C cannot fall within ABC' or upon a side. It follows then that a longer side of one triangle, say AC', must cut in a point M a shorter side, BC, of the other triangle, and the point M must lie within each of the segments AC' and BC. Hence $\angle BAC'<\angle BAC\leqq\angle ABC<\angle ABC'$, and therefore $BM<AM$. On MA lay off $MD=MB$, and on MC lay off $ME=MC'$, and draw DE. The two triangles MBC' and MDE have equal areas and $DE=BC'$. It will be shown that E lies within the segment MC. We have $AC+CM+MB=AD+DM+MC'+C'B$, and, since $MB=DM$, $MC'=ME$, $C'B=DE$, this equality reduces to $AC=AD+DE+(ME-CM)$. If $ME\geqq CM$ we would have the length of the broken line $ADEC$ equal to the length of the unbroken line AC, which is impossible. Hence $CM>ME$. It now follows that the area of ABC exceeds the area of ABC' by the area of the quadrilateral $ADEC$. Moreover, $BC'<BM+MC'=BM+ME<BC$, and from this inequality follows that $AC'>AC$. The theorem will be used in the following form:

If two triangles have bases equal to c and sides $a<b$, $a'\leqq b'$, respectively, such that $a+b=a'+b'$; then, if $a<a'$, the triangle with the side, a, has a smaller area than the triangle with the side a'.

THEOREM II: *Of all triangles having the same length of perimeter the equilateral triangle has the greatest area.*

PROOF: Let a, b, c be the lengths of the sides of a triangle which is not equilateral, and let S be its area; let e be the side of an equilateral triangle having the area E, and such that $3e=a+b+c$. Let a and c be the shortest and longest side, respectively; then $a<e<c$. Consider an isosceles triangle with the base c and the equal sides $a'=\frac{1}{2}(a+b)$, and let its area be S'. If $a=b=a'$, then $S'=S$; but if a and b are unequal, then $a<a'<b\leqq c$. Hence $S'>S$. Since $2a'+c=3e$ and $c>e$, we have $a'<e<c$. Consider now a triangle with base a', one side of length e and the other side of length b' so that $a'+b'=2e$, and let its area be S''. Since $a'<e$, it follows that $e<b'$. Hence in the two triangles S' and S'' having bases of length a' the shorter side of the first, a', is less than the shorter side, e, of the second, and therefore $S''>S'$. We compare finally E and S'' considered as having the base e. From the inequalities above $a'<b'$, $a'<e$. Hence $E>S''$, and $E>S''>S'\geqq S$, and the theorem is proved.

THEOREM III: *If two triangles which circumscribe the same small circle have an angle of one equal to an angle of the other, the one having the shortest adjacent side has the longest adjacent side and the greater area.*

PROOF: Let ABC and $A'B'C'$ be two triangles circumscribing the same small circle with center I, and such that $C=C'$, $CB<CA$, $C'B'\leqq C'A'$, $CB<C'B'$. Let the triangles be placed so that C and C' coincide, and if in this position $C'B'$ does not fall along CB we shall replace one triangle by its symmetric triangle so that this will be the case. Let L and M be the points of contact of CB and CA,

and N and N' the points of contact of AB and $A'B'$. Let $\angle LIM=\gamma$, $\angle MIN=\alpha$, $\angle NIL=\beta$, $\angle MIN'=\alpha'$, $\angle N'IL=\beta'$. Then $\alpha+\beta=\alpha'+\beta'$. In the right triangles LIB and LIB', the common side IL is less than $90°$ and $180°>LB'>LB$; hence $\frac{1}{2}\beta<\frac{1}{2}\beta'$ and therefore $\alpha'<\alpha$. In the right triangles MIA and MIA' we have again the common side $MI<90°$, $\angle MIA>\angle MIA'$; therefore $MA>MA'$ or $CA>C'A'$. Hence AB and $A'B'$ meet in a point P which lies within each segment, and it is easily shown that $PN=PN'$. In the triangles BIN and BIP with the common side BI and the angle at B in common, $\angle BIN=\frac{1}{2}\beta<\frac{1}{2}\beta'=\angle BIP$, and hence P lies within the segment NA. We show in a similar manner that P lies within the segment $N'B'$. We have then $AN=AM>A'M=A'N'\geqq B'L=B'N'$. Hence $AP=AN-NP>B'N'-PN'=B'P$. Also $A'N'=A'M\geqq B'L>BL=BN$, and hence $A'P=A'N'+N'P>BN+NP=BP$. Thus in the two triangles $PA'A$ and PBB' with equal angles at P, $A'P>BP$ and $AP>B'P$. Hence the area of the first triangle is greater than the area of the second, and from this follows at once that the area of ABC is greater than the area of $A'B'C$ or of $A'B'C'$. The angles α, β, γ for the triangle ABC will be designated as its central angles.

COROLLARY: *Of two triangles circumscribing the same circle and having a central angle of one equal to a central angle of the other, the one with the smallest of the remaining four central angles has the greatest central angle and the greater area.*

This follows at once from the above since $\beta<\beta'\leqq\alpha'<\alpha$ and $\gamma=\gamma'$.

THEOREM IV: *If two triangles circumscribe the same small circle and have an angle of one equal to an angle of the other, the one having the smallest adjacent side has the longest adjacent side and the greater perimeter.*

PROOF: The assumptions here and the proof of the first part are the same as in the proof of III, and it remains to prove that the perimeter of ABC is greater than the perimeter of $A'B'C'$. We have $\angle A'IA=\frac{1}{2}(\alpha-\alpha')$, $\angle B'IB=\frac{1}{2}(\beta'-\beta)$, and therefore $\angle A'IA=\angle B'IB$, since $\alpha+\beta=\alpha'+\beta'$. In the triangle AIB, $IN<90°$ and $AB<180°$. Hence AI produced meets ANB produced in A_1 so that $AIA_1=ANA_1=180°$, and since $NB<NA_1<180°$, we have $IB<IA_1$, or $AI+IB<AI+IA_1=180°$. Lay off on MA the lengths $MB_1=LB$ and $MB_1'=LB'$, and draw IB_1 and IB_1'. Then in the triangle AB_1I, $B_1I+IA<180°$, and $\angle AIA'=\angle B_1'IB_1\leqq\frac{1}{2}\angle AIB_1<90°$. Therefore $AA'>B_1'B_1=BB'$* In the two triangles ABC and $A'B'C$ the part $CM+CL$ is common and the corresponding remaining parts of the perimeters are $2(MA+LB)$ and $2(MA'+LB')$. But $MA+LB=MA'+A'A+LB'-BB'>MA'+LB'$. Hence the perimeter of ABC is greater than the perimeter of $A'B'C'$.

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* This theorem has been proposed as problem 3331 [1928, 321] in this Monthly, and a proof will appear later. The theorem is true whether the sum of the angles of the triangle is more than, equal to, or less than 180°; but in the last two cases the condition on the sides is of course omitted and the proof is much simpler.

COROLLARY: *Of two triangles circumscribing the small circle and having a central angle of one equal to a central angle of the other, the one with the smallest of the four remaining central angles has the greatest of the four and the greater perimeter.*

THEOREM V. *Of all triangles circumscribing the same small circle, the equilateral triangle has the smallest area and the shortest perimeter.*

PROOF: Let ABC be a triangle which is not equilateral and which circumscribes a given small circle, and let its area be S and its perimeter, p. Then the central angles α, β, γ are not all equal and we may suppose that the smallest is α and that the largest is γ. Hence $\alpha < 120° < \gamma$. Consider the circumscribing triangle with a central angle α and $\beta' = \gamma' = 180° - \frac{1}{2}\alpha$. Then $\beta' = \gamma' > 120° > \alpha$. If $\beta = \gamma$, the two triangles are congruent and the area S' and the perimeter p' of the second are the same as for the first. If $\beta \neq \gamma$, $\beta < \gamma$ and since $\beta + \gamma = 2\beta' = 2\gamma'$, we have $\beta < \beta'$ and $\gamma > \gamma' = \beta' > \beta$. Hence $S > S'$ and $p > p'$. Now consider the triangle with central angle $\beta'' = \beta' > 120°$, $\gamma'' = 120°$, $\alpha'' = 240° - \beta'' < 120°$, and let S'' and p'' denote its area and perimeter. The two triangles S' and S'' have the central angle β' and of the remaining four angles $\alpha < \gamma'$, $\alpha'' < \gamma''$, and $\alpha < \alpha''$. The last inequality follows form $\alpha'' + \gamma'' = \alpha + \gamma'$, $\gamma'' = 120° < \gamma'$. Hence $S' > S''$ and $p' > p''$. Now the circumscribing equilateral triangle has $\alpha''' = \beta''' = \gamma''' = 120°$. Let its area be E and its perimeter $3e$. Then S'' and E have $\gamma'' = \gamma''' = 120°$, and of the four remaining angles $\alpha'' < 120° < \beta''$; hence $S'' > E$ and $p'' > 3e$ and $S \geqq S' > S'' > E$ and $p \geqq p' > p'' > 3e$, and the theorem follows.

Some special forms of proof: In spherical geometry the properties of polar triangles and the measure of the area of a triangle enable us to prove the following theorem:

THEOREM VI: *Of all triangles inscribed in a given small circle on the surface of a sphere the equilateral triangle has the greatest area and the greatest perimeter.*

PROOF: Let S be the area of a triangle with the angles A, B, C and the opposite sides a, b, c, inscribed in a circle with the center I and of radius $R < 90°$. Let E be the area of the inscribed equilateral triangle with the equal angles ϕ and the equal sides e. The polar triangle of S has the angles $180° - a$, $180° - b$, $180° - c$; the sides $180° - A$, $180° - B$, $180° - C$; and the area S'. The polar of E has the equal angles $180° - e$, the equal sides $180° - \phi$, the area E'. The polar triangle of S circumscribes the circle with center I and with the radius $90° - R$ and similarly for the polar of E. Hence by V we have $S' > E'$ or $540° - (a + b + c) > 540° - 3e$. Therefore $3e > a + b + c$. Also by the same theorem $540° - (A + B + C) > 540° - 3\phi$ or $3\phi > A + B + C$. Hence $E > S$.

In Euclidean geometry of the plane the area of a triangle is equal to the product of the radius of the inscribed circle and one half of the perimeter. Hence in this geometry we can infer[1] II from V or V from II.

[1] This is shown in the article, *Maximum and minimum areas in geometry*, School Science and Mathematics, vol. 28 (1928), pp. 710–716.

A GENERALIZATION OF THE ORTHOPOLE THEOREM

By R. GOORMAGHTIGH, Bruges, Belgium

1. If A_1, B_1, C_1 be the projections of the vertices A, B, C of the triangle ABC on a straight line Δ, the perpendiculars from A_1 on BC, from B_1 on CA, and from C_1 on AB are concurrent at a point called the orthopole[1] of Δ for the triangle ABC.

More generally, when from A, B, C straight lines are drawn making with Δ equal angles θ at α, β, γ, the straight lines drawn from α, β, γ and making angles equal to $\pi-\theta$ with BC, CA, AB, respectively, are also concurrent.[2]

The orthopole theorem has further also been generalized as follows:

Let A_1', B_1', C_1' be the points dividing AA_1, BB_1, CC_1 in the same ratio; then the perpendiculars from A_1', B_1', C_1' on BC, CA, AB respectively, are concurrent.

We shall now deal with a new theorem containing these properties and also other known theorems in the geometry of the triangle.

2. Take Δ as x-axis, the y-axis being any perpendicular on Δ, and let (x_1,y_1) (x_2,y_2), (x_3,y_3), be the coordinates of A, B, C; call

$$X_1 = x_2 - x_3, \quad X_2 = x_3 - x_1, \quad X_3 = x_1 - x_2,$$

$$Y_1 = y_2 - y_3, \quad Y_2 = y_3 - x_1, \quad Y_3 = y_1 - y_2,$$

Let now $A\alpha$, $B\beta$, $C\gamma$ be the straight lines drawn through A, B, C, and making with Δ the angle θ at α, β, γ; if $\tan\theta = l$, the coordinates of α are $(x_1 - l^{-1}y_1, 0)$ and those of the point α' dividing $A\alpha$ so that $\alpha'A:\alpha A = \mu$ are $x_1 - l^{-1}\mu y_1$, $(1-\mu)y_1$.

The equation to the straight line d_a drawn through α' and making the angle ϕ with BC is:

$$[y - (1-\mu)y_1] = m(x - x_1 + l^{-1}\mu y_1), \tag{1}$$

and, if $\tan\phi$ is denoted by k,

$$m - (Y_1/X_1) = k[1 + (mY_1/X_1)].$$

Hence (1) becomes

$$\begin{aligned}(2)\quad p_1x + q_1y + r_1 \equiv (kX_1 + Y_1)x + (kY_1 - X_1)y \\ + (1 - \mu + l^{-1}\mu k)y_1X_1 + (l^{-1}\mu - k + k\mu)y_1Y_1 - kxX_1 - x_1Y_1 = 0.\end{aligned}$$

If β' and γ' are the points dividing $B\beta$ and $C\gamma$ so that

$$\beta'B:\beta B = \gamma'C:\gamma C = \mu,$$

[1] Neuberg, *Nouvelle Correspondance Mathématique* (1875), p. 189; Desboves, *Questions de Géométrie* (1875), p. 241; Gallatly, *The Modern Geometry of the Triangle*, 2nd ed., p. 46. On the properties of the orthopole, see my own paper in the Tôhoku Mathematical Journal, vol. 30 (1926), pp. 77–125.

[2] J. A. Third, Proceedings of the Edinburgh Mathematical Society (1913), p. 17; Neuberg, Mathesis (1914), p. 89.

the equations to the straight lines d_b, d_c drawn through β' and γ', and making with CA and AB the angle ϕ are similar to (2).

The area σ of the triangle formed by d_a, d_b, d_c is given by the formula

$$2\sigma = \frac{|\,p_1\; q_1\; r_1\,|^2}{(p_2q_3 - p_3q_2)(p_3q_1 - p_1q_3)(p_1q_2 - p_2q_1)}.$$

But

$$p_2q_3 - p_3q_2 = p_3q_1 - p_1q_3 = p_1q_2 - p_2q_1 = (k^2+1)\sum x_1Y_1 = 2(k^2+1)S,$$

S being the area of ABC.

Further if to the last row of the determinant $|p_1\, q_1\, r_1|$ is substituted the sum of the three rows, the determinant becomes

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ 0 & 0 & \Phi \end{vmatrix} = (p_1q_2 - p_2q_1)\Phi$$

where

$$\Phi = (1 - \mu + l^{-1}\mu k)\sum y_1X_1 - \sum x_1Y_1 = 2(2 - \mu + l^{-1}\mu k)S.$$

Therefore

$$\sqrt{\frac{\sigma}{S}} = (2 - \mu + \mu\tan\phi\cot\theta)\cos\phi.$$

Hence the following theorem:

Let $A\alpha$, $B\beta$, $C\gamma$ be the straight lines drawn through the vertices A, B, C, of the triangle ABC and making with a given straight line Δ, the same angle θ at α, β, γ. If α', β', γ' are the points dividing $A\alpha$, $B\beta$, $C\gamma$ in the same ratio, so that

$$\alpha'A:\alpha A = \beta'B:\beta B = \gamma'C:\gamma C = \mu,$$

the lines drawn through α', β', γ' and making with BC, CA, AB the same angle ϕ, form a triangle similar to ABC, and, for given angles θ and ϕ and a given ratio μ, the ratio of similitude of the two triangles is the same for any position of Δ in the plane of ABC.

The value of the ratio of similitude is

$$(2 - \mu + \mu\tan\phi\cot\theta)\cos\phi.$$

3. The orthopole theorem corresponds to $\mu = 1$, $\theta = \phi = \pi/2$; similarly the second theorem in paragraph 1 corresponds to $\mu = 1$, $\phi = \pi - \theta$. If $\phi = \theta$, the ratio of similitude is $2\cos\phi$, for any value of μ. When $\phi = 0$, the ratio of similitude is $2 - \mu$; hence the following theorem:

Let $A\alpha$, $B\beta$, $C\gamma$ be three parallels drawn through the vertices of ABC and cutting a given straight line at α, β, γ. If α', β', γ' be the points dividing $A\alpha$, $B\beta$, $C\gamma$, in the same ratio so that

$$\alpha' A : \alpha A = \beta' B : \beta B = \gamma' C : \gamma C = \mu,$$

then the ratio of similitude of the triangle formed by the parallels to BC, CA, AB drawn through α', β', γ' *and the triangle ABC is* $2-\mu$.

When $\alpha' \equiv \alpha$, $\beta' \equiv \beta$, $\gamma' \equiv \gamma$, these two triangles are equal.[1]

THEOREMS ON TOTAL REPRESENTATIONS AS SUMS OF SQUARE OR TRIANGULAR NUMBERS

By E. T. BELL, California Institute of Technology

1. We give further instances of a curious type of theorem considered in a previous note.[2] To recall what was proved, let $E(n)$, $O(n)$ denote respectively the total numbers of representations of the positive integer n as a sum of an even, an odd number of squares with roots $\lesseqgtr 0$, the order of the squares in each representation being essential. It was shown that $E(2n) > O(2n)$, $O(2n+1) > E(2n+1)$.

The following definitions will be required. The nth triangular number, where n is an integer >0, is $n(n+1)/2$. If for every pair of co-prime integers a, $b>0$, the function $g(n)$, single-valued and finite for all integers $n>0$, has the property $g(ab)=g(a)g(b)$, $g(n)$ is said to be factorable. Neither of $O(n)$, $E(n)$ is factorable.

2. Let $Q_r(n)$ denote the number of representations of the integer $n>0$ as a sum of r squares with roots $\lesseqgtr 0$, and $T_r(n)$ the number of representations of n as a sum of r triangular numbers. Write

$$Q(n) \equiv Q_1(n) - \tfrac{1}{2}Q_2(n) + \tfrac{1}{3}Q_3(n) - \cdots + (-1)^{n-1} n^{-1} Q_n(n),$$

$$T(n) \equiv T_1(n) - \tfrac{1}{2}T_2(n) + \tfrac{1}{3}T_3(n) - \cdots + (-1)^{n-1} n^{-1} T_n(n).$$

Neither $Q_j(n)$ nor $T_j(n)$ for n arbitrary and j odd, >1, or j even, >8, is expressible as a polynomial in the real divisors of n alone. Further, the algebraical complexity of the functions expressing $Q_j(n)$, $T_j(n)$ increases rapidly with j. The following is thus unexpectedly simple.

Theorem I. *If* $2^c (c \geqq 0)$ *is the highest power of* 2 *that divides* n, *each of the functions*

$$\frac{(-1)^{n-1}}{2} Q(n), \quad \frac{2^c\, T(n)}{3-2^{c+1}}$$

is equal to the sum of the reciprocals of all the odd divisors of n, *and hence each is factorable.*

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[1] Neuberg, Wiskundig Tydschrift (1913–1914), p. 79, Thébault, Journal de Mathématiques Élémentaires (1913–1914), p. 121. The analogon for the tetrahedron had already been given by Neuberg in Mathesis (1891), p. 50.

[2] This Monthly, vol. 30 (1923), p. 441.

Assuming this for a moment we have at once the following analogues of the theorems on $O(n)$, $E(n)$. Write

$$Q'(n) \equiv Q_1(n) + \tfrac{1}{3}Q_3(n) + \tfrac{1}{5}Q_5(n) + \cdots,$$

$$Q''(n) \equiv \tfrac{1}{2}Q_2(n) + \tfrac{1}{4}Q_4(n) + \tfrac{1}{6}Q_6(n) + \cdots,$$

$$T'(n) \equiv T_1(n) + \tfrac{1}{3}T_3(n) + \tfrac{1}{5}T_5(n) + \cdots,$$

$$T''(n) \equiv \tfrac{1}{2}T_2(n) + \tfrac{1}{4}T_4(n) + \tfrac{1}{6}T_6(n) + \cdots.$$

Theorem II. *$Q'(n)$ is greater or less than $Q''(n)$ according as n is odd or even, and similarly for $T'(n)$, $T''(n)$.*

The next states several further consequences of theorem I, all of which can be verified at a glance from the assumed truth of theorem I.

Theorem III. *If a, b are coprime,*

$$Q(ab) = \tfrac{1}{2}(-1)^{(a-1)(b-1)}Q(a)Q(b), \quad T(ab) = T(a)T(b).$$

If $n, \alpha, \beta, \gamma, \cdots, \delta$ are arbitrary integers >0, and $p, q, \cdots, r$ are g distinct odd primes, and if c is an arbitrary integer $\geqq 0$,

$$Q(2n) = (-1)^nQ(n), \quad Q(2^\alpha) = -2, \quad Q(1) = 2,$$

$$Q(2^cp^\beta q^\gamma \cdots r^\delta) = \frac{(-1)^{n-1}}{2^{g-1}}Q(p^\beta)Q(q^\gamma)\cdots Q(r^\delta),$$

$$T(2^c) = \frac{3-2^{c+1}}{2^c},$$

$$T(2^cp^\beta q^\gamma \cdots r^\delta) = T(2^c)T(p^\beta)T(q^\gamma)\cdots T(r^\delta).$$

To prove theorem I, and hence all, it will be sufficient to prove the part referring to $Q(n)$, as that for $T(n)$ proceeds in exactly the same way from the appropriate identity, which will be indicated. It is more interesting, however, to outline the proof of a more general theorem, from which the part relating to $Q(n)$ follows immediately.

3. Let $f(x)$ be single-valued and finite for integer values of x, and such that $f(n)=f(-n)$ for all integers n. Write

$$F_j(n) \equiv \sum f(n_1 + n_2 + \cdots + n_j),$$

where the summation refers to all integers $n_i \lesseqgtr 0 (i=1, \cdots, j)$ such that

$$n_1^2 + n_2^2 + \cdots + n_j^2 = n,$$

and write

$$F(n) \equiv F_1(n) - \tfrac{1}{2}F_2(n) + \tfrac{1}{3}F_3(n) - \cdots + \frac{(-1)^{n-1}}{n}F_n(n).$$

Denote by $\sigma(n)$ the sum of all the divisors of n.

Theorem IV. *The summation on the right referring to all pairs (t, τ) of integers $t>0$, $\tau>0$, τ odd, such that $n=t\tau$,*

$$F(n) = -1/n[\{1+(-1)^n\}\sigma(\tfrac{1}{2}n)f(0) + 2\sum(-1)^t\tau f(t)].$$

To see first that this implies the $Q(n)$ part of theorem I, take $f(x)=1$ for all integer values of x, as clearly is permissible under the definition of $f(x)$. A short reduction of the resulting right hand member gives the required relation.

To prove theorem IV, observe that we may take $f(n)\equiv\cos nx$, where x is a parameter, for all integers n, and get a true theorem provided theorem IV is true. But conversely, if the cosine form of the theorem is an identity in x, we can infer the general form as stated.[1] The cosine form, however, follows by a straightforward reduction from the identity.[2]

$$\log(1+\sum{}' q^{n^2}) = \log\theta_3 + 2\sum\frac{(-1)^n q^n}{n(1-q^{2n})}\,(1-\cos 2nx),$$

where Σ' refers to $n=\pm 1, \pm 2, \pm 3, \cdots$, Σ to $n=1, 2, 3, \cdots$, and

$$\log\theta_3 = \sum[\log(1-q^{2n}) + 2\log(1+q^{2n-1})],$$

where Σ refers to $n=1, 2, 3, \cdots$. The expansions are valid for q, x suitably restricted, and similarly for the series obtained by expanding the logarithms by the logarithmic series. Comparison of coefficients of like powers of q in the result gives the stated cosine identity, as can be easily verified.

There is a similar but more complicated generalization of the $T(n)$ part of theorem I. Omitting this, we need only state the classic identity which implies the theorem as stated:

$$1+\sum_{n=1}^{\infty} q^{n(n+1)/2} = \Pi(1-q^n)\Pi(1+q^n)^2,$$

from which, by taking logarithms and expanding, the result follows.

ON CERTAIN FUNCTIONAL RELATIONS

By MORGAN WARD, California Institute of Technology

1. *Introductory problem.* If $y=f(x)$ is an analytic function of x for $0\leqq|x|<r$ and if $f(0)=0, f'(0)\neq 0$ so that

$$y = a_1x + a_2x^2 + a_3x^3 + \cdots \qquad (a_1\neq 0), \tag{1}$$

then the inverse function $x=f^{-1}(y)$ is also analytic for $0\leqq|y|<\rho=\rho(r)$ and vanishes with y.

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[1] This is a simple instance of what was called paraphrase in several previous papers, e.g., Transactions of the American Mathematical Society, vol. 22 (1921), p. 1.

[2] See almost any text on elliptic functions, e.g., Tannery-Molk, vol. 3, p. 116.

Suppose that the function $f^{-1}(x)$ is identical with[1] $f(x)$, so that

$$x = a_1 y + a_2 y^2 + a_3 y^3 + \cdots . \tag{2}$$

The coefficients $a_1, a_2, a_3, \cdots$, must then satisfy certain algebraic conditions. These conditions express the fact that the result of substituting for the successive powers of x in (1) their expressions in terms of y from (2) must reduce to the identity $y = y$. In particular, we see that $a_1^2 = 1$. There are two totally different cases according as $a_1 = +1$ or $a_1 = -1$. If $a_1 = +1$, the remaining coefficients $a_2, a_3, \cdots$, all vanish; if however $a_1 = -1$, the situation is more complicated. We find that

$$a_3 = -a_2^2, \quad a_5 = 2a_2^4 - 3a_2a_4$$

$$a_7 = -13a_2^6 + 18a_2^3a_4 - 4a_2a_6 - 2a_4^2$$

$$a_9 = 145a_2^8 - 221a_2^5a_4 + 35a_2^3a_6 + 50a_2^2a_4^2 - 5a_2a_8 - 5a_4a_6$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$x^n = (-y)^n \left\{ 1 - na_2y + \frac{n(n+1)}{2!}a_2^2y^2 - \left(\frac{n(n-1)(n+4)}{3!}a_2^3 + na_4\right)y^3 \right.$$
$$\left. + \left(\frac{n(n-3)(n+2)(n+7)}{4!}a_2^4 + n(n+2)a_2a_4\right)y^4 - \cdots \right\}$$

By a somewhat lengthy induction, we can establish the following theorem:

Theorem 1: *Given*

$$y = f(x) = a_1x + a_2x^2 + a_3x^3 + \cdots$$
$$x = f(y) = a_1y + a_2y^2 + a_3y^3 + \cdots \qquad (a_1^2 = 1)$$

Then if $a_1 = 1$,

$$a_n = 0 \quad (n = 2, 3, 4, \cdots,)$$

But if $a_1 = -1$,

$$a_{2n+1} = P_n(a_2, a_4, \cdots, a_{2n}) = P_n(-a_2, -a_4, \cdots, -a_{2n}),$$

where P_n *is a uniquely determined polynomial in* $a_2, a_4, \cdots, a_{2n}$ *with integral coefficients.*[2]

The following result is immediate.

Theorem 2: *The necessary and sufficient condition that* x *and* y *be related as in theorem* 1 *is that there exist an analytic function* $F(x, y)$ *of* x *and* y *satisfying the conditions,*

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[1] A simple example of such a function is $y = x/(x-1) = -x - x^2 - x^3 - \cdots$.

[2] I have not yet succeeded in obtaining the explicit expression for P_n.

$$F(x,y) = F(y,x,) = F(0,0) = 0\,; \quad F_x(0,0) \neq 0.$$

2. *Extended problem.* Suppose that $y=\Phi(a, b; x)$ is a function of x and the two real parameters a and b which satisfies the following conditions:
(i) *$\Phi(a, b; x)$ is an analytic function of x on and within the square $\mathfrak{S}$ bounded by the lines $a=\pm h$, $b=\pm h$ in the $a\cdot b$ plane for $0\leqq |x|<r=r(h)$;*
(ii) *$\Phi(a, b; x)$ vanishes with x throughout $\mathfrak{S}$.*

We may consequently write

$$y = \Phi(a,b\,;\,x) = \phi_1(a,b)x + \frac{\phi_2(a,b)x^2}{2!} + \frac{\phi_3(a,b)}{3!}x^3 + \cdots,$$

where

$$\phi_n(a,b) = (\partial^n\Phi/\partial x^n).$$

If, moreover, $\phi_1(a, b)\neq 0$ in $\mathfrak{S}$, the inverse function $x=\Phi^{-1}(y)$ exists for $0\leqq |y|<\rho(h)$. Let us assume finally that
(iii) $\Phi^{-1}(x)=\Phi(b, a; x)$ for $0\leqq |x|<r(h)$ throughout $\mathfrak{S}$. We shall then have

$$x = \Phi(b,a\,;y) = \phi_1(b,a)y + \frac{\phi_2(b,a)y^2}{2!} + \frac{\phi_3(b,a)y^3}{3!} + \cdots$$

and as in section 1 it is necessary that

$$\phi_1(a,b)\cdot\phi_1(b,a) = 1. \tag{3}$$

Let us determine some of the properties of functions which satisfy the conditions (i), (ii), and (iii). We shall refer to such functions as "Φ-functions."

3. *Canonical form for functions.* From (3) we see that $\phi_1(a, b)$ and $\phi_1(b, a)$ can never vanish in $\mathfrak{S}$ and must both be of the same sign in $\mathfrak{S}$. If we write

$$y = |\,\phi_1(a,b)\,|^{1/2}v, \quad x = |\,\phi_1(b,a)\,|^{1/2}u,$$

the series defining Φ in section 2 become either

$$\text{(I)} \qquad \begin{aligned} v &= u + \psi_2u^2 + \psi_3u^3 + \cdots \\ u &= v + \psi_2'v^2 + \psi_3'v^3 + \cdots \end{aligned}$$

if $\phi_1(a, b)$ and $\phi_1(b, a)$ are positive in $\mathfrak{S}$, or

$$\text{(II)} \qquad \begin{aligned} v &= -u + \psi_2u^2 + \psi_3u^3 + \cdots \\ u &= -v + \psi_2'v^2 + \psi_3'v^3 + \cdots \end{aligned}$$

if $\phi_1(a, b)$ and $\phi_1(b, a)$ are negative in $\mathfrak{S}$; where in both cases

$$\text{(4)} \qquad \begin{aligned} \psi_n &= \psi_n(a,b) = \frac{\phi_n(a,b)}{n!}\,|\,\phi_1(b,a)\,|^{(n+1)/2} \\ \psi_n' &= \psi_n(b,a) = \frac{\phi_n(b,a)}{n!}\,|\,\phi_1(a,b)\,|^{(n+1)/2} \end{aligned}$$

$$(n = 2, 3, \cdots,).$$

If we write $a=b$ in (I) and (4), we see from the first part of theorem 1 that

$$\phi_n(a,a) = 0. \qquad (n = 2,3, \cdots ,).$$

From (3) and (4) follows

Theorem 3. *If $y=F(a, b; x)$ is any Φ-function whose coefficients are polynomials in a and b, then the coefficient of every power of x in F save the first is divisible by $a-b$.*

The two canonical forms of Φ-function in (I) and (II) show us that we have a correspondence with the two types of solution of $y=f(x)$, $x=f^{-1}(y)$ in theorem 1. Let us call Φ-functions of the first type "proper functions" and Φ-functions of the second type "improper functions." $y=x$ is the simplest proper function, but in contrast to the first part of theorem 1, we have a theorem analagous to theorem 2 for both proper and improper functions. We shall confine our statement to the former type of function in the canonical form (I).

Theorem 4. *The necessary and sufficient condition that v be a proper Φ-function of u is that u and v be connected by an implicit relation of the form*

$$u - v + F(a,b\ ; u,v) - F(b,a\ ; v,u) = 0, \tag{5}$$

where for sufficiently small positive values of $|u|$ and $|v|$, $F(a, b; u, v)$ is an analytic function of both u and v in some region $\mathfrak{R}$ in the ab-plane which includes the origin, and where

$$F(a,b\ ; 0,0) = F(b,a\ ; 0,0),$$
$$F_x(a,b\ ; 0,0) = F_x(b,a\ ; 0,0),$$
$$F_y(a,b\ ; 0,0) = F_y(b,a\ ; 0,0),$$

for all values of a and b in $\mathfrak{R}$.

In fact these conditions allow us to substitute for v a series in u with undetermined coefficients Ψ_n which we know will be convergent, and to determine the $\Psi_n=\Psi_n(a, b)$ by equating the coefficients of u, u^2, u^3, $\cdots$, to zero in the resulting identity; in particular, we shall have $\Psi_1=1$. Now if instead we substitute for u a series in v with undetermined coefficients Ψ_n', the equations determining Ψ_n' are obtained from those determining Ψ_n by merely inter-changing a and b, so that $\Psi_n'(a, b)=\Psi_n(b, a)$ and v is a proper function of u. Conversely, if (I) holds, by halving and subtracting the two series we obtain

$$u - v + \tfrac{1}{2}\sum_{n=2}^{\infty}\psi_n(a,b)u^n - \tfrac{1}{2}\sum_{n=2}^{\infty}\psi_n(b,a)v^n = 0,$$

where, by our definition of a Φ-function, $\frac{1}{2}\sum_{n=2}^{\infty}\psi_n(a, b)\, u^n$ satisfies all the conditions imposed upon $F(a, b, u, v)$ in the theorem.

5. *Example.* As an illustration of a proper Φ-function, suppose temporarily that a, b are real, but never zero. Consider the relation

$$(1 + bx)^a = (1 + ay)^b. \tag{6}$$

By the binomial theorem, if $|x|<1/|b|$, $|y|<1/|a|$

$$x - y + \frac{(a-1)bx^2}{1\cdot 2} + \frac{(a-1)(a-2)b^2x^3}{1\cdot 2\cdot 3} + \cdots$$
$$- \frac{(b-1)\cdot ay^2}{1\cdot 2} - \frac{(b-1)(b-2)a^2y^3}{1\cdot 2\cdot 3} - \cdots = 0,$$

so that

$$x - y + F(a,b\,;x,y) - F(b,a\,;y,x) = 0,$$

where

$$F(a,b\,;x,y) = \sum_{n=1}^{\infty} \frac{(a-1)(a-2)\cdots}{(n+1)!}(a-n)b^n x^{n+1}.$$

Now $F(a, b; x, y)$ satisfies all the conditions of theorem 4, even when a, b are zero, so that we have

$$y = x + J(a,b\,;x)\,; \quad x = y + J(b,a\,;y),$$

where we easily see that

$$J(a,b\,;x) = \sum_{n=1}^{\infty} \frac{(a-b)(a-2b)\cdots(a-nb)x^{n+1}}{(n+1)!}. \tag{7}$$

Moreover if $\lambda \neq 0$,

$$\lambda J\left(\lambda a,\lambda b\,;\frac{x}{\lambda}\right) = J(a,b\,;x)$$

and if $ab \neq 0$,

$$1 + ax + aJ(a,b\,;x) = (1+bx)^{a/b}.$$

We can define $J(a, b; x)$ by the series (7) and then prove that $x+J(a, b; x)$ is actually a proper Φ-function. This has been done by O. Jezek.[1] We conclude with a few easily proved but curious properties of the function $J(a, b; x)$. If

$$x = t + J(b,ab\,;t) \qquad \text{and} \qquad y = t + J(a,ab\,;t),$$

then

$$x + J(ab,b\,;x) = y + J(ab,a\,;y)\,;$$
$$y - x = J(a,b\,;x)\,; \quad x - y = J(b,a\,;y).$$

If

$$x = t - J(a,abc\,;t) + J(b,abc\,;t) + J(c,abc\,;t),$$
$$y = t + J(a,abc\,;t) - J(b,abc\,;t) + J(c,abc\,;t),$$
$$z = t + J(a,abc\,;t) + J(b,abc\,;t) - J(c,abc\,;t),$$

then

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[1] O. Jezek, *Ueber die Reihenumkehrung*, Wiener Sitzungsberichte Zweite Abteilung, vol. XCIX (1890), pp. 191–203.—See also Whittaker and Watson, *Modern Analysis*, 3rd edition, p. 147, example 14.

$$x - y = J(a,b\,;y+z),\ y - x = J(b,a\,;x+z)$$
$$y - z = J(b,c\,;z+x), \qquad z - y = J(c,b\,;y+x),$$
$$z - x = J(c,a\,;x+y), \qquad x - z = J(a,c\,;z+y),$$
$$J(abc,c\,;x+y) = J(abc,a\,;y+z) = J(abc,b\,;z+x).$$

GENERALIZATIONS IN GEOMETRY AS SEEN IN THE HISTORY OF DEVELOPABLE SURFACES

By FLORIAN CAJORI, University of California

"The mathematicians of the eighteenth century would have been astonished to a high degree, had they been told that there exist developable surfaces which are not ruled surfaces." Perhaps this passage from the pen of Picard[1] surprises many mathematicians even of the present time; it challenges the historian to endeavor to trace the evolution of ideas. The result alluded to is no less surprising to us than was to Euler in the eighteenth century the fact that i^i, where $i=\sqrt{(-1)}$, has a real value. In a letter to Goldbach, Euler showed his interest by computing this value to ten decimal places. Picard's statement is no ess surprising than the declaration about integral numbers made by Galileo in the seventeenth century: "Neither is the number of squares less than the totality of all numbers, nor the latter greater than the former."

Period of Primitive Intuition

Aristotle remarked that "a line by its motion produces a surface."[2] When this line was a straight line, ruled surfaces would result, which clearly included the cone and cylinder. But Aristotle's statement does not necessarily carry the implication that there are ruled surfaces which can be spread out upon a plane. Nevertheless, early students of geometry must have recognized as intuitionally evident the fact that, without stretching or tearing, the curved surface of cylinders and cones could be unbent upon a plane. Explanations of this property are not generally given. We have found the developed surface of a right cone drawn as the sector of a circle, in a practical work on mensuration,[3] without any novelty being claimed for it. In the same treatise the cylinder is described as being "in form of a Rolling stone used in Gardens," an expression conveying the picture of a surface rolled over a plane so that all its points are brought into coincidence with the plane.

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[1] Émile Picard, *La science moderne et son état actuel*, Paris, p. 53.

[2] Aristotle, *De Anima*, I, 4, 409, a4; T. L. Heath's *Thirteen Books of Euclid*, vol. 1, 2nd edition (1926), p. 170.

[3] William Hawney, *The Complete Measurer*, ninth Edition (1755), p. 159. See also p. 154. (First edition, London, 1717).

The French writer on stone-cutting, A. F. Frézier,[1] considered the application to the plane of curved surfaces of the *oblique* circular cylinder and cone. He considered certain skew surfaces but did not distinguish accurately between developable surfaces and general ruled surfaces.

First Generalization—Infinitesimal Analysis used in the Treatment of Developable Surfaces

The first critical studies of developable surfaces were made by Leonhard Euler and Gaspard Monge. The two investigators approached the subject about the same time, but Euler's paper received earlier publication, in 1772. It is noteworthy that at this time Euler was blind. The title of his paper, "On solids whose surface may be spread out upon a plane,"[2] shows that surfaces were not yet looked upon as distinct entities, but as boundaries of solids. Euler asks himself the question, are there other solids than the cylinder and cone, whose surfaces can be unfolded upon a plane? (quorum superficiem itidem in planum explicare liceat nec ne?). He gives the topic three different treatments. The first is purely analytic, the second is geometric and trigonometric, the third considers surfaces formed by rays of light enveloping the shadow cast by an opaque body illuminated by a luminous disk.

In the first method he assumes that an infinitesimal right triangle whose right vertex is x, y, z passes into a congruent right triangle in the plane whose right vertex is t and u. Euler tacitly assumes, as is to be expected of eighteenth century mathematicians, that the differential and integral calculus can be applied to the problem in hand, that, in other words, there exist in the geometry of these surfaces limiting values called derivatives. He takes the vertices of the right triangle in the plane to be t, u; $t+dt, u$; $t, u+du$. The coordinates of the vertices of the corresponding right triangle in the surface are x, y, z; $x+ldt$, $y+mdt$, $z+ndt$; $x+\lambda du$, $y+du$, $z+\nu du$. Euler obtains six equations which on his assumptions are the necessary and sufficient analytical conditions that the surface be developable upon the plane, viz.,

$$(dl/du) = (d\lambda/dt), \quad (dm/du) = (d\mu/dt),$$

$$(dn/du) = (d\nu/dt), \quad l^2 + m^2 + n^2 = 1, \quad \lambda^2 + \mu^2 + \nu^2 = 1, \quad l\lambda + m\mu + n\nu = 0,$$

where the parentheses indicate partial derivatives, and where l, m, n, λ, μ, ν are certain unknown functions of t and u, the determination of which is a "problem by itself considered most difficult, but whose solution will be shown further on in most elegant manner."

This solution is accomplished in the course of the second or *geometric* treat-

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[1] Amédée François Frézier, *La théorie et pratique de la coupe des pierres et des bois*, Strasbourg, 1737–39. Our information on Frézier is drawn from C. Wiener, *Lehrbuch der Darstellenden Geometrie*, vol. 1, Leipzig, 1884, p. 23, 24.

[2] L. Euler, *De solidis qvorvm svperficiem in planvm explicare vicet*, Novi commentarii academiae scientiarvm imperiatis Petropolitanae, Tom. XVI, pro anno 1771, Petropoli, 1772, p. 3–34.

ment of developable surfaces other than cones and cylinders. Euler takes a sheet of paper and draws non-intersecting straight lines upon it which are not parallel nor all directed to a common point. He considers the possibility of folding the sheet along these straight lines so as to form a surface. He says: "In this sheet it is possible to draw straight lines Aa, Bb, Cc, etc. at pleasure, such that none of them are parallel nor all converging to a fixed point, provided that they nowhere intersect each other, as shown in Fig. 1. However that sheet is bent along the straight lines, it is always possible to conceive of a solid which fits that bent sheet. From this it follows that besides prismatic and pyramidal bodies there are any number of other kinds of bodies which may be covered in this manner by that sheet, and whose surface may accordingly be unfolded upon a plane."

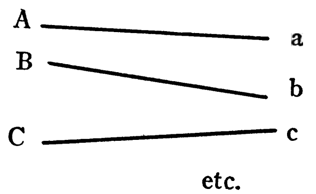

Fig. 1.

We quote further:

"Let us now increase to infinity [the number of] those lines Aa, Bb, Cc, etc. so that our solid acquires a surface everywhere curved, as our problem demands according to the law of continuity. And now it appears at once, that the surface of such bodies should be so constituted that from any point in it at least one straight line may be drawn which lies wholly on this surface; although this condition alone does not exhaust the requirements of our problem, for it is necessary also that any two proximate straight lines lie in the same plane and therefore meet unless they are parallel."

Euler explains that the points of intersection of pairs of neighboring straight lines form on the surface a twisted curve of double curvature, and that, conversely, given any twisted curve, one may derive from it a developable surface formed by the successive tangents of that curve. He establishes the analytic relation between the coordinates t, u, v of a point on the given twisted curve and the coordinates x, y, z of a corresponding point on the resulting developable surface. He connects the formulas thus obtained with the results of his first investigation involving the undetermined functions l, m, n, λ, μ, ν, by showing that $dl:d\lambda = dm:d\mu = dn:d\nu = -\cos\omega:\sin\omega$, where ω is an angle in his geometric figure. "Accordingly, if we examine carefully these things, we discern certain paths along which we may run down the direct solution of that most difficult problem."

About the same time, and independently of Euler, the subject of developable surfaces was investigated by Gaspard Monge, the creator of descriptive geometry. His earliest publication on such surfaces appeared at Paris in 1785; he discussed them repeatedly in later writings. Monge's treatment is less analyti-

cal than that of Euler and more nearly the result of direct contemplation of space relations. He starts with a curve of double curvature (twisted curve). Through a point on it he draws a plane perpendicular to a tangent line to the curve at that point. Similarly through a second point on the curve, infinitely near (infiniment proche) to the first, he draws a second plane. The two planes intersect in a line. Drawing in the same manner planes through all "consecutive" points (par tous ses points consecutifs), the lines of intersection of "consecutive" planes form a surface. This surface, he says, enjoys the general property of being developable upon a plane, as do conical and cylindrical surfaces, without overlapping and break of continuity (sans duplicature et sans solution de continuité). Monge observes also that a tangent line to the twisted curve, moving along that curve, generates a developable surface. He then proceeds to determine the equation of a developable surface from the equations of a twisted curve.

The general impression which we received from reading Euler and Monge is that Euler made a more searching inquiry into the nature of surfaces which can be spread out upon a plane. By considering also the opposite operation of taking a sheet of paper and folding it loosely, he came very near to a still broader generalization.

Second Generalization. Imaginary Surfaces.

It might seem desirable, in the treatment of real surfaces, to remain altogether in the realm of the real. But experience has shown this to be impossible of attainment in a general exposition. In the general development of higher geometry as well as the theory of surfaces in particular, the use of imaginary and ideal elements became inevitable. These are creations affording economy of thought and elegance of exposition. Confinement in geometry to real magnitudes definitely located in finite space frequently necessitates the consideration of special cases, which merge beautifully into one general case when imaginaries and elements at infinity are admitted. Two ellipses may intersect in four real points, provided their eccentricities are not zero. In the special case when the eccentricities are zero, the curves are circles and cannot have more than two real points of intersection. For the sake of generality, the geometer speaks of two imaginary *circular points* of the line at infinity. Similarly for spheres. An early leader in this movement toward generalization was the Frenchman Poncelet, in the first decennia of the nineteenth century.

Even in geometrical problems which arose in the eighteenth century, the use of imaginary relations could not be avoided. We illustrate this remark by the case of a minimum surface arising in Lagrange's problem[1] to determine a connected surface bounded by a given closed curve, so that the area of this surface shall be a minimum. Lagrange was led to a partial differential equation of the second order, for which later Monge[2] found a general solution. "Accord-

[1] J. Lagrange, *Essai d'une nouvelle méthode . . . .*, Miscellanea Taurinensia,, vol. 2 (1760–1761); *Oeuvres de Lagrange*, vol. 1 (1877), p. 356.

[2] G. Monge, Mémoires de l'académie r. d. sciences pour 1784, p. 118 et Suppl. p. 536.

ing to the nature of the case," says Lie,[1] "Monge's general integral was affected by imaginaries and consequently for a long time it was possible to find only very particular real minimal surfaces. Bonnet[2] [about 70 years later] was the first to give a general method for finding all real minimal surfaces · · ·." In deriving all real surfaces satisfying Lagrange's problem, Bonnet made liberal use of imaginaries, which, after rendering valiant service, were gotten rid of, by the use of conjugate functions $F(x+iy)$ and $F(x-iy)$.

The use of imaginaries and minimal lines led to unexpectedly new results in developable surfaces. Let the coordinates of the points on a straight line in space be represented by $x=a+At$, $y=b+Bt$, $z=c+Ct$, where a, b, c is a fixed point, and t is a parameter. This straight line passes through the point a, b, c, and through the point $a+A$, $b+B$, $c+C$, as appears by letting t be 0 and 1. There is a point on the line for every value of t. The distance from the point a, b, c to any other point x, y, z on the line is $\sqrt{[(x-a)^2+(y-b)^2+(z-c)^2]}$. If we substitute in this the values for x, y, z, the expression takes the form $t[A^2+B^2+C^2.]^{1/2}$ When A, B, C are all real, and $t>0$, this distance is positive and not zero. In the case of imaginary values of A, B, C, giving rise to imaginary lines, it is possible to have $A^2+B^2+C^2=0$. For example, we may take $A=3$, $B=4$, $C=5i$. Thereby we are led to the so-called "minimal lines," whose lengths are zero for any value of the parameter t. Thus, for real values of A, B, C, one obtains real straight lines whose lengths vary with t; only imaginary straight lines can be minimal, all having their lengths "null."

Suppose next that in place of the fixed point a, b, c, we take the variable point moving along a curve in space, which has its three coordinates $\phi(s)$, $\chi(s)$, $\psi(s)$, where the parameter s is the length of arc on that curve. We then obtain

$$x = \phi(s) + At, \quad y = \chi(s) + Bt, \quad z = \psi(s) + Ct$$

as the equations of a cylinder. If, as before, A, B, C are made to satisfy the equation $A^2+B^2+C^2=0$, then the cylinder is imaginary and composed of minimal lines. It has been shown that to every point in this imaginary cylinder there corresponds a point in the plane such that corresponding arcs on the cylinder and in the plane are of equal length. Thus, it appears that *the imaginary cylinder is developable upon a plane*[3]—a result which would have astonished most of the eighteenth century geometers.

Minimal lines are of course a special type of "minimal curves," a name introduced by Sophus Lie.[4] But the analytical expressions representing minimal curves occur much earlier, in the writings of Monge, Legendre, Enneper and Weierstrass. It has been found that the discussion of geodesics upon a surface is much simplified by referring the surface to null lines as parametric curves.[5]

[1] S. Lie, Mathematische Annalen, vol. 14 (1879), p. 331.

[2] Ossian Bonnet, Comptes Rendus, vol. 37, Paris (1853), pp. 529–533.

[3] Georg Scheffers, *Einführung in die Theorie der Curven*, vol. 1 (1901), p. 288; See also V. Kommerell u. K. Kommerell, *Raumkurven u. Flächen*, vol. 2 (1903), p. 183.

[4] S. Lie, Mathematische Annalen, vol. 14 (1879), p. 337.

[5] A. R. Forsyth, *Lectures on Differential Geometry*, Cambridge (1912), p. 76.

Third Generalization. Transcending the Limitations of Infinitesimal Analysis

The theory of surfaces was enriched dring the nineteenth century by the great researches of Gauss, Weierstrass, Bonnet, Darboux and others, which are greatly admired for the generality of the results. Nevertheless, the close of that century brought the astonishing revelation that there existed surfaces applicable to the plane which are not ruled surfaces. The new generalization, due to H. Lebesgue, is no less uncanny than is Peano's famous "space-filling curve." We give in his own words[1] the description of one of Lebesgue's surfaces as that is not ruled, yet is applicable to the plane: "To obtain [such] surfaces that are not ruled I take an analytic developable one and upon it an analytic curve C, not geodesic. One knows that there exists another developable surface passing through C, such that one can make the two developables applicable to a plane in such a manner that to each point of C, whether considered as belonging to the one or to the other of the two surfaces, there corresponds one and the same point in the plane. The curve C divides the first developable in two pieces, A, B, also the second into two pieces A', B'.

"Two of the four surfaces (A, A'), (A, B'), (B, A'), (B, B') are applicable to the plane without tearing or duplication and are indeed such that one can detach from them a finite piece enjoying the above property and containing an arc of C. This C is then a singular line. As before, one may pass from this one singularity to an infinite number of singularities and one thus obtains surfaces applicable to the plane, yet not containing any segment of a straight line."

In this compact statement it is hard to picture the effect of an infinite number of repetitions of the process described.

Lebesgue gives a second mode of variation, the unlimited repetition of which leads to an unruled surface that is applicable to the plane: "In general, if $y=f(x)$ is a curve, limited in its variation so that its total variation from x_0 to x_1 is $K|x_0-x_1|$, K being constant, the surface generated by this curve when rotated about Oy, is applicable to the plane."

It is worth while to supplement these passages by the comment of Picard:[2] "According to general practice, we suppose in the preceding analysis, as in all infinitesimal geometry of curves and surfaces, the existence of derivatives which we need in the calculus. It may seem premature to entertain a theory of surfaces in which one does not make such hypotheses. However, a curious result has been pointed out by Mr. Lebesgue (*Comptes Rendus*, 1899 and thesis); according to which one may, by the aid of continuous functions, obtain surfaces corresponding to a plane, of such sort that every rectifiable line of the plane has a corresponding rectifiable line of the same length of the surface, nevertheless the surfaces obtained are no longer ruled. If one takes a sheet of paper, and crumples it by hand, one obtains a surface applicable to the plane and made up of a

[1] Comptes Rendus, vol. 128 (1899), p. 1502–1505; a second article of Lebesgue on this topic is found in Annali di matematica (Brioschi), 3rd series, vol. 7, Milan (1902), p. 324.

[2] Émile Picard, *Traité d'analyse*, 3rd edition, vol. 1, Paris (1822), p. 555, foot-note.

finite number of pieces of developable surfaces, joined two and two by lines, along which they form a certain angle. If one imagines that the pieces become infinitely small, the crumpling being pushed everywhere to the limit, one may arrive at the conception of surfaces applicable to the plane and yet not developable [the envelope of a family of planes of one parameter] and not ruled."

The crumpling of a sheet of paper reminds one of Euler's process of fold-along straight lines. Euler's procedure was regular, systematic—too much so to yield the results of Lebesgue.

A RELATION BETWEEN POLAR CONICS AND OSCULANT CONICS OF A NODAL CUBIC

By FRANC C. EARHART

Brill,[1] in developing the theory of involutions on rational curves, considered that all projective properties of a curve are given by properties of certain involutions of groups of points on this curve. In the geometric development of this theory use has been made of certain covariant systems of curves of lower order. Study[2] and Jolles[3] have treated these systems analytically and the latter has given to them the name "osculants."[4]

The osculants of rational curves are defined as follows:[5] If the parametric point equations of a rational curve are

$$x_i = f_i(t,\tau), \quad i = 1,2,3, \tag{1}$$

where f_i are binary forms of order n in the homogeneous parameter t/τ, then the equations

$$x_i = \left(t_1 \frac{\partial}{\partial t} + \tau_1 \frac{\partial}{\partial \tau}\right) f_i, \tag{2}$$

obtained by taking the first polars of f_i with respect to (t_1,τ_1), represent the first osculant of (1) at the point t_1/τ_1. Likewise polarizing (2) with respect to t_2/τ_2, we obtain

$$x_i = \left(t_2 \frac{\partial}{\partial t} + \tau_2 \frac{\partial}{\partial \tau}\right)\left(t_1 \frac{\partial}{\partial t} + \tau_1 \frac{\partial}{\partial \tau}\right) f_i,$$

which is the first osculant of (2) and a second osculant of (1).

The process may be continued until the f_i's are completely polarized.

Osculants at the point t_1 touch the curve there; all first osculants touch the

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[1] Mathematische Annalen, vol. 20 (1882), p. 335.

[2] *Über die Raumcurve vierter Ordnung zweiter Art*, Sitzungsberichte der Königlichen Sächsische Gesellschaft der Wissenschaften, vol. 38 (1886).

[3] *Theorie der Osculanten, etc.*, Habilitationsschrift, Aachen (1886).

[4] Journal für die reine und angewandte Mathematik, vol. 101 (1887), p. 300.

[5] Winger, *Projective Geometry* (1923) pp. 386–387.

stationary tangents of the curve, and the $(n-1)$st osculant is the tangent to the curve.[1]

It is the purpose of this paper to study a relation between the first polars and the first osculants of rational cubics.

The linear polar of a point on a curve is the tangent at the point and thus coincides with the linear osculant at that point. But, in general, the polar conic of a point on a cubic is not identical with the osculant conic at that point.

A canonical form for the parametric equations of the rational nodal cubic is[2]

$$x = 3t^2, \qquad y = 3t, \qquad z = t^3 + 1, \tag{3}$$

the flex parameters being -1, $-\omega$, $-\omega^2$, the nodal parameter 0, and the line $z=0$ the line of flexes, or in the homogeneous form:

$$x = 3t^2\tau, \qquad y = 3t\tau^2, \qquad z = t^3 + \tau^3. \tag{4}$$

Then the osculant curve at the point t_1 is given by the equations:

$$x = t^2 + 2t_1t, \qquad y = 2t + t_1, \qquad z = t_1t^2 + 1. \tag{5}$$

Eliminating t in (3), we obtain the equation of the cubic: $x^3+y^3-3xyz=0$. The polar conic of x_1, y_1, z_1 is

$$x_1(x^2 - yz) + y_1(y^2 - xz) - z_1xy = 0,$$

or, substituting for x_1, y_1, z_1, their respective values from (3),

$$3t_1^2(x^2 - yz) + 3t_1(y^2 - xz) - (t_1^3 + 1)xy = 0.$$

To determine the intersections of the osculant conic and the polar conic, substitute for x, y, z, their respective values from (5) which yields an equation of the fourth degree in t: $t(t-t_1)^2=0$. Therefore $t=\infty$, 0, t_1, t_1 are parameters of the points of intersection of the polar and the osculant conics. t_1 gives the point of contact at which each is known to be tangent to the cubic.

When $t=0$, $x:y:z=0:t_1:1$.

When $t=\infty$, $x:y:z=1:0:t_1$. The equation of the line joining the points $t=0$ and $t=\infty$ is:

$$t_1^2x + y - t_1z = 0.$$

Differentiating with respect to t_1 and eliminating t_1 between the two equations, the envelope is shown to be the conic: $4xy-z^2=0$. This form would suggest the possibility of identifying this envelope with:

1) The polar conic of a special point.
2) The poloconic of a special line.
3) A covariant conic of the cubic.

The absence of the term z^3 in the cubic precludes 1). The poloconic of a line $ax_1+by_1+cz_1=0$, with respect to the cubic $x^3+y^3-3xyz=0$, is found to be[3]

$$(a^2 - 4bc)x^2 + (b^2 - 4ac)y^2 + c^2z^2 - 2(ab + 2c^2)xy - 2bcyz - 2acxy = 0.$$

[1] Thomsen, American Journal of Mathematics, vol. 32 (1910), p. 207.

[2] Winger, *Projective Geometry*, p. 368.

[3] White, *Plane Cubic Curves*, p. 47.

Comparing this with the conic $4xy-z^2=0$,

$$(6) \qquad a^2 - 4bc = 0, \qquad b^2 - 4ac = 0, \qquad bc = 0, \qquad ac = 0,$$
$$c^2 = 1, \quad 2(ab + 2c^2) = 4,$$

$c \neq 0$ since that would imply also $a=b=0$. When $a=b=0$, the conditions in (6) are satisfied. Therefore $4xy-z^2=0$ is the poloconic of the line $z=0$. Hence the following theorem:

The envelope of the line joining the two points of intersection (not the point of tangency) of the first polar and the first osculant at a point t_1 of a nodal cubic is the poloconic of the line of flexes.

It is known that the corresponding points A and B on the Hessian are also the points of contact of the two tangents which can be drawn from a point on the curve. Winger has discussed the envelope of the line joining two such points in his study of involutions on the rational cubic.[1] This envelope, the Cayleyan of the original cubic, is a tri-tangent conic of the Hessian which Winger has called "the involution conic of the node." Hence the above theorem may be stated also in the following forms:

The envelope of the line joining the two points of intersection (not the point of tangency) of the first polar and the first osculant of a point t_1 of a nodal cubic is the Cayleyan of the cubic.

The envelope of the line joining the two points of intersection (not the point of tangency) of the first polar and the first osculant of a point t_1 of a nodal cubic is the involution conic of the node of the Hessian.

Since the points of intersection of the osculant and polar conics of the cuspidal cubic are absorbed by the point t_1 and the cusp, there is no envelope problem.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

DISCUSSIONS

I. A GRAPHICAL DERIVATION OF CRAMER'S RULE

By J. P. BALLANTINE, University of Washington

Consider first the case of two equations in two unknowns:

$$(1) \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 &= a_{13}, \\ a_{21}x_1 + a_{22}x_2 &= a_{23}. \end{aligned}$$

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[1] Winger, *Projective Geometry*, p. 383.

Considering the vectors $A_1=(a_{11}, a_{21})$, $A_2=(a_{12}, a_{22})$, $A_3=(a_{13}, a_{23})$, the solution of (1) is equivalent to the determination of two numbers x_1, x_2, such that

$$A_1 x_1 + A_2 x_2 = A_3. \tag{2}$$

Let the vectors A_1, A_2, A_3 be as illustrated in Fig. 1. The parallelograms A_1A_2 and A_3A_2 are completed and denoted by P_{12} and P_{32}. The vector A_1x_1 is determined by the following two conditions: (1) It is a multiple of A_1; and

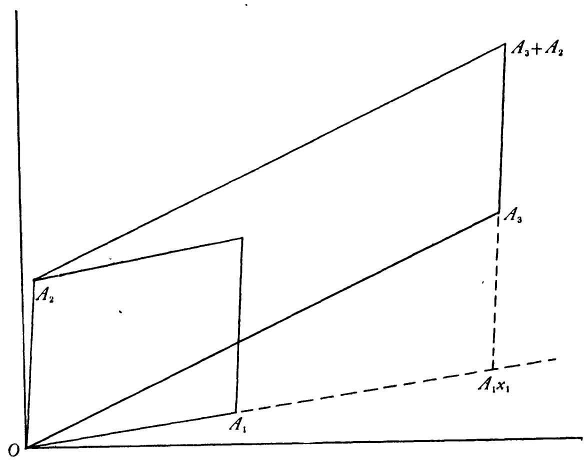


FIG. 1.

(2) if a proper multiple of A_2 is added to it, the sum is A_3. The first of these conditions restricts A_1x_1 to the line from O to A_1, and the second restricts it to the line from A_3 to A_3+A_2.

The value of x_1 is thereby determined as the ratio A_1x_1/A_1. This in turn is seen to be equal to the ratio between the altitudes of P_{32} and P_{12}, the altitude in both cases being taken on the common side, namely the line of the vector A_2. This in turn is seen to be equal to the ratio between the areas of P_{32} and P_{12}. Since these areas are given by certain two row determinants, we have the familiar formula for x_1 in terms of determinants.

The above derivation of Cramer's rule for the case $n=2$ appears at first to be dependent on the fact that $n=2$, and hence not capable of extension. For $n \geqq 3$, one could not plot the vectors except in perspective, and then the ratio of A_1x_1/A_1 would be affected. A closer examination, however, of the above derivation shows that the diagram did not enter into the proof. Let us illustrate by considering the case $n=3$.

The system of equations to be solved is:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= a_{14}, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= a_{24}, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= a_{34}. \end{aligned} \tag{3}$$

The vectors A_1, A_2, A_3 are defined as (a_{11}, a_{21}, a_{31}), (a_{12}, a_{22}, a_{32}), (a_{13}, a_{23}, a_{33}),

namely the columns of the matrix of coefficients, and $A_4=(a_{14}, a_{24}, a_{34})$, the set of right hand members. Instead of (2), we have

$$A_1x_1 + A_2x_2 + A_3x_3 = A_4. \tag{4}$$

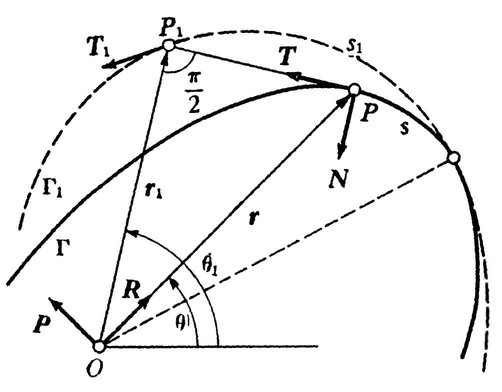
The parallelopiped obtained by completing the vectors A_i, A_j, A_k shall be denoted by P_{ijk}. The vector A_1x_1 is determined by the following two conditions: (1) It is a multiple of A_1, and (2) if a proper linear combination of A_2 and A_3 is added to it, the sum is A_4. The first of these conditions restricts A_1x_1 to the line from O to A_1, and the second restricts it to a plane, namely the plane passing through the point A_4, and parallel to the lines of the vectors A_2 and A_3. Excluding the case in which Cramer's rule does not hold, the determinant of the coefficients does not vanish, the area of P_{123} is not zero, and A_1 is not a linear combination of A_2 and A_3. Hence the above line is not parallel to the above plane, and they intersect at a point, thus uniquely determining the vector A_1x_1.

The value of x_1 is thereby determined as the ratio A_1x_1/A_1. This in turn is seen to be equal to the ratio between the altitudes of P_{423} and P_{123}, the altitudes in both cases being taken as the altitudes on the common face, namely the face whose two edges are the lines of the vectors A_2A_3. This in turn is seen to be equal to the ratio between the volumes of P_{423} and P_{123}. Since these volumes are equal to certain well known three row determinants, the value of x_1 has been obtained, and Cramer's rule derived.

II. A Note on Pedals of Plane Polar Curves

By H. K. Justice, University of Cincinnati

The usual method of obtaining the polar equation of the first positive pedal of any plane polar curve, with regard to the pole, requires the elimination of three quantities from four equations.



The purpose of this discussion is to present a rather interesting special method of deducing parametric equations of the pedal curve by expressing each of the coördinates of a point on the pedal as a function of either of the coördinates of the corresponding point on the original curve, through integrals involving the radius of curvature of the latter.

Let the point $P(r, \theta)$ traverse the arc, s, of the given curve, Γ, in such a way that the radius vector, $\boldsymbol{r}=OP$, rotates counterclockwise, in the positive sense determined by a unit binormal vector, $\boldsymbol{B}$, directed toward the reader. This establishes a positive direction for the unit tangent vector, $\boldsymbol{T}=d\boldsymbol{r}/ds$. Let $\boldsymbol{R}$ be a unit vector in the direction of $\boldsymbol{r}$, and define other unit vectors $\boldsymbol{N}$ and $\boldsymbol{P}$, according to the right hand rule, by the equations,

$$\boldsymbol{N} = \boldsymbol{B} \times \boldsymbol{T}, \quad \boldsymbol{P} = \boldsymbol{B} \times \boldsymbol{R}.$$

Designate by κ and ρ the curvature and radius of curvature of Γ, at P. Characterize, by the subscript 1, the symbols associated with the pedal, Γ_1, at the corresponding point $P_1(r_1, \theta_1)$. Evidently $\boldsymbol{r}_1=\boldsymbol{r}\cdot \boldsymbol{NN}$. Also[1]

$$\boldsymbol{T}_1 = \frac{d\boldsymbol{r}_1}{ds_1} = \frac{d\boldsymbol{r}_1}{ds}\frac{ds}{ds_1} = \frac{d}{ds}(\boldsymbol{r}\cdot \boldsymbol{NN})\frac{ds}{ds_1},$$

which, by virtue of Frenet's formula, $d\boldsymbol{N}/ds = -\kappa \boldsymbol{T}$, becomes

$$\boldsymbol{T}_1 = -\kappa(\boldsymbol{r}\cdot\boldsymbol{NT} + \boldsymbol{r}\cdot \boldsymbol{TN})\frac{ds}{ds_1}\cdot$$

Differentiating the obvious identity, $\boldsymbol{r}_1 = -r_1\boldsymbol{N}$, we obtain

$$\boldsymbol{T}_1 = -\frac{dr_1}{ds_1}\boldsymbol{N} - r_1\frac{d\boldsymbol{N}}{d\theta_1}\frac{d\theta_1}{ds_1} = -\frac{dr_1}{ds_1}\boldsymbol{N} + r_1\frac{d\theta_1}{ds_1}\boldsymbol{T}.$$

Hence

$$-\frac{dr_1}{ds_1}\boldsymbol{N} + r_1\frac{d\theta_1}{ds_1}\boldsymbol{T} = -\kappa(\boldsymbol{r}\cdot\boldsymbol{NT} + \boldsymbol{r}\cdot \boldsymbol{TN})\frac{ds}{ds_1}\cdot \tag{1}$$

Taking components along $\boldsymbol{T}$, we see that

$$r_1\frac{d\theta_1}{ds_1} = -\kappa r\boldsymbol{R}\cdot\boldsymbol{N}\frac{ds}{ds_1} = -\kappa r\boldsymbol{R}\times \boldsymbol{B}\cdot \boldsymbol{T}\frac{ds}{ds_1} = \kappa r\boldsymbol{P}\cdot \boldsymbol{T}\frac{ds}{ds_1},$$

from which it follows, by virtue of the equation,

$$\boldsymbol{T} = r\boldsymbol{P}\frac{d\theta}{ds} + \frac{dr}{ds}\boldsymbol{R}, \tag{2}$$

obtained by differentiating the identity, $\boldsymbol{r}=r\boldsymbol{R}$, that

$$d\theta_1 = \frac{\kappa r^2}{r_1}d\theta = \frac{\kappa r^2}{r_1 r'(\theta)}dr. \tag{3}$$

Similarly, resolving (1) in the direction of $\boldsymbol{N}$, we find that

[1] The writer acknowledges indebtedness to Professor Louis Brand, who has previously employed similar vector methods in the solution of other problems in differential geometry.

$$\frac{dr_1}{ds_1} = \kappa r \mathbf{R} \cdot \mathbf{T} \frac{ds}{ds_1},$$

which, with (2), yields the known result,

$$dr_1 = \kappa r dr = \kappa r r'(\theta) d\theta. \tag{4}$$

Thus the equations,

$$r_1 + C_1 = \int \frac{r}{\rho} dr = \int \frac{r}{\rho} r'(\theta) d\theta, \tag{5}$$

$$\theta_1 + C_2 = \int \frac{r^2}{\rho r_1 r'(\theta)} dr = \int \frac{r^2}{\rho r_1} d\theta, \tag{6}$$

obtained by integrating (4) and (3) respectively, express the coördinates, (r_1, θ_1), of a point on the pedal curve, as functions of either r, or θ, as a parameter. The constants of integration, C_1 and C_2, may be determined from the condition that the points (r, θ) and (r_1, θ_1) obviously coincide for ordinary extremes of r.

Example: Find the pedal of the curve $r^a = b^a \cos a\theta$, whose radius of curvature is $\rho = b^a r^{1-a}/(a+1)$.

From (5) and (6),

$$r_1 + C_1 = \int \frac{rdr}{\rho} = \frac{a+1}{b^a} \int r^a dr = \frac{r^{a+1}}{b^a} = b \cos^{(a+1)/a} a\theta,$$

$$\theta_1 + C_2 = \int \frac{r^2 d\theta}{\rho r_1} = \int \frac{(a+1) r^2 b^a}{b^a r^{1-a} r^{a+1}} d\theta = (a+1) \int d\theta = (a+1)\theta.$$

When $\theta = 0$, r has the maximum value b. Hence when $\theta = 0$, $r_1 = r = b$, and $\theta_1 = \theta = 0$. Therefore $C_1 = C_2 = 0$. By eliminating θ from the above equations, we deduce the polar equation of the pedal curve in the form,

$$r_1^{a/(a+1)} = b^{a/(a+1)} \cos \frac{a\theta_1}{a+1},$$

which obviously represents a member of the original family, and is obtainable from the equation of the latter by changing a to $a/(a+1)$. Interesting special cases result from specific numerical substitutions for a.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College of the City of New York.

All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Ames, J. S. and *Murnaghan, F. D.* Theoretical Mechanics. An Introduction to Mathematical Physics. Boston, Ginn and Co., 1929. ix+462 pages. $5.00.

Rietz, H. L. and *Crathorne, A. R.* College Algebra, Third Edition. New York, Henry Holt and Co., 1929.

"The book has been thoroughly revised again. ··· The exercises and problems have been [almost] completely changed. ··· The chapter on probability is new. ··· A new chapter on compound interest and annuities has been added."

Ford, Lester R. Automorphic Functions. New York, The McGraw-Hill Book Co., 1929. xii+334 pages. $4.50.

Coble, Arthur B. Algebraic Geometry and Theta Functions. American Mathematical Society Colloquium Publications, Vol. X. New York, 1929. viii+282 pages.

"An amplification of the Colloquium lectures delivered at Amherst in September, 1928 under the title, "The determination of the tri-tangent planes of the space sextic of genus four."

Schorling, Raleigh and *Clark, John R.* Modern Mathematics, New Edition. Yonkers, N. Y., The World Book Co., 1929. Seventh School year, xiv+274 pages; Eighth School Year, xiv+306 pages.

Campbell, J. W. An Introduction to Mechanics. Boston, Houghton, Mifflin Co., 1929. xiv+384 pages. $3.50.

Reynolds, Joseph B. Elementary Mechanics. New York, Prentice-Hall, Inc., 1929. viii+250 pages. $2.50.

The first of these two texts on mechanics presupposes elementary calculus, while the second is based only on secondary school algebra, geometry, and trigonometry.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Numerische Infinitesimalrechnung. By Martin Lindow. Ferd. Dümmlers, Berlin, 1928. 176 pages.

Interpolation. By J. F. Steffensen. The Williams and Wilkins Company, Baltimore. Md., 1927. 248 pages. $8.00.

Mathematical books belong to a number of different categories. Among them are monographs of original research; expositions of work scattered throughout the literature, bringing into available form and into a given language the original work of others; and, appearing at rare intervals, books doing both, making available what is already known, filling in the gaps and extending the theory.

Numerische Infinitesimalrechnung is of the second type, a useful, well arranged, clearly written tract on interpolation, in German. *Interpolation* by Professor J. F. Steffensen is in the last class, being written so clearly that one does not realize while first reading it how much novel material the book contains.

Each of these books has the virtue of having an honest introduction that gives the key to the subject matter and mode of approach.

Dr. Lindow says in his foreword: "Pure mathematics stresses absolute rigor and generality. But this is often unsuited to the practical examinations of a concrete case. That which is of the greatest use for the physicist, chemist, and engineer requires no proof." Dr. Lindow does not pretend to give a rigorous or novel mathematical treatment, but by a multiplicity of examples and clear explanations of the use of formulas, supplemented by tables of coefficients appearing in formulas frequently used in interpolation, he tries to be of the utmost use to the computer. The book should be especially commended for these tables. The general scope is given by the following headings: interpolation, numerical differentiation, numerical integration and numerical handling of differential equations. The scope of *Interpolation* is nearly coextensive except for the chapter on the "Calculus of Symbols." The topics discussed are interpolation, numerical differentiation, construction of tables, summation processes, mechanical quadrature, numerical integration of differential equations, interpolations with several variables, mechanical quadrature and the calculus of symbols.

Professor Steffensen, in the preface and introduction of *Interpolation*, gives the following outline of the purpose of the book:

"Formulas and methods are developed on the assumption that the function under consideration is a polynomial, and thereafter applied to functions which are certainly not polynomials." "Attempts have been made, now and then, to present the subject of interpolation, adopting the point of view that only such approximative formulas are to be included for which it is possible to derive a remainder-term simple enough to permit the calculation of limits to the error involved in the formula." "The number of formulas with workable remainder-terms has lately increased so much, that although further development is still possible and desirable, a fresh attempt should be made at writing a text-book on the aforesaid lines." "I wish it to be understood that the book is meant as a text-book, and not as a hand-book or encyclopedia on the subject. To carry through with consistency the point of view which appears to me to be the only tenable one, has been my principal aim." "The mathematical equipment required in order to master the book is very small. A knowledge of the first principles of the differential and integral calculus should be sufficient." "In practice, it is a very general custom to derive formulas of interpolation on the assumption that the function with which we have to deal is a polynomial of a certain degree. . . . If it is applied to a polynomial of higher degree or to a function which is not a polynomial, nothing whatever is known about the accuracy obtained. . . . If we have to deal with a numerical calculation, it is not sufficient to know that an approximation is obtainable; what we want to know, is how close is the approximation actually obtained."

It is a pleasure to say that after stating his intention, Professor Steffensen hews to the line. The distinction between interpolation when a remainder term

is known and interpolation when the result is merely a guess is emphasized throughout; for instance, on page 53: "The result of an interpolation under such circumstances must be considered as an hypothesis, and not as a mathematically proved fact."

The consistency with which this purpose is carried out is seen from the fact that such formulas as Newton's, Stirling's, Everett's, Bessel's and Gauss's and Lagrange's interpolation formulas, the formulas of numerical differentiation, inverse interpolation and mechanical quadrature, and such formulas as Laplace's, Gauss's, Euler's, and Lubbock's and Woolhouse's summation formulas are all given with remainder terms. A great effort has been made to make these remainder terms as applicable and simple as possible. This has met with striking success. Even the formulas of interpolation in more than one variable, of mechanical quadrature, and of numerical integration of differential equations are supplied with remainder terms.

As a striking warning to our intuition the following example is developed: Consider $1/(1+x^2)$ in the interval $-5 \leqq x \leqq 5$; divide the interval into $5v$ equal intervals, where $v=4k-1$. It would seem to one's intuition that a polynomial going through the end points of these intervals would approximate the function more and more closely as k increases. However, it is found that the maximum error becomes infinite exponentially with increasing k.

The whole book is written on a different plan from most discussions of interpolation, being a worthy continuation of Markoff's work. It can be read with pleasure by the mathematician as the proofs are elegant and rigorous, the hypotheses are clearly stated and in no place is intuition used instead of brains, as in many works on applied mathematics. Moreover, the forbidding symbolism usual in the discussion of this subject is minimized. The problems are well chosen to illustrate the theory, though perhaps those in Dr. Lindow's book more completely indicate the difficulties the computer may meet.

It would be unfair to *Interpolation* not to speak of its physical aspects. It is printed on paper which is pleasant to handle and in clear, attractive type.

Rarely does the reviewer read a book which is mathematically as completely satisfying as *Interpolation* and at the same time one which a good senior could read, except in rare spots, with only moderate difficulty and with great profit. No one who has much to do with interpolation either theoretical or applied could afford to do without Professor Steffensen's book, while *Numerische Infinitesimalrechnung* will prove of use to many computers.

M. H. INGRAHAM

Mengenlehre. By E. Kamke. Berlin, Walter de Gruyter & Co., 1928. 159 pages.

The subject matter of this excellent little book is the general theory of aggregates, with particular reference to transfinite numbers. It is divided into four chapters, as follows: I, the definition of such terms as sum, divisor, etc., and the elementary theorems on enumerable sets; II, the usual theorems on

cardinal numbers and their arithmetic; III, order-types in general, including the identification of the number-continuum with its order-type; IV, the ordinal numbers and the associated aleph numbers. The book is rich in illustrations and applications to the point-set theory. The author follows the traditional treatment of the subject, using Cantor's definition of an aggregate and ignoring the controversy over the axiom of choice; at the end, however, he gives a brief account of the well-known paradoxes. The book covers an astonishing amount of ground for its size, and is clearly and simply written. It should be very useful as an introduction to the subject and it is just the thing for the beginner in the theory of functions of real variables, who needs the commonly accepted theorems as tools, but does not desire to plough through the difficult philosophical questions involved. Harassed directors of undergraduate mathematical clubs may also find something to interest them.

WALLACE A. WILSON

Six-Place Tables, a selection of tables of squares, cubes, square roots, cube roots, fifth roots and powers, circumferences and areas of circles, common logarithms of numbers and of the trigonometric functions, the natural trigonometric functions, natural logarithms, exponential and hyperbolic functions, and integrals. With explanatory notes by Edward S. Allen. Third Edition. McGraw-Hill Book Company, New York, 1929. $1.50.

The scope of this little book—found by experiment to be almost literally a vest-pocket edition—is well indicated by the complete title, as given above. There are six-place tables of logarithms of numbers, and of the trigonometric functions and their logarithms. Natural logarithms occupy one page, and exponential and hyperbolic functions two pages. Table I, occupying sixteen pages, gives certain functions of all integers up to 1000, namely the square, cube, square root, cube root, reciprocal, and circumference and area of circle having the given radius. Table III, curiously enough, seems superfluous as well as incorrect. It tabulates circumferences and areas of circles for radii increasing by eighths of a unit up to 100; and the entries, given to six significant figures, are based on the value 3.1416 for π, and therefore are in many cases in error in the last figure. The same data as given in table I are correctly determined, circumferences to six significant figures and areas to eight.

There is a list of about 150 integrals, with a simple summary of the method of partial fractions.

R. A. J.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3389. *Proposed by J. V. Uspensky.*

Let the function $F(s)$ be defined as follows:

$$F(s) = \frac{1}{(n-1)!}\sum_{\nu=0}^{\nu=s}(-1)^{\nu}\frac{n(n-1)\cdots(n-\nu+1)}{\nu!}(s-\nu)^{n-1},$$

n being a positive integer and s a positive variable. Show that for every n which is very large the following equation holds:

$$\int_0^n s^{-1}F(s)ds = 2n^{-1} + n^{-2}\theta_n,$$

where θ_n remains bounded when n increases indefinitely.

3390. *Proposed by Paul Wernicke, Washington, D. C.*

With the altitudes of a triangle $t_i = A_iB_iC_i$ as sides construct a consecutive triangle t_{i+1} in the series of $t_i(\cdots -1, 0, 1, \cdots)$. Compare the areas of the triangles t_i in the series. Under what condition does the construction become impossible?

3391. *Proposed by Otto Dunkel, Washington University.*

The ratio of the shortest diagonal to a side of an ordinary regular polygon of 19 sides satisfies an equation of the 9th degree with rational coefficients. The remaining roots are the corresponding ratios with alternately plus and minus signs for the remaining regular polygons of star form. By a known theorem the roots of this equation may be obtained by solving first a cubic with rational coefficients and then solving three other cubics whose coefficients are rational functions of the roots of the first cubic. Derive with as little computation as possible a set of such equations.

3392. *Proposed by V. M. Spunar, Chicago, Ill.*

Two points M and N are taken on the sides AB and AC, respectively, of the triangle ABC; and then the point P is taken on the line MN. If these points are chosen so that $BM/MA = AN/NC = MP/PN$, find the locus of P.

3393. *Proposed by Emma M. Gibson, Central High School, Springfield, Mo.*

Suppose a solid is altered so that straight lines remain straight lines and parallel lines parallel. Show that a parallelogram remains a parallelogram, a straight line is stretched to the same extent at all points, parallel straight lines are equally stretched, and an ellipse becomes an ellipse. I.C.S. 1902.

SOLUTIONS

3342 [1928, 445]. *Proposed by R. Goormaghtigh, La Louvière, Belgium.*

Let α, β, γ, δ be the points where the straight lines AP, BP, CP, DP, meet the faces of the tetrahedron $ABCD$; the perpendiculars dropped from the vertices A, B, C, D on the lines joining respectively α, β, γ, δ to the orthocentre H meet the corresponding faces of the tetrahedron $ABCD$ in four coplanar points. The plane passing through these four points is perpendicular to PH.

Note by the Editors: This problem is a generalization of Problem 3228 [1926, 525 and 1928, 42]. See also the solution of 3258 [1928, 210].

Solution by Nathan Altshiller-Court, University of Oklahoma.

The orthocenter H of the given tetrahedron $ABCD$ is the center of the sphere (H) with respect to which the tetrahedron is self polar. The polar plane of α with respect to (H) passes through A and is perpendicular to αH, hence this plane contains the perpendicular line $A\alpha'$ dropped from A to the line αH. The trace α' of $A\alpha'$ in the plane BCD is thus conjugate to α with respect to (H).

The polar plane of α' with respect to (H) contains the two points A, α conjugate to α' with respect to (H); hence this plane contains also the point P, which is collinear with A and α. Similarly for the points β', γ', δ' analogous to α'. Consequently the four points α', β', γ', δ' lie in the same plane, namely, the polar plane of P with respect to (H), which plane is therefore perpendicular to PH.

Note: By a strange coincidence the proposer and the present writer have simultaneously and quite independently of each other stated and proved the proposition in the plane of which the proposition under discussion is an immediate extension. The plane proposition appeared in the April, 1928, issues of "Mathesis" (p. 173) and of this "Monthly" (p. 210).

Also solved by the Proposer.

Note by the Editors: In the proof above, the sphere (H) is not real if H lies within the tetrahedron. In this case the device used in the note [1928, 211] may be employed to complete the proof.

3343 [1928, 446]. *Proposed by J. V. Uspensky.*

Show that

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_{2n\pi}^{\infty} \frac{\sin z}{z} dz = \pi - \frac{\pi}{2} \log 2\pi$$

and that

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_{n\pi}^{\infty} \frac{\sin z}{z} dz = \frac{\pi}{2} - \frac{\pi}{2} \log \pi.$$

Solution by Harry Langman, Arverne, L. I.

The first sum may be written

$$A = \sum_{n=1}^{\infty} \frac{1}{n} \int_{2\pi}^{\infty} \frac{\sin nz}{z} dz = \sum_{n=1}^{\infty} \frac{1}{n} \int_{\pi}^{\infty} \frac{(-1)^n \sin nz}{z+\pi} dz$$

$$= - \int_{\pi}^{\infty} \frac{dz}{z+\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nz}{n}$$

$$= - \sum_{k=1}^{\infty} \int_{(2k-1)\pi}^{(2k+1)\pi} \frac{dz}{z+\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nz}{n}.$$

For values of z between $(2k-1)\pi$ and $(2k+1)\pi$ the sum $\sum_{n=1}^{\infty}(-1)^{n+1} \sin nz/n$ becomes $z/2 - k\pi$. We have then:[1]

$$A = - \sum_{k=1}^{\infty} \int_{(2k-1)\pi}^{(2k+1)\pi} \frac{dz}{z+\pi} \left(\frac{z}{2} - k\pi \right)$$

$$= -\frac{1}{2} \sum_{k=1}^{\infty} \Big[z - (2k+1)\pi \log (z+\pi) \Big]_{(2k-1)\pi}^{(2k+1)\pi}$$

$$= \frac{\pi}{2} \sum_{k=1}^{\infty} \big[(2k+1) \log (k+1) - (2k+1) \log k - 2) \big] = \frac{\pi}{2} \lim_{r\to\infty} u_r,$$

where

$$u_r = \sum_{k=1}^{r} \big[(2k+1) \log (k+1) - (2k+1) \log k - 2 \big]$$

$$= (2r+1) \log (r+1) - 2 \sum_{1}^{r} \log k - 2r.$$

Applying Stirling's theorem, we have:

$$\lim_{r\to\infty} e^{u_r} = \lim_{r\to\infty} \frac{(r+1)^{2r+1}}{e^{2r}(r!)^2} = \lim_{r\to\infty} \frac{(r+1)^{2r+1}}{e^{2r}} \cdot \frac{e^{2r}}{2r\pi r^{2r}} = \lim_{r\to\infty} \frac{1}{2\pi} \left(1 + \frac{1}{r} \right)^{2r+1} = \frac{e^2}{2\pi},$$

whence $A = \pi - \frac{1}{2}\pi \log 2\pi$.

The second sum may similarly be written

$$B = - \int_{0}^{\infty} \frac{dz}{z+\pi} \sum_{k=1}^{\infty} \frac{(-1)^{n+1} \sin nz}{n} = \pi - \frac{\pi}{2} \log 2\pi - \int_{0}^{\pi} \frac{dz}{z+\pi} \cdot \frac{z}{2},$$

which easily reduces to $\frac{1}{2}\pi - \frac{1}{2}\pi \log \pi$.

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[1] Chrystal, *Algebra*, II, 308.

3344 [1928, 446]. *Proposed by B. F. Yanney, Wooster, Ohio.*

Two altitudes of a given triangle and the side from whose extremities these altitudes are drawn meet in collinear points the corresponding sides of the orthic triangle of the given triangle.

Solution by H. A. Do Bell, New York State College for Teachers.

In the given triangle, ABC, let the altitudes from A, B, C intersect the opposite sides in D, E, F, respectively, and denote the orthocenter by H. It will be shown first that the altitudes AD, BE, and the side AB meet the corresponding sides of the orthic triangle in collinear points. The procedure is to select two triangles which are perspective from a center, thereby satisfying Desargues's Theorem. The triangles DEF and BAH are perspective from C. Therefore, if we set $(DE)\ (BA)=X$; $(DF)\ (BH)=(DF)\ (BE)=Y$; $(EF)\ (AH) =(EF)\ (AD)=Z$; the points X, Y, Z are collinear.

The other two cases of combining two of the three altitudes are shown similarly using centers A and B.

The property of perpendicularity which the altitudes possess plays no role in the above solution, but the fact that they are concurrent at H is all important. Hence, similar theorems can be formulated by using the medians (meeting in the centroid) and also by using proper angle-bisectors (meeting in the in-center and three ex-centers).

Also solved by Rufus Crane, E. D. Eaves, L. W. Johnson, Harry Langman, J. H. Neelley, and A. Pelletier.

3345 [1928, 446]. *Proposed by B. F. Finkel, Drury College.*

A mill wheel of radius a revolves so that its rim has a velocity v, and drops of water are thrown off from the rim. Find the envelope of the paths of the drops.

Solution by P. S. Dwyer, Antioch College.

Assume the origin to be at the center of the wheel, that the rotation is positive, and that the angular displacement of a point on the wheel is α. We then have the parametric equations,

$$x = a\cos\alpha - vt\sin\alpha,\quad y = a\sin\alpha + vt\cos\alpha - \tfrac{1}{2}gt^2,$$

which locate at any time, t, the position of a drop starting from any point $(a\cos\alpha,\ a\sin\alpha)$ on the circumference of the wheel. If we eliminate the time parameter from these equations we get

$$(1)\qquad y = a\csc\alpha - x\cot\alpha - \tfrac{1}{2}gv^{-2}\csc^2\alpha(a\cos\alpha - x)^2,$$

which is the family of parabolas traversed by the various drops.

Differentiating with respect to α, we find

$$(2)\qquad 0 = v^{-2}\csc^2\alpha(a\cos\alpha - x)(v^2 - ga\csc\alpha + gx\cot\alpha).$$

(a) When $a \cos\alpha - x = 0$, $y = a \sin\alpha$, and we have the equation of the wheel's circumference, $x^2+y^2=a^2$.

(b) $v^{-2}csc^2\alpha=0$ is inapplicable.

(c) The last factor gives

$$x = a \sec\alpha - v^2g^{-1}\tan\alpha. \tag{3}$$

Substituting (3) in (1) and simplifying, we get

$$y = \tfrac{1}{2}v^2g^{-1} - \tfrac{1}{2}gv^{-2}[(v^4g^{-2}+a^2)\tan^2\alpha - 2av^2g^{-1}\sec\alpha\tan\alpha]. \tag{4}$$

Squaring x in equation (3), we find

$$x^2 - a^2 = (v^4g^{-2}+a^2)\tan^2\alpha - 2av^2g^{-1}\sec\alpha\tan\alpha\ ; \tag{5}$$

and eliminating α from (4) and (5), we find as the equation of the envelope:

$$x^2 = -\ 2v^2g^{-1}[y - \tfrac{1}{2}gv^{-2}(v^4g^{-2}+a^2)].$$

The envelope is a parabola having its vertex at a distance of $\frac{1}{2}g(v^2g^{-2}+a^2v^{-2})$ above the center of the wheel.

Also solved by W. B. Campbell, William Hoover, Enrique Linares, and Paul Wernicke.

3346 [1928, 446]. *Proposed by Frank Irwin, University of California.*

Show that, if $P_1P_2 \cdots P_nP_1$ be any polygon of n sides, the broken line whose segments are parallel and equal to P_1P_k, P_2P_{k+1}, $\cdots$, P_nP_{k-1} $(P_{n+i}=P_i)$ in order will close. Generalize.

Again let S_i be the middle point of P_iP_{i+1}, $i=1, 2, \cdots, n$, $(P_{n+1}=P_1)$; show that the broken line whose sides are parallel and equal to P_1S_k, P_2S_{k+1}, $\cdots$, P_nS_{k-1} in order will close. Generalize.

Solution by the Proposer.

The propositions are quite obviously true when once stated, so that their only interest is in the propositions themselves. They hold in three, or n, dimensions.

Let α_i be the vector from any origin, O, to P_i. We shall generalise our first proposition as follows: *Let Q_1, Q_2, $\cdots$, Q_n be the n points P_i in any order, and similarly R_1, R_2, $\cdots$, R_n. Then the broken line formed with the vectors Q_1R_1, Q_2R_2, $\cdots$, Q_nR_n will close*; for since the vector $P_iP_j=\alpha_j-\alpha_i$, we see that in the vector sum, $Q_1R_1+Q_2R_2+\cdots$, each α_i will occur once with a plus sign and once with a minus sign and the vector sum will be zero.

To generalise the second proposition, *let S_i lie on P_iP_{i+1} and let $P_iS_i/P_iP_{i+1} = l$, $i=1, 2, \cdots, n$. Let Q_1, Q_2, $\cdots$, Q_n be, as before, the points P_i in any order, and let R_1, R_2, $\cdots$, R_n now be the points S_i in any order. Then again the broken line formed with the vectors, Q_1R_1, Q_2R_2, $\cdots$, will close*; for the vector $OS_i = (1-l)\alpha_i + l\alpha_{i+1}$, so that the vector sum, $Q_1R_1+Q_2R_2$, $\cdots$, will equal

$$(1-l)(\alpha_1+\alpha_2+\cdots) + l(\alpha_1+\alpha_2+\cdots) - (\alpha_1+\alpha_2+\cdots) = 0.$$

A familiar case of this proposition is that a triangle may be constructed with sides equal and parallel to the medians of any triangle.

We may generalize further by using in place of the points Q_i (in the last proposition) points T_i, where T_i lies on P_iP_{i+1} and $P_iT_i/P_iP_{i+1}=m$. The last proposition is the special case $m=0$. Further generalisations readily suggest themselves.

Also solved by Harry Langman, Enrique Linares, A. Pelletier, J. Rosenbaum, and Paul Wernicke.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

The Fine Memorial Mathematics Hall, which will be erected at Princeton University at a cost of $400,000 in memory of the late Henry B. Fine, for many years a professor of mathematics and dean of science, will be started in the near future.

The New York Academy of Sciences offers its A. Cressy Morrison prize "No. 1" for 1930 for a paper on solar and stellar energy. Compteition is open to all.

At Hunter College of the City of New York, exceptional lecture courses on mathematical statistics will be given in the evening session during 1929–30. The department of mathematics has been fortunate in securing the part time services of Mr. Arne Fisher, well-known statistician of the Western Union Company. Mr. Fisher has agreed to offer, in conjunction with the regular curricular work, two such courses.

Assistant Professor E. C. Bower, of the department of astronomy and mathematics at Ohio Wesleyan University, has been appointed Martin Kellogg fellow at the Lick Observatory of the University of California, with academic residence at Berkeley.

At Ohio State University, Professor Alfred Lande, of the University of Tübingen, has been appointed visiting professor of theoretical physics for the autumn and winter quarters of 1929–30. Dr. L. H. Thomas, of Trinity College, Cambridge, has been appointed visiting assistant professor of theoretical physics for the autumn, winter, and spring quarters.

Dr. W. L. Ayres has been appointed to an assistant professorship at the University of Michigan.

Professor D. P. Bartlett, of the Massachusetts Institute of Technology, has retired.

Professor R. D. Carmichael has been appointed administrative head of the department of mathematics at the University of Illinois, as successor to Professor E. J. Townsend, who has retired.

Assistant Professor P. D. Edwards, of Ball Teachers College, has been promoted to an associate professorship.

Mr. W. I. Foster, of Rochester Junior College, has been appointed assistant professor of mathematics at Northern Montana School.

Dr. B. F. Kimball, of Cornell University, has been appointed to an assistant professorship at the University of New Hampshire.

Assistant Professor E. E. Libman, of the University of Illinois, has resigned to accept a position in the marine and air craft engineering department of the General Electric Company.

Associate Professor T. A. Pierce, of the University of Nebraska, has been promoted to a professorship.

Assistant Professor S. A. Schelkunoff, of the State College of Washington, has resigned to accept a position in the Bell Telephone Laboratories, New York City.

Mr. C. K. Sherer, of the University of Nebraska, has been appointed head of the department of mathematics at Texas Christian University.

Assistant Professor G. W. Smith, of the University of Kansas, has been promoted to an associate professorship.

Dr. Roxana H. Vivian has been appointed professor of mathematics at Hartwick College, Oneonta.

Mr. E. H. Wells, of Princeton University, has been appointed to an assistant professorship at the University of New Hampshire.

Dr. Edgar W. Woolard, of the George Washington University, has been promoted to an assistant professorship of mathematics.

Dr. Harvey A. Zinszer, professor of physics and acting professor of mathematics at Hanover College, has been appointed professor of physics and astronomy at the Kansas State Teachers College.

The following appointments to instructorships are announced:

Connecticut College, Miss Grace Shover.

Newark College of Engineering, Mr. J. H. Fithian.

New York University, Dr. D. A. Flanders, Mr. E. H. Johnson, Mr. A. S. Peters.

Ohio State University, Dr. P. M. Swingle.

Pennsylvania State College, Dr. Leo Zippin.

Professor F. C. Kent, of the Oregon State Agricultural College, died June 11, 1929.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of America the sum of .... Dollars, to be known as the .... Fund, and to be used

for[2] {Endowment—the income only of which may be expended.
Special Projects—for which both principal and income may be expended.

Witness: .... Signature ....

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.
[2] Indicate which one of the two purposes is desired, and omit the other.

The Association needs funds for scientific publications and for the promotion of scientific activities.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Thirteenth Summer Meeting of the Association, Boulder, Colorado, August 26-27, 1929.

Fourteenth Annual Meeting, Des Moines, Iowa, December 31, 1929, January 1, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1929.

ILLINOIS, Carthage, Ill., May 3-4.

INDIANA, Culver Military Academy, May 3-4.

IOWA, Fairfield, Iowa, April 26-27.

KANSAS, Topeka, Kansas, February 2.

KENTUCKY, Lexington, Ky., April 13.

LOUISIANA-MISSISSIPPI, Lafayette, La., April 12-13.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, May 4.

MICHIGAN, Ann Arbor, Mich., March 16.

MINNESOTA, St. Paul, Minn., May 11.

MISSOURI, Kansas City, Mo., November 16

NEBRASKA.

OHIO, Columbus, Ohio, April 4.

PHILADELPHIA, University of Pennsylvania, November 30.

ROCKY MOUNTAIN, Greeley, Colo., April 12-13.

SOUTHEASTERN, Macon, Ga., April 19-20.

SOUTHERN CALIFORNIA, University of Redlands, March 9.

TEXAS, Houston, Texas, Jan. 26.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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THE THIRTEENTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The thirteenth summer meeting of the Mathematical Association of America was held, by invitation, at the University of Colorado, Boulder, Colo., on Monday and Tuesday, August 26–27, 1929, in conjunction with the summer meeting and colloquium of the American Mathematical Society. Two hundred thirty-nine were present at the meetings, including the following one hundred twenty-one members of the Association.

E. F. ALLEN, Bryn Athyn, Pa.

CLARA L. BACON, Goucher College
W. D. BATEN, Ann Arbor, Mich.
SUZAN R. BENEDICT, Smith College
W. J. BERRY, George Washington University
R. W. BRINK, University of Minnesota
JACK BRITTON, University of Colorado
MYRTLE C. BROWN, Teachers College, Denton, Tex.
R. E. BRUCE, Boston University
H. E. BUCHANAN, Tulane University
J. E. BURNAM, Simmons College
W. H. BUSSEY, University of Minnesota

W. D. CAIRNS, Oberlin College
C. C. CAMP, University of Nebraska
E. W. CHITTENDEN, University of Iowa
A. G. CLARK, Colorado Agricultural College
ABRAHAM COHEN, Johns Hopkins University
TERESA COHEN, Pennsylvania State College
E. E. COLYER, Hays (Kans.) State Teachers College
LENNIE P. COPELAND, Wellesley College
D. R. CURTISS, Northwestern University

OSCAR DAHLENE, University of Alabama
D. R. DAVIS, University of Oregon
H. T. DAVIS, Indiana University
ALICE C. DEAN, Rice Institute
W. W. DENTON, University of Michigan
L. L. DINES, University of Saskatchewan
E. L. DODD, University of Texas

C. M. ERIKSON, Michigan State Normal College
J. R. EVERETT, Colorado School of Mines

J. C. FITTERER, Colorado School of Mines
L. R. FORD, Rice Institute
W. B. FORD, University of Michigan
T. C. FRY, Bell Telephone Laboratories, New York, N. Y.

M. G. GABA, University of Nebraska
J. L. GIBSON, University of Utah
CORNELIUS GOUWENS, Iowa State College
F. L. GRIFFIN, Reed College
LOIS W. GRIFFITHS, Northwestern University

SIDNEY HACKER, University of Colorado
LAURENCE HAMPTON, University of Oklahoma
J. A. HARDIN, Centenary College
W. L. HART, University of Minnesota
E. R. HEDRICK, University of California at Los Angeles
CORA B. HENNEL, Indiana University
GERTRUDE A. HERR, Iowa State College
H. C. HICKS, Texas Tech. College
T. H. HILDEBRANDT, University of Michigan
GOLDIE P. HORTON, University of Texas
JEWELL C. HUGHES, University of Arkansas
J. W. HURST, Montana State College
C. A. HUTCHINSON, University of Colorado

M. H. INGRAHAM, University of Wisconsin

DUNHAM JACKSON, University of Minnesota

A. J. KEMPNER, University of Colorado
CLARIBEL KENDALL, University of Colorado
J. H. KINDLE, University of Cincinnati
H. W. KUHN, Ohio State University

C. G. LATIMER, University of Kentucky
D. N. LEHMER, University of California at Berkeley
O. C. LESTER, University of Colorado
A. J. LEWIS, University of Denver
C. F. LEWIS, Kansas State Agric. College
G. H. LIGHT, University of Colorado
W. V. LOVITT, Colorado College
R. G. LUBBEN, University of Texas

S. L. MACDONALD, Colorado State Agric. College
ELSIE MCFARLAND, University of Oklahoma
A. S. MCMASTER, University of Colorado
G. E. MARCH, South Dakota State School of Mines
E. L. MICKELSON, New Mexico State Teachers College
R. R. MIDDLEMISS, University of Colorado
W. E. MILNE, University of Oregon
U. G. MITCHELL, University of Kansas
THIRZA A. MOSSMAN, Kansas State Agric. College
R. L. MOORE, University of Texas
C. A. MURRAY, West Texas State Teachers College

W. L. NELSON, University of Colorado
C. V. NEWSON, University of New Mexico
J. A. NYSWANDER, University of Michigan

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E. J. PURCELL, University of Colorado

E. D. RAINVILLE, University of Colorado
SUSAN M. RAMBO, Smith College
A. W. RANDALL, Prairie View (Tex.) State College
G. E. RAYNOR, University of Oklahoma
B. L. REMICK, Kansas State Agric. College
R. G. D. RICHARDSON, Brown University
D. H. RICHERT, Bethel College
H. L. RIETZ, University of Iowa
J. H. ROBERTS, University of Texas
T. G. RODGERS, New Mexico Normal University
W. H. ROEVER, Washington University

MARY S. SABIN, East High School, Denver
W. A. SHEWHART, Bell Telephone Laboratories, New York, N. Y.
C. H. SISAM, Colorado College
E. B. SKINNER, University of Wisconsin
L. L. SMAIL, Lehigh University
CLARA E. SMITH, Wellesley College
G. W. SMITH, University of Kansas
VIRGIL SNYDER, Cornell University
C. C. SPOONER, Northern (Mich.) State Teachers College
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E. B. STOUFFER, University of Kansas

A. HELEN TAPPAN, Western College
H. G. TITT, Huron College
M. O. TRIPP, Wittenberg College
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MARION B. WHITE, Carleton College
G. T. WHYBURN, Stanford University
R. M. WINGER, University of Washington

The registration figures indicate the great success of this first meeting of the Association in the far West. The attendance was beyond the expectations, almost beyond the hopes, of those in charge, for the membership in the Rocky Mountain region is quite scattered. However the wide publicity given to the meetings and the natural attractions of Colorado resulted in a most gratifying attendance. The majority of members of the Rocky Mountain Section attended the meetings of the Association and the Society and repeatedly gave expression to the inspiration which came to them from the presence of members from other regions; and, in turn, those who had not known the institutions and mathematics teachers of Colorado and neighboring states found much pleasure and profit in making these new acquaintances.

The visiting members were comfortably cared for in several fraternity houses except for a few who found it quite feasible to live in their tents at the Boulder Municipal Camp. All took their meals at the University Cafeteria which held over for the week following the close of the summer quarter. The elastic hours and convenient grouping at the tables made possible much informal visiting such as is always a feature of the mathematics meetings.

About forty of the visitors were present at Colorado Springs on the Sunday preceding the meetings for a visit about the campus and buildings of Colorado College and a pleasant tea at the home of Professor and Mrs. Sisam. The ladies associated with the department of mathematics at Boulder gave a reception on Monday afternoon in the Women's Building especially for the visiting ladies; since the men of the group were also cordially invited, this took the character of a general reception to inaugurate the meetings.

Following the first colloquium lecture and session of the Society Tuesday afternoon, the visitors went in university busses or in their own automobiles to the University Camp, situated twenty-eight miles north-west of Boulder at an

elevation of 9,600 feet. Dinner was served here, and then the guests repaired to a camp-fire while tables were cleared and benches were rearranged for the more formal part of the evening. For this feature Professor Kempner first called on Dean Lester who welcomed the visitors on behalf of the University. Professor Hedrick then spoke as president of the Society, expressing our pleasure and the value of our meetings in this region, and emphasizing the wide-spread and outstanding character of the Society's work. Professor Cairns, representing the Association, told of its present activities, mentioning particularly the thought and planning being devoted to the Carus Monographs, the forthcoming Chace publication of the Ahmes Papyrus and the contemplated reestablishment of the Bibliotheca Mathematica. More definite announcements will be made within a few weeks concerning the Association's publications. Secretary Cairns proposed a vote expressing the appreciation of the visitors to the University authorities and mathematics faculty for their genial hospitality, to Professor Sisam and the program committee for the formation of a strong program, and to Professor Hutchinson and the other members of the recreations committee for their extensive plans for the comfort of the visitors and for the various outings of the week. This was a noteworthy feature of the Boulder meetings, for the recreation facilities of the university, so ably directed by Professor Hutchinson as a valuable adjunct of the summer quarter, were carried over for an extra week solely for the entertainment of the mathematicians. The motion of appreciation was carried heartily by a rising vote. Professor Emeritus DeLong gave an interesting series of reminiscences of the earlier years of the University and of the department of mathematics. Professor Hutchinson as the last speaker described more in detail the recreations department, its methods, purposes and ideals.

Wednesday was an open day and was devoted to an excursion into the mountains by university and private automobiles. The drive was made by the beautiful South St. Vrain Canyon and Estes Park village to a point near Horseshoe Falls, where lunch was served in the open by the university. In the afternoon the drive was continued from an elevation of about 8,500 feet to the summit of Fall River Pass, an elevation of 11,797 feet, well above the timberline. Clouds sweeping over the pass, a rainstorm a mile or two to the south, and numerous snowball fights made this trip noteworthy to many of the group not used to the mountain experiences. The party returned to Boulder by way of North St. Vrain Canyon in ample time for dinner. Other shorter trips were made by smaller groups during the week. Many planned their vacations so as to make a stay in the mountains for two or three weeks preceding or following the meetings.

The American Mathematical Society held its thirty-fifth summer meeting and thirteenth colloquium from Tuesday to Friday, with lectures by Professor R. L. Moore of the University of Texas on "Point set theory." More than ninety were present at the colloquium lectures. Sessions for the reading of papers were held on Tuesday afternoon, Thursday morning and afternoon and Friday morning.

The Mathematical Association held sessions on Monday afternoon and Tuesday morning, Professor Rietz presiding at the first session and Professor Bussey at the second session. The program was arranged by a committee consisting of Professors C. H. Sisam (chairman), E. W. Chittenden, E. E. DeCou, T. M. Putnam and S. W. Reaves. Abstracts of some of the papers are given, numbered in accordance with the numbers of the papers.

First Session of the Association

(1) "The undergraduate mathematical curriculum in a liberal arts college," by Professor F. L. Griffin, Reed College.

(2) "Differential equations as a foundation for electrical circuit theory," by Dr. T. C. Fry, Bell Telephone Laboratories, New York City.

(3) "Preliminary tests in mathematics for college freshmen," by Professor W. L. Hart, University of Minnesota.

1. The paper of Professor Griffin will appear in an early issue of the Monthly.

In comment on Professor Griffin's paper, Professor Hedrick stated that a considerable variation from the suggested scheme must be made to accord with the taste of the instructor and the wishes or the abilities of the students. He would like some one to undertake a condensed course covering modern geometry and descriptive geometry and perhaps including projective geometry; there is much material of a general nature in these fields that should be known to prospective teachers. We need also to make the values of mathematics evident in quarters where this appeal is not now felt; we shall convince the world of the importance of mathematics by actually making it useful in other fields.

2. The foundation for the widespread use of complex quantities in dealing with electrical circuit theory lies in certain simple properties of differential equations, and can be explained much more effectively in an elementary course in that subject than in one devoted to the more technical aspects of engineering. Because of this, Dr. Fry urged that the laying of such a foundation be made a part of courses on differential equations when a part of the class is composed of technical students and that the importance of the subject to these students be emphasized whenever possible.

The paper will appear in a later issue of the Monthly.

3. Professor Hart presented the results of a statistical investigation concerning 600 freshmen in classes in mathematics in the College of Arts at the University of Minnesota in the year 1926–27. One half of this group had had two units of mathematics in high school and the remainder had had two and one-half units. The data for the investigation consisted of the following items: (1) the performance of the students on taking the two examinations in mathematics of the Iowa series of placement tests, before the opening of college; (2) the subsequent performance of the students in their first course in college mathematics; (3) the scores obtained by some of the students on the general college ability rating scale used at the University of Minnesota. On the basis of the

results presented, conclusions were drawn concerning the preliminary and the placement plan which might be adopted for a non-homogeneous group of freshmen such as had been considered.

In comment upon the coefficient of correlation .6 Professor Gibson said that his experience is that in general a coefficient as large as this is of some significance, but that in this case it is probably not high enough to be significant, that in his judgment the scoring of college ability is defective. The availability of a correlation coefficient between the grades of the first and second examinations and the grade in the course is most to be desired except for the administrative difficulty of shifting pupils from one section to another better suited to their ability. We must usually fall back on placement tests such as are given at the various universities.

SECOND SESSION OF THE ASSOCIATION

(4) "Some aspects of ordinary differential equations," retiring presidential address, by Professor W. B. FORD, University of Michigan.

(5) "Factorization of numbers," by Professor D. N. LEHMER, University of California.

(6) "The application of groups to geometry," by Professor R. M. WINGER, University of Washington.

4. Professor Ford outlined at some length the Fuchs theory of ordinary differential equations and contrasted with this a general theory as developed by Dini. This theory is allied to the theory of integral equations and covers solutions not comprised by the old theory, besides formulating the theory for the solutions in a much more general form. It is to be hoped that Professor Ford will make this presentation available for the members of the Association, particularly since the integral equation methods and related theory play so prominent a part at present in the study of wave mechanics.

5. Professor Lehmer described a set of factor stencils which is being prepared by the Carnegie Institution of Washington, for the purpose of facilitating the finding of factors of numbers as high as two billion and a half. The theory on which the process rests, viz., certain theorems on quadratic residues, was also presented and illustrated. A fuller notice will appear soon in the Bulletin of the American Mathematical Society.

This supplements the well-known list of primes published by the Carnegie Institution in 1914 and other work done by Professor Lehmer and his son, D. H. Lehmer. See this Monthly for March 1928, pp. 114–121.

6. This paper limits itself to applications of finite groups in the binary and ternary domains. First symmetry is shown to be a metric aspect of group theory and the maximum symmetry of algebraic curves is considered. The geometry of configurations is also closely related to that of collineation groups. Every group has an invariant configuration and conversely certain configurations completely define allied finite groups. The two major problems in the theory of self-projective curves are mentioned: (1) the determination of the

complete system of invariant curves of a given group; (2) the determination of all varieties of groups that may leave a curve of given order invariant. Finally the special problem of self-projective rational curves is considered. In S_{n-1} the group on the parameter is one of the five species of binary (regular body) groups, while the group on the points is an n-ary group isomorphic with the binary group. The interplay between the geometry of the two associated groups is one of the interesting phases of the problem for rational curves.

This paper will appear in an early issue of the Monthly.

Meeting of the Board of Trustees

Eight trustees were present at the meetings on Monday evening and Tuesday noon.

The following twenty-six persons were elected to membership on applications duly certified:

To Individual Membership

J. B. Adkins, Ph.B. (Chicago). Teacher, Culver Military Acad., Culver, Ind.
E. F. Allen, B.S. (Acad. of the New Church). Instr., Academy of the New Church, Bryn Athyn, Pa.
Brother Aurelius, A.M. (Catholic Univ. of Amer.). Teacher, St. Joseph's College High School, Bardstown, Ky.
Jack Britton, A.B. (Clark). Inst., Univ. of Colorado, Boulder, Colo.
Alice Bromwell, A.M. (Nebraska). Instr., Monticello Seminary, Godfrey, Ill.
June F. Constantine, B.S. (Minnesota). Research Asst., Coll. of Educ., Univ. of Minnesota, Minneapolis, Minn.
W. J. Ettinger, B.S. in M.E. (Lewis Inst.). Research Engr., Edison Electric Appliance Co., Chicago, Ill.
Mary Ewin, A.B. (George Washington). Grad. Student, George Washington Univ., Washington, D. C.
Sidney Hacker, A.B. (Colorado). Part-time Instr., Univ. of Colorado, Boulder, Colo.
David Kevles, A.B. (Pennsylvania). 862 N. Marshall St., Philadelphia, Pa.
W. W. McCormick, B.S. (Geneva). Instr., Math. and Physics, Geneva Coll., Beaver Falls, Pa.
Charlotte L. McFall, A.M. (Chicago). Prof., West Virginia State College, Institute, West Va.
R. E. McPherson, M.S. (Chicago). Teacher, Glenn High School, Terre Haute, Ind.
G. E. March, B.S. (S. Dak. State School of Mines). Asst. Prof., South Dak. State School of Mines, Rapid City, S. Dak.
Florentina Mathias, A.M. (Ohio State). Teacher, Chillicothe High School, Chillicothe, Ohio.
Sigurd Mundhjeld, A.B. (Concordia, Moorhead, Minn.). Instr., Waldorf Coll., Forest City, Iowa.
H. A. Perkins, A.B. (Colby). Chairman Math. Group, Hampton Inst., Hampton Institute, Va.
E. W. Ploenges, A.M. (Michigan). Prof., Kansas Wesleyan Univ., Salina, Kans.
M. F. Rosskopf, A.B. (Minnesota). Teaching Asst., Univ. of Minnesota, Minneapolis, Minn.
E. A. Saibel, Ph.D. (Mass. Inst. of Tech.). Instr., Math. and Mech., Univ. of Minnesota, Minneapolis, Minn.
Samuel Silberfarb, Ph.D. (Chicago). Asst. Prof., Univ. of Akron, Akron, Ohio.
Martha L. Smith, A.B. in Educ. (Virginia Union Univ.). Asst. Prof., Virginia Union Univ., Richmond, Va.
Marguerite Edna Stagner, B.S. (Iowa State). Teacher, High School, Glenham, S. Dak.
T. R. C. Wilson, C.E. (Purdue). Senior Engineer, Forest Products Lab., Madison, Wis.
Patrick Youtz, M.S. (Chicago). Instr., Bucknell Univ., Lewisburg, Pa.
Sister Yvonne, A.M. (Minnesota). St. Joseph's Acad., St. Paul, Minn.

The trustees gave further consideration to various routine matters of Association business and to the possibility of an improvement in the method of nominating candidates for offices in the Association.

W. D. Cairns, *Secretary-Treasurer*

THE MAY MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION

The twenty-fifth regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held at the George Washington University, Washington, D. C., on Saturday, May 4, 1929. Sessions were held in the morning and in the afternoon; Professor C. C. Bramble, Chairman of the Section, presided at both sessions.

Forty-four persons attended the meeting, including the following thirty-three members of the Association: O. S. Adams, W. J. Berry, L. M. Blumenthal, C. C. Bramble, Paul Capron, Tobias Dantzig, Alexander Dillingham, J. A. Duerksen, J. T. Erwin, P. J. Federico, Michael Goldberg, W. M. Hamilton, F. E. Johnston, L. M. Kells, W. D. Lambert, A. E. Landry, C. L. Leiper, F. D. Murnaghan, O. J. Ramler, C. H. Rawlins, Jr., J. N. Rice, A. W. Richeson, H. M. Robert, Jr., R. E. Root, J. B. Scarborough, W. F. Shenton, John Tyler, C. E. Van Orstrand, W. J. Wallis, Paul Wernicke, C. H. Wheeler, 3d, E. W. Woolard, Oscar Zariski.

During the intermission between the morning and the afternoon sessions, those attending the meeting were entertained at luncheon by the Washington members. Preceding the reading of papers at the afternoon session, a brief business meeting was held, at which a vote of thanks was passed in appreciation of the hospitality of the local members and of the provisions made by the University for holding the meeting. The following officers were elected: *Chairman*, Professor W. F. Shenton, American University; *Secretary*, Edgar W. Woolard, George Washington University; *Members of the Executive Committee*, Professor H. M. Robert, Jr., U. S. Naval Academy, and Professor Florence P. Lewis, Goucher College. Professor Gwinner offered several suggestions as to policy concerning the conduct of the Section, for the consideration of the members and the new Executive Committee.

The following six papers were presented:

1. "Vector operations in projective geometry," by Professor Tobias Dantzig, University of Maryland.

2. "Some industrial engineering curves," by Professor Harry Gwinner, University of Maryland.

3. "The invalidity of a certain method of computing a probable error," by Professor J. B. Scarborough, U. S. Naval Academy.

4. "Some interesting formulae for the constant pi," by J. A. Duerksen, U. S. Coast and Geodetic Survey.

5. "On Einstein's new theory," by Professor F. D. Murnaghan, Johns Hopkins University.

6. "Comment on the problem of three listening posts," by Professor Paul Capron, U. S. Naval Academy.

Abstracts of some of these papers follow:

1. It is well known that the relations of projective geometry can be made

independent of all considerations of Cartesian metrics; it is less known that these relations are such as to permit the application, practically without restrictions, of the principles of vector algebra. This paper sought to show that vector analysis, so successfully used in differential geometry, can render just as signal a service in projective geometry. Examples were given to illustrate how the classical identities of vector algebra become theorems of projective geometry, and how a great number of geometrical constructions can be directly and intrinsically described by this method.

5. The essential features of the older generalized Theory of Relativity were described, and a brief exposition of the new form and the respects in which it differs from the preceding, were given.

6. A comment on the problem of three fixed stations in a straight line, at each of which a record is made of the time at which the sound of the discharge of a distant gun arrives: This problem is given in several texts as a problem in the intersection of two hyperbolas of known transverse axes with one of their three collinear foci in common. It was shown that consideration of the third hyperbola makes a simple solution possible that gives a brief computation for the desired position with no danger of mistaking an intersection of but two hyperbolas for the true position. The formulas obtained were seen to be readily usable for any order of succession of the arrival of the sound at the different posts, and to give ready means of comparing the accuracy of a plot of the actual curves with the plot of the asymptotes. It was also shown that there is always one false position; and criteria were given for the occurrence of two or of three such. The formula is useful in devising problems of any desired character; extension of the use of the formula to analogous problems concerning ellipses was made evident.

EDGAR W. WOOLARD, *Secretary*

THE ANNUAL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association was held at Rice Institute, Houston, Texas, on Saturday, January 26, 1929. Professor G. T. Whyburn presided. The attendance was forty-three, including the following twenty members of the Association: J. H. Binney, L. W. Blau, A. A. Blumberg, H. E. Bray, Alice C. Dean, J. L. Dorroh, H. J. Ettlinger, G. C. Evans, L. R. Ford, E. Garza, H. Halperin, E. O. Lovett, E. R. C. Miles, W. L. Porter, W. A. Rees, W. T. Reid, J. H. Roberts, W. G. Smiley, Jr., P. H. Underwood, G. T. Whyburn.

The following members of the Association functioned as the local committee on arrangements: L. R. Ford, chairman, H. E. Bray and Alice Dean. The program committee consisted of G. C. Evans, chairman, G. T. Whyburn, and H. J. Ettlinger. The following was the program of the day:

1. "Generalized Vandermonde determinants and their applications to sym-

metric functions," E. R. Heineman, Texas Technological College (by invitation).

2. "A generalization of Hadamard's theorem on the absolute value of a determinant," E. F. Bechenbach, Rice Institute (by invitation).

3. "A fundamental continuity theorem in algebra," by W. T. Reid, University of Texas.

4. "Some properties of mortality curves," by L. R. Ford, Rice Institute.

5. "Some topics relating to the foundations of geometry," by J. L. Dorroh, University of Texas.

6. "Rates of foreign exchange," by G. C. Evans, Rice Institute.

7. "The polar form of a second order linear differential system," by H. J. Ettlinger, University of Texas.

8. "A report of the committee on requirements for teachers of mathematics," by Professor F. W. Sparks, Texas Technological College.

9. "Results of tests in mathematics given to freshman physics students," by L. W. Blau, University of Texas.

10. "Query: What should be the contents of the freshman mathematics course?," by C. R. Sherer, Texas Christian University. Discussion.

Abstracts of these papers follow:

1. A generalized Vandermonde determinant is obtained from the ordinary Vandermonde determinant by permitting the indices to take any set of values. The Vandermonde matrix is defined to be the Vandermonde determinant with the nth powers of its variables added as an extra row. By successively blocking out each of the first n rows of this matrix, we obtain n determinants which can be called secondary Vandermonde determinants. The ordinary Vandermonde determinant, which we get by omitting the last row of this matrix, will be called the principal Vandermonde determinant. It can be shown that every generalized Vandermonde determinant is expressible as a determinant-function of the principal and secondary Vandermonde determinants. These results, in conjunction with a theorem of Muir's, give a method for expressing any integral, rational, symmetric function in terms of elementary symmetric functions.

2. For definite Hermitian determinants Mr. Bechenbach extends the result of Frisch that $|\Delta| \leqq M^n$ or $|\Delta| \leqq \pi_{i=1}^{n} |\alpha_{ii}|$.

3. Mr. Reid discussed the theorem that the roots of an algebraic function are continuous functions of the coefficients. He pointed out that one must give an interpretation of the definition of continuity in this case.

Weber in his "Lehrbuch der Algebra" states the following theorem: "The roots of an algebraic equation are continuous functions of the coefficients." If the equation cannot be solved algebraically, the statement that each root of the equation is a continuous function of the coefficients has little meaning unless a method is given for discriminating between the roots of the equation. This question is considered and also, since the proof given by Weber is erroneous, a proof of the following theorem is given:

If $P_m(z) \equiv z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$ is a polynomial which has roots

$r_1, r_2, \cdots, r_k$ with corresponding multiplicities $\alpha_1, \alpha_2, \cdots \alpha_n$, $\alpha_1+\alpha_2+\cdots+\alpha_n = m$, then for every $\epsilon>0$, there exists a $\delta_\epsilon>0$ such that if $\bar{P}_m(z)\equiv z^m+\bar{a}_{m-1}z^{m-1} + \cdots +\bar{a}_1 z+a_0$ is a polynomial such that $\sum_{i=0}^{m-1} |a_i-\bar{a}_i|<\delta_\epsilon$, then corresponding to each root r_i of $P_m(z)$ there exist α_i roots $\beta_1^{(i)}, \beta_2^{(i)}, \cdots, \beta_{\alpha_i}^{(i)}$ of $\bar{P}_m(z)$ such that $|r_i-\beta_j^{(i)}|<\epsilon$, $(j=1, 2, \cdots \alpha_i)$.

4. Professor Ford presented some elementary properties of mortality curves which seem to have escaped notice hitherto:

(1) At its maximum, the expectation of life is the reciprocal of the force of mortality.

(2) At its maximum, the life annuity is the reciprocal of the sum of the force of mortality and the force of interest.

(3) The most probable or least probable moment of death occurs when the derivative of the force of mortality equals the square of the force of mortality.

(4) A sum to be paid at death within a short interval of constant length is most or least expensive when the logarithmic derivative of the force of mortality equals the sum of the force of interest and the force of mortality.

5. In his paper, "Sets of metrical hypotheses for geometry" (Transactions of the American Mathematical Society, vol. 9 (1908), pp. 487–512), R. L. Moore has discussed the relationship between certain sets of his axioms and the groups I–IV used in Hilbert's *Grundlagen der Geometrie.* It is indicated, in the present discussion, how some recent proofs show that the groups of I–IV of Hilbert's axioms are satisfied in a space satisfying the sets O and C of Moore's axioms if his definition for the congruence of angles is used.

6. Professor Evans replaces the algebraic system of Cournot[1] governing rates of foreign exchange by corresponding differential equations.

7. Professor Ettlinger explained the general linear homogeneous system of a second order in the polar form and showed that when the rectangular coordinates are replaced by amplitude and angle functions many of the results of Sturm can be obtained directly from the properties of these functions.

8. During the afternoon session the secretary read a report of a committee of which F. W. Sparks is chairman, appointed at the 1928 meeting for the purpose of considering ways and means of improving the teaching of mathematics in Texas high schools. The committee presented the following resolutions:

(1) That prospective teachers of high school mathematics be urged to take a minimum of three years of college mathematics.

(2) That the State Board of Education be requested to place one-half year of algebra in the curriculum of the fourth year of the high school.

(3) That the general aim of the reorganization of high school mathematics courses be to conform with the recommendations of the College Entrance Examination Board and of the National Committee on Mathematical Requirements.

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[1] Cournot, *The Mathematical Principles of Wealth.*

(4) That the teachers colleges be requested to provide not only for better mathematical equipment of prospective teachers of mathematics but also for competent instruction in methods of teaching mathematics.

Professors Evans, Underwood, and Ettlinger commented at length on these resolutions. By a vote of the Section, Resolution (2) was adopted unanimously and the committee was requested to continue its study of the work in hand.

9. Mr. L. W. Blau, instructor in Physics in the University of Texas, presented some tests in arithmetic and elementary plane geometry given to freshman students at the University of Texas.

10. Prof. C. R. Sherer, head of the mathematics department at Texas Christian University, Fort Worth, presented the query, "What should be the contents of the freshman mathematics course?" This question was discussed from many angles by Professor Bray of Rice Institute, by Professor Ettlinger of the University of Texas, and by Professor Halperin of Texas A. & M. College.

Those in attendance were at Cohen House for lunch as guests of Rice Institute. The section voted a resolution of thanks to the members of Rice Institute for their splendid hospitality. Dr. G. T. Whyburn was re-elected chairman for a period of two years.

H. J. ETTLINGER, *Secretary*

ON THE DENSITY OF AN OBLATE SPHEROIDAL PLANET AND THE MOTION OF A SATELLITE

By LOUIS ALLEN HOPKINS, University of Michigan.

In his dissertation on *Periodic orbits about an oblate spheroid*, W. D. MacMillan,[1] proceeding by methods due to Poincaré, secured the following simple formula for the advance of the line of apsides of the planet per revolution, viz.,

$$\alpha = 360°\left[\frac{3b^2}{10a^2}\mu^2(1 + e^2 + \cdots)\right]. \tag{1}$$

In this expression b is the polar radius of the planet, μ the eccentricity of all meridian cross sections, a the mean distance of the satellite, and e the eccentricity of its orbit. In making application of his theory to the motion of the fifth satellite of Jupiter, MacMillan found a wide difference between the computed and the observed values of α. However he supposed that the density of Jupiter was uniform. The object of this paper is to discuss consequences of distributions of density which are homogeneous in concentric oblate spheroidal shells and which are biquadratic functions of the distance from the center. Applications

[1] W. D. MacMillan, Transactions of the American Mathematical Society, vol. 11 (1910) and F. R. Moulton, *Periodic Orbits* (Carnegie Institution of Washington, Publication No. 161), chapter 4.

will be made to Jupiter and V, Mars and the two satellites, and to the Earth and Moon.

Density and Mass of an Oblate Spheroid

We assume the following formula for the density:

$$\rho = \rho_0\left[1 - \frac{\beta}{b^2}(1 - \mu^2 \cos^2 \phi)r^2 + \frac{\gamma}{b^4}(1 - \mu^2 \cos^2 \phi)^2 r^4\right], \tag{2}$$

where ρ_0 denotes the density at the center, b the polar radius, μ the eccentricity of the meridian cross sections, ϕ the latitude of a point, and r its distance from the center, while β and γ are parameters whose values may be determined in each application. The polar equation of all meridian cross sections of a concentric similar shell is

$$r^2 = \frac{c^2}{(1 - \mu^2 \cos^2 \phi)},$$

where c is the polar radius. The density at this shell is therefore[2]

$$\rho = \rho_0(1 - \beta z^2 + \gamma z^4), \tag{3}$$

where $z = c/b$ and for our physical problem, $0 \leqq z \leqq 1$. At the surface of the planet, denoting the density by σ, we have

$$1 - \beta + \gamma = \frac{\sigma}{\rho_0}\cdot \tag{4}$$

The mass of the planet with density (2) is obtained at once from

$$M = \int_{-\pi/2}^{\pi/2}\int_0^r\int_0^{2\pi} \rho r^2 \cos \phi d\phi dr d\theta,$$

where θ is the longitude and the upper limit on the second integration is

$$r = b(1 - \mu^2 \cos^2 \phi)^{-1/2}.$$

We thus find

$$M = \frac{4\pi\rho_0 b^3}{1-\mu^2}\left(\frac{1}{3} - \frac{1}{5}\beta + \frac{1}{7}\gamma\right).$$

But from the observations of the planets, we know their total mass and dimensions and consequently their average density δ, i.e.,

$$M = 4\pi\delta b^3/3(1 - \mu^2).$$

Equating the two values of M, we obtain

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[2] This expression is in harmony with Eddington's work on polytropic gas spheres contained in his book, *The Internal Constitution of the Stars* (Cambridge). From the work of Emden, he develops the density in infinite series in the distance from the center. He shows that the coefficient of the first power vanishes but it is not difficult to prove that no odd power occurs in his series.

$$\frac{1}{3} - \frac{1}{5}\beta + \frac{1}{7}\gamma = \frac{\delta}{3\rho_0} \cdot \tag{5}$$

In physical problems we would expect $\sigma \geqq 0$, $\rho_0 > 0$, $\delta > 0$, and

$$\sigma \leqq \rho_0, \quad \delta \leqq \rho_0.$$

Thus if in equation (4), $\sigma = 0$, and again $\sigma = \rho_0$, the values of β and γ are restricted to the area in a $\beta\gamma$-plane bounded by the lines

$$1 - \beta + \gamma = 0, \quad - \beta + \gamma = 0.$$

In Fig. 1, these are the lines FH and AK respectively. Similarly, if in (5) we put $\delta = \rho_0$ and again $\rho_0 \to \infty$, the values of β and γ must lie between the lines

$$-\frac{1}{5}\beta + \frac{1}{7}\gamma = 0, \quad \frac{1}{3} - \frac{1}{5}\beta + \frac{1}{7}\gamma = 0,$$

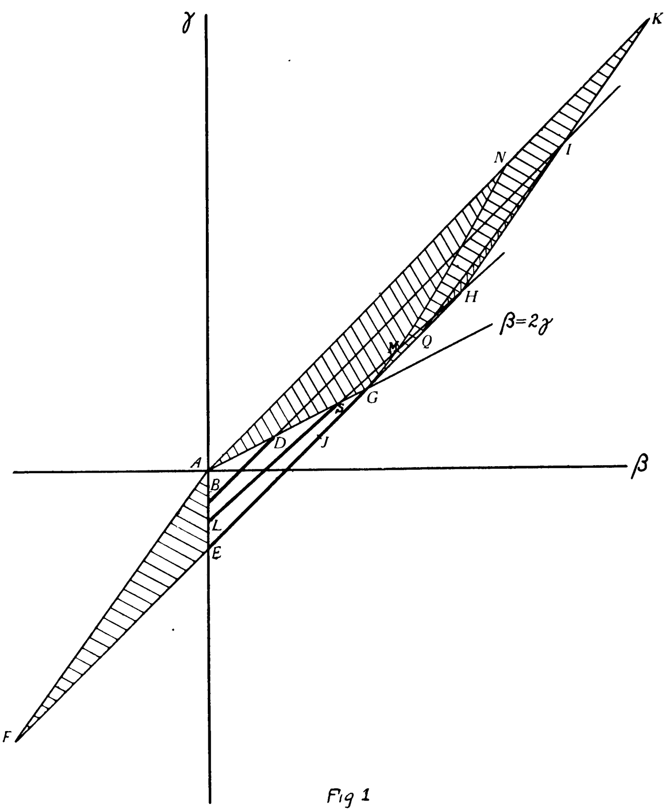

Fig 1

i.e., the lines FA and HK. The values of β and γ are therefore confined to the parallelogram $AKHF$ determined by the four lines above. The point K corresponds to an infinite density throughout the mass; F to a zero density everywhere in the body; H to an infinite density at the center and a zero density at the

surface; and A to a constant density. The points K and F must be excluded in the applications.

As we proceed along the polar axis of the planet the ratio w, of the density ρ at a point and the density ρ_0 at the center, may be written, from (3), $w=1-\beta z^2+\gamma z^4$.

The maximum and minimum values of w will occur when

$$dw/dz = -2\beta z + 4\gamma z^3 = 0, \text{ or when } z = 0, \text{ and } z^2 = \beta/2\gamma.$$

In the physical problems we would expect the greatest density to be at the center and therefore

$$d^2w/dz^2 = -2\beta + 12\gamma z^2 < 0$$

for $z=0$, or $\beta>0$.

Thus the region AEF of the parallelogram in which β is negative is excluded from consideration. The minimum value of w is

$$w = \frac{(4\gamma - \beta^2)}{4\gamma},$$

and occurs when $z^2=\beta/2\gamma$. Should this happen within the planet, i.e., when $\beta/2\gamma<1$, the corresponding values of β and γ must lie above the line $\beta=2\gamma$. But in this case, w will be positive in the area $AGMNA$, i.e., to the left of the parabola $\beta^2=4\gamma$, and negative to the right. Thus, if we exclude negative densities we must delete the area $NMGHKN$. But should we postulate that the least density of the planet occurs at the surface then $\beta/\gamma \geqq 0$, i.e., we must exclude all that portion of the parallelogram above the line $\beta=2\gamma$. The values of β and γ in the physical problem are therefore limited to the triangle $AEGA$.

Spheroid with density (2)

The potential of a homogeneous oblate spheroid on a distant unit particle is fully developed by Moulton.[1] Here we need, in addition to take into consideration the density distribution (2). The potential V then becomes

$$(6)\quad V = \frac{M}{R} - \frac{\pi}{R^3}\int_{-\pi/2}^{\pi/2}\int_0^r \rho r^4 \cos\phi d\phi dr + \frac{3\pi}{2R^5}(x^2+y^2)\int_{-\pi/2}^{\pi/2}\int_0^r \rho r^4\cos^3\phi d\phi dr$$
$$+ \frac{3\pi z^2}{R^5}\int_{-\pi/2}^{\pi/2}\int_0^r \rho r^4 \sin^2\phi\cos\phi d\phi dr + \cdots.$$

In this expression we use the notation of Moulton who denotes the coordinates of the unit particle by x, y, z, with $R^2=x^2+y^2+z^2$; r is the distance of a point of the spheroid from the center and ϕ is its latitude. There is no difficulty in performing the integration with respect to r after substituting for ρ the density formula (2). Denoting the integrals in turn by A, B, C, we obtain,

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[1] F. R. Moulton, *Celestial Mechanics*, p. 119.

$$A = \rho_0 b^5 F \int_{-\pi/2}^{\pi/2} (1 - \mu^2 \cos^2 \phi)^{-5/2} \cos \phi d\phi,$$

$$B = \rho_0 b^5 F \int_{-\pi/2}^{\pi/2} (1 - \mu^2 \cos^2 \phi)^{-5/2} \cos^3 \phi d\phi,$$

$$C = \rho_0 b^5 F \int_{-\pi/2}^{\pi/2} (1 - \mu^2 \cos^2 \phi)^{-5/2} \sin^2 \phi \cos \phi d\phi,$$

where $F = \frac{1}{5} - \frac{1}{7}\beta + \frac{1}{9}\gamma$. It is evident at once that $A - B = C$. We find that

$$A = \rho_0 b^5 F \frac{2(3 - \mu^2)}{3(1 - \mu^2)^2}, \qquad B = \rho_0 b^5 F \frac{4}{3(1 - \mu^2)^2}.$$

The expression for C is:

$$C = \rho_0 b^5 F \frac{2}{3(1 - \mu^2)}.$$

To write the potential (6) simply, let us use the abbreviation, $E = \frac{1}{3} - \frac{1}{5}\beta + \frac{1}{7}\gamma$. Then

$$M = \frac{4\pi\rho_0 b^3}{1 - \mu^2} E,$$

and

$$V = \frac{M}{R} - \frac{2\pi\rho_0 b^5 F}{3R^3} \cdot \frac{3 - \mu^2}{(1 - \mu^2)^2} + \frac{2\pi\rho_0 b^5 F (x^2 + y^2)}{R^5} \cdot \frac{1}{(1 - \mu^2)^2} + \frac{2\pi\rho_0 b^5 F z^2}{R^5} \cdot \frac{1}{1 - \mu^2} + \cdots.$$

Including second order terms in the expression for the potential, this becomes,

$$V = \frac{M}{R}\left[1 + \frac{b^2}{6R^4} \cdot \frac{F}{E} (x^2 + y^2 - 2z^2)\mu^2 + \cdots \right].$$

Moulton's potential function is obtained by putting, $\beta = \gamma = 0$, i.e., $F = \frac{1}{5}$, $E = \frac{1}{3}$. With this expression for the potential, by substitution in MacMillan's work we find for the motion of the line of apsides:

$$\alpha = 360^\circ \left[\frac{b^2}{2a^2} \cdot \frac{F}{E} \mu^2 (1 + e^2 + \cdots)\right], \tag{7}$$

where a, b, μ and e have the same significance as before.

Pencils of Apsidal and Density Lines

Now in equation (7), if we let

$$\frac{F}{E} = p = \frac{\alpha a^2}{180 b^2 \mu^2 (1 + e^2 + \cdots)},$$

it may be interpreted as a pencil of lines (apsidal lines)

$$\frac{1}{5} - \frac{1}{7}\beta + \frac{1}{9}\gamma = p\left(\frac{1}{3} - \frac{1}{5}\beta + \frac{1}{7}\gamma\right),$$

with pole at $I(14/3, 21/5)$ and parameter p. The particular line of the pencil in each problem is determined by the value of p or, what amounts to the same thing, by the α that is assumed for a satellite. Similarly if we put $q = \delta/3\sigma$ in equations (4) and (5), they combine to form a pencil of lines (density lines)

$$\frac{1}{3} - \frac{\beta}{5} + \frac{\gamma}{7} = q(1 - \beta + \gamma), \tag{8}$$

with pole at $H(10/3, 7/3)$ and parameter q. The values of β and γ in a particular problem can now be found as the coordinates of the intersection of an apsidal line and a density line.

We distinguish three cases: viz., (a) when the distribution of density within the planet contributes to the regression of the line of apsides, i.e., $\alpha < 0$; (b) when the distribution has no effect, i.e., $\alpha = 0$ and (c) when the density within the planet causes the apses to advance, i.e., $\alpha > 0$; These three situations are distinguished according as $p < 0$, $p = 0$, $p > 0$. For $p = 0$, the apsidal line is $\frac{1}{5} - \frac{1}{7}\beta + \frac{1}{9}\gamma = 0$, which is the equation of IQ (Fig. 1). Thus if β and γ are the coordinates of points on this line the corresponding density of planet would permit the satellite to revolve in a stationary ellipse. For $p < 0$, the apsidal line would pass through the area QIH. But the density of the planet within this area somewhere becomes negative. We thus doubly exclude the area.

For $p > 0$, the apsidal lines would lie in the obtuse angle KIQ. But our interest in them is confined to those lines of the pencil which cross the triangle $AGEA$, i.e., $\frac{3}{5} \geqq p \geqq \frac{1}{3}$.

Application to Jupiter and Satellite V

We shall assume that the surface density of Jupiter is negligible, i.e., $\sigma = 0$, and therefore $q \to \infty$. Thus for Jupiter, β and γ are the coordinates of points on the segment EG of the line $1 - \beta + \gamma = 0$. The value of p is therefore confined to the inequality $7/15 \geqq p \geqq 1/3$.

For the fifth satellite of Jupiter, according to H. Struve, the advance of the line of apsides is 916° per year. As the period is $11^h\ 57^m\ 22.^s7$, the advance per revolution is $\alpha = 1.°2494$. With $\mu = .35888$, $a = 112{,}600$, $b = 41{,}394.37$ and $e = .0028$, we find $p = .3988$. The corresponding point on the line EG is J. Thus the

density of Jupiter may be such as to account for more or less or exactly the observed motion of the line of apsides of the satellite.

The mean density of Jupiter is $\delta = 1.34$ and from (5) the values of the central density are readily computed for the points E, J, and G. The laws of density distributions corresponding to these three points are then found to be, respectively,

$$\rho_E = 2.345(1 - z^4),$$
$$\rho_J = 4.135(1 - 1.443z^2 + .443z^4),$$
$$\rho_G = 5.863(1 - 2z^2 + z^4).$$

These densities are represented graphically in Fig. 2 by the curves marked E, J, and G. The range of possible density on any spheroidal shell of Jupiter lies between the ordinates of the curves E and G where the abscissa is the fractional part of the polar radius.

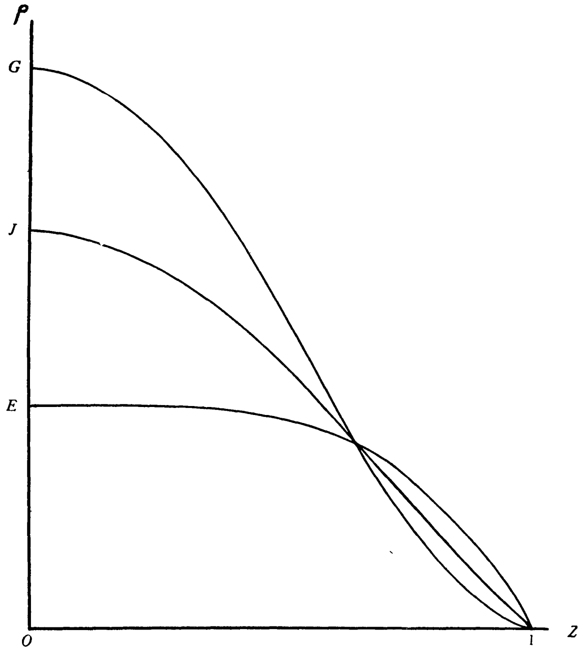

FIG. 2

To gain a clearer concept of the distribution of mass in these problems let us consider the case in which the density accounts for the whole of the motion of the apses of Satellite V. When the density of the planet of polar radius b is considered constant, its mass is

$$M_b = \frac{4\pi\delta b^3}{3(1 - \mu^2)}.$$

The ratio of this mass to a similar one of polar radius c is

$$\frac{M_c}{M_b} = \frac{c^3}{b^3} = z^3.$$

Thus an oblate spheroid of constant density and unit polar radius is divided into four equal masses by spheroids of polar radii

$$z = .6300, \quad z = .7937, \quad z = .9086 \quad \text{and} \quad z = 1.$$

The meridian cross sections of such a body are drawn with the oblateness of Jupiter in Fig. 3. The broken lines correspond to the first three computed values of z. But in the case of a density distribution according to formula (3),

$$\frac{M_c}{M_b} = z^3 \frac{\dfrac{1}{3} - \dfrac{1}{5}\beta z^2 + \dfrac{1}{7}\gamma z^4}{\dfrac{1}{3} - \dfrac{1}{5}\beta + \dfrac{1}{7}\gamma}. \tag{9}$$

For the values of β and γ determined by the fifth Satellite, the planet would be divided into four equal masses by spheroids of polar radii, $z = .4623$, $z = .6137$, $z = .7523$ and $z = 1$. These are presented in Fig. 3, by the full lines.

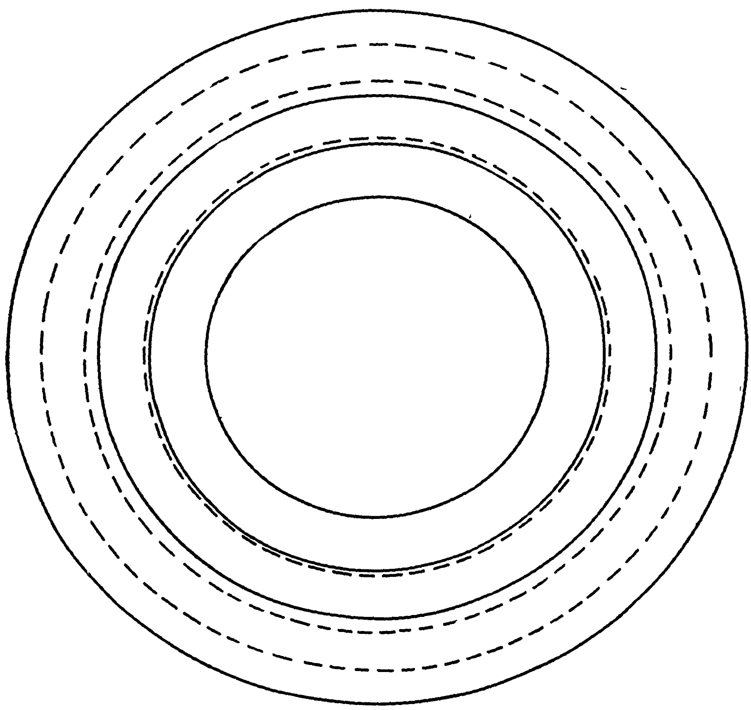

FIG. 3

Application to Mars, Deimos, and Phobos

We know the mean density of Mars, viz., $\delta=3.96$, but its surface density is conjectural. To illustrate the theory developed, let us suppose that the motion of the line of apsides of Deimos can be accounted for by the distribution of density within the planet. The period of Deimos is $30^h\ 17^m\ 54.^s9$ and the apses complete a revolution in about 56 years; thus $\alpha \doteq .^\circ 02204$. The other constants are $a=14{,}580$, $b=2096.4$, $\mu^2=.010499$ (using Lowell's value for the oblateness as 1/190) and $e=.0031$. From these data the apsidal line for Mars is $\frac{1}{5}-\frac{1}{7}\beta+\frac{1}{9}\gamma=.5640(\frac{1}{3}-\frac{1}{5}\beta+\frac{1}{7}\gamma)$. The portion BD of this line (Fig. 1) contained within the triangle $AEGA$ has as extremities (0, $-.3933$) and (.8120, .4060), from which it would follow that

$$2.89 \leqq \sigma \leqq 3.43, \quad \text{and} \quad 4.76 \leqq \rho_0 \leqq 5.77.$$

The period of Phobos is $7^h\ 39^m\ 13.^s851$ and[1] $a=5826.$ and $e=.0217$. With $p=.5640$ as computed for Deimos the value of $\alpha=.^\circ 1381$, or the line of apsides turns through an angle $158^\circ\ 7'\ 22''.8$ per year. This result is in close agreement with the value 158° adopted by H. Struve.[2]

Application to the Earth and Moon

The mean density of the earth is $\delta=5.52$ and the surface density,[3] $\sigma=2.71$. The line of apsides of the moon advances[4] $11\times 360^\circ+109^\circ\ 2'\ 2''.52$ per century, so that the advance per revolution is $3.^\circ 0438$, and therefore $p=9167$. The values of β and γ for the earth are the coordinates of the intersection of

$$\frac{1}{3}-\frac{1}{5}\beta+\frac{1}{7}\gamma=\frac{5.52}{3\cdot 2.71}(1-\beta+\gamma),$$

$$\frac{1}{5}-\frac{1}{7}\beta+\frac{1}{9}\gamma=9167\left(\frac{1}{3}-\frac{1}{5}\beta+\frac{1}{7}\gamma\right),$$

the density and apsidal lines respectively. To the degree of accuracy of the computations, $\beta=10/3$ and $\gamma=7/3$, i.e., the point is located at H of the parallelogram of limitation. As a matter of fact more accurate computations show that the point would lie a little to the right above the point H and outside the

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[1] The values of the constants in this paper are mostly taken from the appendix to vol. I of Russell, Dugan, and Stewart, *Astronomy* (Ginn & Co.); but there is evidently a misprint in the eccentricity of the orbit of Phobos.

[2] H. Struve, Memoires de L'Academie de St. Petersburg, vol. 8. The corrections of Hall and Bower, Astronomical Journal, No. 873, do not affect the results of the present study.

[3] Hill uses $\delta=5.67$ and $\sigma=2.7$ in his Memoir No. 44, *On the interior constitution of the earth as respects density.*

[4] The constants for application to the Earth and Moon are taken mostly from Brown's *Tables of the Motion of the Moon.* It is interesting historically to observe that he adopts a value for the motion of the lunar perigee almost identical to that computed by Airy as referred to in Delaunay *Note sur les mouvements du périgée et du noeud de la Lune* in Comptes Rendus, vol. 74, p. 17.

parallelogram. The corresponding central density would be very large numerically and negative. Thus we find our result in harmony with the well known fact that the total motion of the lunar perigee is not due to the ellipticity and density distribution of the earth. The coordinates of the extreme points L and S on the density line within the triangle $AEGA$ for the earth are $(0, -.6447)$ and $(1.6386, .8193)$ and therefore the density ρ_0 at the center of the earth is limited to the inequality, $7.63 \leqq \rho_0 \leqq 14.99$. The two extreme values of p are $p = .6281$ and $p = 4.6429$. We thus find that due to the earth the advance in the line of apsides per revolution of the moon is limited to

$$^\circ.00021 \leqq \alpha \leqq {}^\circ.00154,$$

i.e., the lunar perigee is advanced between $0.^\circ 2788$ and $2.^\circ 0608$ per century due to the ellipticity and density distribution of the earth.

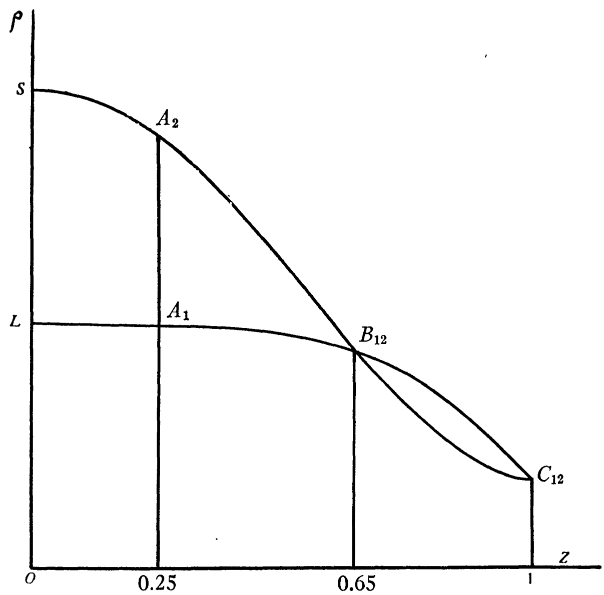

FIG. 4

We have now found that the density at a point of the polar axis of the earth or on the surface of the concentric spheroid through the point is confined between the values of ρ secured from

$$\rho_L = 7.63(1 - .6447z^4), \tag{10}$$

$$\rho_S = 14.99(1 - 1.6386z^2 + .8193z^4), \tag{10'}$$

by substituting for z, the distance measured from the center with the polar radius as unity. These equations are represented graphically in Fig. 4 by the curves $A_1B_{12}C_{12}$ and $A_2B_{12}C_{12}$, respectively. The density at a point, say $z = .25$,

is then numerically between the ordinates of A_1 and A_2. The extreme densities are equal at B_{12}, i.e., $\rho=6.75$ at $z=.65$, and at C where $\rho=\sigma=2.71$ at $z=1$.

When the values of β and γ used in (10) and (10′) are substituted in (9), it is found that in the case (10), one quarter of the Earth's mass is contained within a spheroid of polar radius .58 and in the case (10′), the same part of the mass has a radius .50.

The Density at $z=\sqrt{\frac{3}{7}}$.

In Fig. 2 the density curves for Jupiter have two common points, i.e., they give a common density at the surface and at one interior point. The same is true in Fig. 4. That this is always true may be seen as follows. Consider two points (β_1, γ_1) and (β_2, γ_2) of any density line, i.e.,

$$
\begin{aligned}
1-\beta_1+\gamma_1 &= \frac{3\sigma}{\delta}\left(\frac{1}{3}-\frac{1}{5}\beta_1+\frac{1}{7}\gamma_1\right), \\
1-\beta_2+\gamma_2 &= \frac{3\sigma}{\delta}\left(\frac{1}{3}-\frac{1}{5}\beta_2+\frac{1}{7}\gamma_2\right).
\end{aligned} \tag{11}
$$

The corresponding densities along the polar axis are

$$
\begin{aligned}
\rho &= \frac{\sigma}{1-\beta_1+\gamma_1}(1-\beta_1 z^2+\gamma_1 z^4), \\
\rho &= \frac{\sigma}{1-\beta_2+\gamma_2}(1-\beta_2 z^2+\gamma_2 z^4).
\end{aligned}
$$

These densities will be equal when

$$
\frac{1-\beta_1 z^2+\gamma_1 z^4}{1-\beta_1+\gamma_1} = \frac{1-\beta_2 z^2+\gamma_2 z^4}{1-\beta_2+\gamma_2}.
$$

The roots of this equation are $z^2=1$, corresponding to the surface of the planet and

$$
z^2 = \frac{(\beta_1-\beta_2)-(\gamma_1-\gamma_2)}{(\gamma_1-\gamma_2)+(\beta_1\gamma_2-\beta_2\gamma_1)}.
$$

From equations (11) we readily find,

$$
\beta_1\gamma_2-\beta_2\gamma_1 = \frac{5(\delta-\sigma)}{3\sigma-5\delta}(\gamma_1-\gamma_2),
$$

and

$$
\beta_1-\beta_2 = \frac{5}{7}\cdot\frac{7\delta-3\sigma}{5\delta-3\sigma}(\gamma_1-\gamma_2),
$$

from which the second root is reduced to $z^2=\frac{3}{7}$. Thus with the distribution (2), the density of planets at $z=\sqrt{\frac{3}{7}}$, is the same for all points of a density line and may be determined with the accuracy of the surface density.

In this study, the computations have been made and the figures drawn by Miss Gretchen Mullison, Research Assistant.

SOME PROPERTIES OF CORRELATIVE VERTEX LINES IN A PLANE TRIANGLE

By VLADIMIR KARAPETOFF, Cornell University

Theorem 1. *The three correlative vertex lines in a triangle always intersect in one point.*

Proof: By definition, let a straight line passing through the vertex of a triangle be called a vertex line. Let α, β and γ be three arbitrary scalar parameters associated with the sides opposite the vertices A, B, and C, respectively. Let the direction AL be determined in the following manner: Take $AK = AB/\gamma$ and $AM = AC/\beta$; construct a parallelogram on these lines and let its diagonal be called AL. If now a line from B and a line from C be constructed in the same manner, using the three quantities α, β, and γ in the cyclic order, then these three lines by definition are called correlative. It is required to prove that any three correlative lines intersect in one point, O. Several well-known theorems may be deduced as special cases of this general theorem.

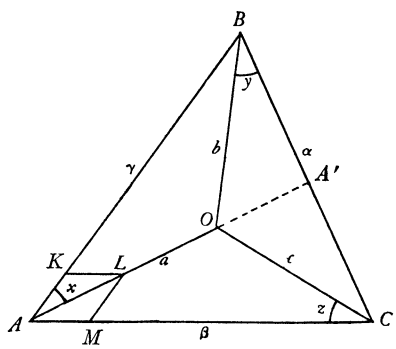

FIG. 1

Instead of starting with a given triangle we begin with an origin O, and with three arbitrary vectors, $OA = \boldsymbol{a}$, $OB = \boldsymbol{b}$, and $OC = \boldsymbol{c}$, drawn from O. These vectors determine the triangle, and the condition that the three vectors lie in the same plane is

(1) $$\boldsymbol{a}\alpha + \boldsymbol{b}\beta + \boldsymbol{c}\gamma = 0,$$

where α, β, and γ are known quantities for given $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$. Of course, α, β, and γ may all be multiplied by some arbitrary scalar quantity without changing the problem, because it is their ratio only that determines the triangle.

It will be proved now that α, β, γ, defined by (1), are identical with the coefficients defined in Fig. 1. For this purpose add $(-\beta\boldsymbol{a} - \gamma\boldsymbol{a})$ to both sides of equation (1) and transfer $\boldsymbol{a}\alpha$ to the right. Then equation (1) becomes:

$$(\boldsymbol{b} - \boldsymbol{a})\beta + (\boldsymbol{c} - \boldsymbol{a})\gamma = -\boldsymbol{a}(\alpha + \beta + \gamma)$$

or

$$(\boldsymbol{b} - \boldsymbol{a})/\gamma + (\boldsymbol{c} - \boldsymbol{a})/\beta = -\boldsymbol{a}(q/\beta\gamma), \tag{2}$$

where $q = \alpha + \beta + \gamma$.

But $(\boldsymbol{b} - \boldsymbol{a})/\gamma = AK$ and $(\boldsymbol{c} - \boldsymbol{a})/\beta = AM$. Thus, (2) states the fact that the vertex line AL is in the direction of the given vector $\boldsymbol{a}$. By cyclic rotation we can prove that the two other corresponding vertex lines coincide with the given vectors $\boldsymbol{b}$ and $\boldsymbol{c}$, and this proves the theorem.

In specific cases it is only necessary to prove that it is possible to assign definite values to α, β, and γ. Then such vertex lines will intersect in one point.

Special Cases

(a) *Medians.* Here $\alpha = \beta = \gamma$, so that the medians intersect in one point. Incidentally, $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$ being equal to zero, we thus prove that it is always possible to construct a new triangle using the medians of a given triangle as its sides.

(b) *Bisectors of the angles.* Here α, β, γ, are proportional to the lengths of the corresponding sides of the triangle. Hence the bisectors intersect in one point. Equation (1) in this case expresses an intrinsic relationship involving only the lengths of the sides and the distances from the vertices to the center of the inscribed circle.

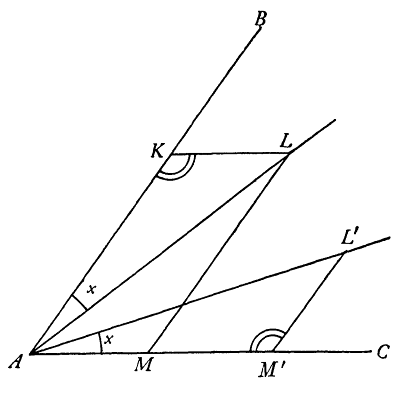

FIG. 2

(c) *Coffin's case.* Let O be an arbitrary point within the triangle and let the vertex lines form angles x, y, z with the sides, as shown in Fig. 1. Let now three other vertex lines be drawn in such a way as to form the same angles with the adjacent sides; for example, the new vertex line from A forms an angle x with AC instead of AB (Fig. 2). It is required to prove that the three new vertex line intersect in one point. This theorem was suggested by Dr. J. G. Coffin as an exercise in vector analysis, in a private letter to the writer.

Since by supposition the first three vertex lines intersect in one point, they form a corresponding set, so that for them there are definite values of α, β,

and γ. It is only necessary to prove that there are definite values, α', β', γ', for the second set. Let AL (Fig. 2) be the given vertex line for which by assumption, $AK=AB/\gamma$ and $AM=AC/\beta$. Let AL' be the new vertex line so that $M'L'=AB/\gamma'$ and $AM'=AC/\beta'$. From the similar triangles we get

$$\frac{AC/\beta'}{AB/\gamma'}=\frac{AB/\gamma}{AC/\beta},$$

or

$$AB^2/(\gamma\gamma') = AC^2/(\beta\beta').$$

Thus, the ratio of β' to γ' is determined by the ratios β/γ and AB/AC. Similarly, for the vertex B we can write $AB^2/(\gamma\gamma')=BC^2/(\alpha\alpha')$, so that the ratio of α' to γ' is also known. Therefore, the three new lines also form a corresponding set and consequently intersect in one point.

(d) *The altitudes.* In this case $\boldsymbol{a}$ is perpendicular to $(\boldsymbol{b}-\boldsymbol{c})$, so that $(\boldsymbol{b}-\boldsymbol{c})\cdot\boldsymbol{a}=0$, or $\boldsymbol{b}\cdot\boldsymbol{a}=\boldsymbol{c}\cdot\boldsymbol{a}$, and by analogy $\boldsymbol{b}\cdot\boldsymbol{a}=\boldsymbol{b}\cdot\boldsymbol{c}$.

Let this product of the two segments of each altitude be equal to k^2, so that $\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{b}\cdot\boldsymbol{c}=\boldsymbol{c}\cdot\boldsymbol{a}=-k^2$.

Form a scalar product of equation (1) with $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, respectively. The following three equations are then obtained:

$$\alpha a^2-(\beta+\gamma)k^2=0,\quad \beta b^2-(\gamma+\alpha)k^2=0,\quad \gamma c^2-(\alpha+\beta)k^2=0.$$

The quantity k^2 may be eliminated from these equations and the resultant two equations solved for α/γ and β/γ. Thus, there are definite ratios for α, β, γ, for the altitudes of a given triangle, so that the directions of the altitudes form a set of corresponding vertex lines and therefore intersect in one point. This proposition also follows directly from the next theorem.

Theorem 2. *Any three correlative vertex lines divide the opposite sides of the triangle into parts proportional to the parameters α, β, γ, belonging to the adjacent sides.*

Proof: Referring to Fig. 1, let A' be the intersection of the vertex line AO with the side BC; and let $m=BA'/A'C$. By a cyclic substitution, for the other two vertex lines and sides let $n=CB'/B'A$ and $p=AC'/C'B$.

Then, for the triangle OBA' we have:

$$OA'=-s\boldsymbol{a}=\boldsymbol{b}+(\boldsymbol{c}-\boldsymbol{b})m/(1+m)$$

where s is a scalar. Hence,

$$s\boldsymbol{a}+\boldsymbol{b}/(1+m)+\boldsymbol{c}m/(1+m)=0. \tag{3}$$

Comparing this expression with equation (1) we find that $m=\gamma/\beta$. Analogously, $n=\alpha/\gamma$ and $p=\beta/\alpha$, which proves the proposition.

Comparing the coefficients of $\boldsymbol{a}$ and $\boldsymbol{b}$ in equations (3) and (1) we find that $\alpha/s=\beta(1+m)$, or using expression $m=\gamma/\beta$ for m, $s=\alpha/(\beta+\gamma)$.

For medians, $\alpha=\beta=\gamma$; $m=n=p=1$; and $s=1/2$; hence the medians trisect each other. For bisectors of angles, α, β, γ are proportional to the lengths of the corresponding sides, so that the equation $m=\gamma/\beta$ expresses the familiar

theorem that the bisector of an angle divides the opposite side into parts proportional to the other two sides.

From $m=\gamma/\beta$, $n=\alpha/\gamma$, $p=\beta/\alpha$, we have $mnp=1$, which means that any three vertex lines intersect in one point if they divide the opposite sides in ratios the product of which is equal to unity. Thus, for the altitude from vertex A, $m=AB \cos B/AC \cos C$.

Writing similar expressions for the other two sides, we find that m, n, and p satisfy the equation $mnp=1$, so that the three altitudes intersect in one point.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

DISCUSSIONS

I. A TRIPLY COMPOSITE HOMOGENEOUS POLYNOMIAL

By E. T. BELL, California Institute of Technology

1. From a paper in the Transactions of the American Mathematical Society for April, 1929, I transcribe the following definition of composition, as stated by Professor O. C. Hazlett (who gives full references to Dickson and other writers.)

"If $f(x)$ is any homogeneous polynomial in the n variables $x_1 \cdots, x_n$, such that

$$f(x)f(\xi) = F(X), \tag{1}$$

where the X's are bilinear functions of the x's and ξ's,

$$X_k = \sum_{i,j}\gamma_{ijk}x_i\xi_j \quad (k = 1, \cdots, n), \tag{2}$$

then $f(x)$ is said to admit the composition (2)."

Readers of Professor Hazlett's illuminating paper will note that in the proofs of the theorems, as in the fundamental papers by Dickson, two conditions are essential: $f(x)$ shall not be expressible in fewer than n variables; the functions X effecting the composition shall be bilinear in the variables of the forms compounded.

Some years ago I worked out a considerable part of a theory of composition in which both of these conditions are abandoned. The complete theory in this extended sense can be given when and only when the problem implied in (1), (2) is completely solved, which has not yet been done. In the meantime it may be of interest to give the results in the algebraic number case, which are fairly complete; this I hope to do in a paper to appear elsewhere. All of the specific

instances, except in the simplest case, involve very long formulas when written out fully; hence no examples of the straightforward general theory in the algebraic case are included in the paper mentioned. It may therefore be of interest to state the formulas for the simplest case of what I have called multiple composition, which is as follows.

If n is a composite integer >2, which has a divisor $d\geqq 2$, we consider those homogeneous polynomials of degree n in n independent variables having the following properties:

(A) The polynomials admit composition in the sense (1), (2), and are not expressible by a linear transformation on the variables, in fewer than n variables.

(B) Each of the polynomials can be written as a homogeneous polynomial in d variables; the d new variables are homogeneous polynomials, with rational coefficients, in the original variables, and are of degree n/d in those variables.

(C) The same as (B), with d, n/d interchanged throughout.

(D) Each of the polynomials in (B), (C), considered as a function of the new variables, has the property (A).

(E) The generalization of (A)$-$(D) in which $n=n_1n_2\cdots n_s$, $n_1>1$, $n_2>1, \cdots, n_s>1$. There are then 2^s-1 compositions of the kind described.

It will be sufficient to state the formulas for the case $s=2$, $n_1=n_2=2$. Let the u's, v's, x's, y's, z's, w's be independent variables, and p, q, r, s parameters. Write

$$g(u,v)\equiv u^2+puv+qv^2,\quad h(u,v)\equiv u^2+ruv+sv^2\,;$$

$$Q(u_1,v_1,u_2,v_2)\equiv u_1u_2-qv_1v_2,$$

$$P(u_1,v_1,u_2,v_2)\equiv u_1v_2+v_1u_2+pv_1v_2,$$

$$S(u_1,v_1,u_2,v_2)\equiv u_1u_2-sv_1v_2,$$

$$R(u_1,v_1,u_2,v_2)\equiv u_1v_2+v_1u_2+rv_1v_2.$$

Let a suffix x, y, z, or w denote partial differentiation with respect to the suffix, thus $g_z(z,w)\equiv\partial g(z,w)/\partial z$, etc. Write

$$G'(x,y,z,w)\equiv g(x,y)-sg(z,w),$$

$$G''(x,y,z,w)\equiv xg_z(z,w)+yg_w(z,w)+rg(z,w),$$

$$H'(x,y,z,w)\equiv h(x,z)-qh(y,w),$$

$$H''(x,y,z,w)\equiv xh_y(y,w)+zh_w(y,w)+ph(y,w).$$

Let a suffix 1, 2, or 3 on any of the above functions refer to variables x_i, y_i, z_i, $w_i (i=1, 2, 3)$; thus $G_i'\equiv G'(x_i, y_i, z_i, w_i)$, etc. Write

$$x_3\equiv x_1x_2-qy_1y_2-sz_1z_2+qsw_1w_2,$$

$$y_3\equiv x_1y_2+y_1x_2+p(y_1y_2-sw_1w_2)-s(z_1w_2+w_1z_2),$$

$$z_3\equiv x_1z_2+z_1x_2+r(z_1z_2-qw_1w_2)-q(y_1w_2+w_1y_2),$$

$$w_3\equiv x_1w_2+w_1x_2+y_1z_2+z_1y_2+p(y_1w_2+w_1y_2)+r(z_1w_2+w_1z_2)+prw_1w_2.$$

Then, as may be verified directly,

$$\text{(I)} \qquad G'_3 = S(G'_1, G''_1, G'_2, G''_2), \qquad G''_3 = R\,(G'_1, G''_1, G_2', G_2''),$$

$$H_3' = Q(H_1', H_1'', H_2', H_2''), \quad H_3'' = P(H_1', H_1'', H_2', H_2'')\ ;$$

$$\text{(II)} \quad g(H'(x,y,z,w),\ H''(x,y,z,w)) = h(G'(x,y,z,w),\ G''(x,y,z,w)).$$

Denote either of the functions in (II) by $f(x, y, z, w)$. Then $f(x, y, z, w)$, homogeneous of degree 4 in x, y, z, w, has the triple composition as indicated below.

$$\text{(III)} \qquad \begin{aligned} f(x_1,y_1,z_1,w_1)f(x_2,y_2,z_2,w_2) &= f(x_3,y_3,z_3,w_3), \\ &= g(H_1',H_1'')g(H_2',H_2'') = g(H_3',H_3''), \\ &= h(G_1',G_1'')h(G_2',G_2'') = h(G_3',G_3''). \end{aligned}$$

Examples in which the subsidiary forms g, h, $\cdots$, are of different degrees are perhaps more striking. The simplest is that in which g is of degree 2, h of degree 3. As the formulas of composition are rather long, we shall not transcribe them. The above is, so far as I know, the first instance in the literature of a multiply composable form. The situation defined in $(A)-(E)$ is here stated for the first time.

II. A Theorem About Maximum and Minimum Points

By Carl Gundersen, Oklahoma A. and M. College

Theorem: *If $f(x)$ is a continuous function with continuous derivatives, and if $f'(x')=0$, and $f''(x')=0$ at the same time as it, $[f''(x')]$, is a minimum, then $f(x')$ is also a minimum.*

Proof: Since $f''(x)$ has a minimum point at x', and $f''(x')=0$, $f''(x)$ must be positive near $x=x'$. That means that $f'(x)$ is an increasing function both before and after the point $x=x'$, and since $f'(x')=0$, $f'(x)$ is negative *before*, and positive *after* the point $x=x'$; it follows that $f(\mathrm{x})$ is decreasing *before* and increasing *after* the point $x=x'$, which is only another way of saying that $f(x')$ is a minimum point.

This theorem is useful in discussing the case when $f''(x')=0$. Assume that $f^{(n)}(x')$ is positive, and $f^{(n-1)}(x')$, $f^{(n-2)}(x')$, $\cdots$, $f'(x')$ are all zero; then $f^{(n-2)}(x')$ must be a minimum by a well known theorem; $f^{(n-4)}(x')$ must be a minimum by the theorem proved above; again $f^{(n-6)}(x')$, $f^{(n-8)}(x')$, etc. must be minima. Therefore if n is even, $f(x')$ is a minimum; if n is odd, $f'(x')$ is a minimum, and being equal to zero it is positive near $x=x'$, and $f(x)$ must have a point of inflection at $x=x'$.

An analogous theorem holds of course for a maximum.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College of the City of New York

All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Modern Geometry. By Roger A. Johnson. Houghton Mifflin Company, 1929. xiii+319 pages. $3.50.

This is a very interesting book both as regards its content and the manner of presentation. The following chapter headings give a fair idea of the material covered. I, Introduction; II, Similar figures; III, Coaxal circles and inversion; IV, Triangles and polygons; V, Geometry of circles; VI, Tangent circles; VII, The theorem of Miquel; VIII, Theorems of Ceva and Menelaus; IX, Three notable points (orthocenter, circumcenter, and median point of a triangle); X, Inscribed and escribed circles; XI, The nine point circle; XII, Symmedian point and other notable points (isogonic centers, Nagel point); XIII, Triangles in perspective; XIV, Pedal triangles and circles; XV, Shorter topics (cyclic quadrangle, Morley's theorem, etc.); XVI, The Brocard configuration; XVII, Equibrocardal triangles; XVIII, Three similar figures.

We are entirely in sympathy with the aim of this book, which is mainly to give beginners in mathematics something interesting to think about. It is rather the fashion to pooh-pooh this branch of mathematics and to say that the time spent on it could be better devoted to matters of a more arithmetical nature. But it is a commonplace that mathematics is like art, and to really appreciate art one must first get interested in it; the main object of a secondary-school or junior-college teacher who believes in his work is to encourage an interest and curiosity in his students, the actual material he talks about being relatively unimportant. We feel that the book in hand is admirably calculated to arouse and stimulate such an interest. Furthermore the careful formulation of proofs so that they are valid for all types of figures involved furnishes a good example to the beginner of what mathematical reasoning should be.

A reviewer is, we understand, expected to point out certain features of the book in question which are, in his opinion, undesirable. In this case this is a little difficult. There might possibly be more exercises, and we prefer the name Fermat points to isogonic centers and the name Hessian points to isodynamic points. There is an easily supplied omission in the statement of the theorem on p. 221. On the whole we heartily recommend the book and wish it the success it deserves.

FRANCIS D. MURNAGHAN

Gerhard Hessenberg's Vorlesungen über darstellende Geometrie, herausgegeben von E. Salkowski, o. Professor an der technischen Hochschule Charlottenburg. Leipzig, Akademische Verlagsgesellschaft, 1929. 274 pages, 481 figures.

Descriptive Geometry. By Harvey Herbert Jordan and Francis Marion Porter, University of Illinois. Ginn and Co., Boston, 1929. xi+349 pages, 791 figures.

The first book introduces its subject with a practical explanation of the use of drawing instruments, followed by three parts of about equal length, on axonometry, the two picture (h, v) method, and curves and surfaces. The sequence is unusual, but the essential features of each method are clearly and concisely presented, and applied. For an American student it would have been improved by a larger supply of unsolved exercises. The press-work and drawings are excellently well done, and the tone of a thoroughly scientific work is preserved throughout.

The second book presents the subject in accordance with the better traditional American methods, presenting a large number of principles, without putting great emphasis on any, pointing out that different ones should be used for different purposes. The first third of the volume is devoted to the ordinary rectilinear problems in h, v, p presentation; then the principles are applied to curved surfaces. A short well written chapter treats of pictorial drawing, including isometric and perspective representations. The book closes with an excellent list of over 400 exercises, arranged for use as each chapter is completed. As a guide to the subject as an art, this book is particularly well adapted; as an introduction to the science of descriptive geometry, the former is preferable.

VIRGIL SNYDER

Vorlesungen über Differential- und Integralrechnung, Bd. II. Funktionen mehrerer Veränderlichen. By R. Courant. Julius Springer, Berlin, 1929. vii+360 pages.

The second and concluding volume of Courant's lectures on the calculus, devoted to functions of several variables, follows the same plan as the first volume[1] and has the same outstanding merits. To borrow from a review by von Mises, "In Courant we have an author · · · who risks the statement of theorems which do not give the impression to the reader that each application of them is attended by the gravest dangers. In contrast with the present-day style of mathematical writing, in which the essential content is everywhere hidden behind a chaos of secondary details, the book of Courant breathes an entirely different spirit." One should not judge from this that the author ever departs from strict rigor; rather he chooses to state a theorem in a usable form rather than in its utmost generality and refinement. Thus the theorem that $f_{yx}=f_{xy}$ is proved under the hypothesis that both these derivatives are con-

[1] Reviewed in this Monthly, vol. 36 (1929), pp. 96–98.

tinuous in a region; a footnote then shows that we need only assume that one mixed derivative exists and is continuous at the point in question. Again matters of considerable detail or refinement are usually placed in the *Anhänge* that follow each chapter.

The book opens with an introductory chapter on three-dimensional analytical geometry and vector analysis; this is to be studied by the reader as the matters treated therein are needed. The following chapters deal with partial and total derivatives (II); further developments and applications of the same (III); integrals over regions (IV); curve integrals, and the integral theorems of Gauss, Stokes, Green and their significance (V); applications, mostly involving differential equations (VI). The book concludes with a 27-page summary of the principal formulas and theorems in both volumes.

For many theorems both scalar and vector formulations are given. To quote Courant, "Many facts and relations of the differential and integral calculus take on an essentially clearer and simpler aspect when the concepts and notation of vector analysis are employed." The Gibbs cross is used for the vector product, juxtaposition for the scalar. This usage is apparently gaining ground in Germany, being used in recent books by Runge, Blaschke, von Mises, and Fraenkel.

While the author never departs from precise mathematical formulation, interpretation, comparison, and application are constantly in the foreground. These lectures, therefore, should be of great service to all who wish to have a thoroughly modern grasp of the Calculus or who wish to apply it fruitfully in other fields.

LOUIS BRAND

Projektive Geometrie der Ebene unter Benutzung der Punktrechnung. Von Herman Grassmann. Zweiter Band: Ternäres. Zweiter Teil. Mit 259 Figuren im Text. Leipzig und Berlin, B. G. Teubner, 1927. xvi+522 pp.

Increased attention has lately been given the Ausdehnungslehre. This has been due in part to favorable expressions of opinion concerning this subject by mathematicians of renown and in part to the close relation which exists between Grassmann's geometrical algebra and the so-called absolute calculus of Ricci, made prominent by the theory of relativity. It is therefore very fortunate that this interest should be further stimulated by a treatise on projective geometry, based on the Ausdehnungslehre, by Grassmann's son. The present volume is in continuation of two previous volumes by the same author which appeared in 1909 and 1913 respectively.

The principal algorithms on which the calculus of Grassmann is based are the "combinatorial" and the "algebraic" multiplications. The second part of the Ausdehnungslehre of 1862 discusses products of the latter type. "Algebraic" multiplication is relevant to the theory of extensive functions. The simplest case of an extensive function, namely, an integral function of the first degree, leads to the conception of an extensive quotient or lacunary expression with one

space which includes as particular case the matrix as symbolic operator. Geometrically, the importance of an extensive quotient lies in the fact that the concepts of collineation and reciprocity are readily expressed in terms of it. This was explicitly pointed out by Hermann Grassmann[1] the elder in the Ausdehnungslehre of 1862. As one of the editors of Grassmann's works, the son Hermann added an elaborate note amplifying his father's remarks and Hermann Grassmann, Jr.'s studies and subsequent publications, including the volume before us, may be regarded as further expansions of this part of the Ausdehnungslehre (pp. 67–208).

The present work falls into three parts: first, projective geometry; second, "parallel" geometry; third, "right angle" geometry. Projective geometry, and parallel geometry which is treated as incidental to the former (cf. page 380), absorb 313 pages of the book; pages 379–522 are devoted to metrical geometry referred to the "circular points at infinity." In contrast with projective geometry, Grassmann conceives of metrical geometry as consisting of "parallel" geometry and "right-angle" geometry. Perpendicularity is made the basis of linear measurement. The length of a linear segment is defined in terms of a quadratic form (p. 393). Grassmann, Jr.'s treatment of the projective geometry of the plane presents a rich collection of theorems in the proofs of which researches of other investigators, using the ordinary methods of analysis, have been freely used. In particular, the author expresses in the preface his indebtedness to Gundelfinger. Such a relation of Grassmann, Jr.'s research to contemporaneous literature finds a remarkable precedent in his father's work. In fact, Grassmann, Sr. in his Ausdehnungslehre of 1844 states on page 138 (collected works): "Nun finden wir zu dem Wege den wir hier verlangen in der neueren Geometrie mannigfache Vorarbeiten, in unserer Wissenschaft aber ist uns der Weg selbst auf's Vollkommenste vorgezeichnet." Regardless, therefore, of Grassmann, Jr.'s frequent reference to other authors, his work stands for itself.

A central point in the theory of the present work is the concept of a lacunary expression which the author develops on page 171 in connection with "apolar polar systems." This development is easily understood and constitutes a valuable introduction to the theory of lacunary expressions in the Ausdehnungslehre of 1862. It is entirely logical that the algebraic product is introduced (p. 175). To the lacunary expression and the algebraic product correspond the two fundamental forms, the *lacunary form* and the *algebraic* (product) *form*; these are called by the author "Lückenform" and "Potenzform." Corresponding to the concepts "order" and "class" two fundamental theorems concerning algebraic (product) forms are established (pp. 177, 185). A further application is made to curves of the third degree (pp. 289, 378). In this connection, the author expresses his indebtedness to Durège.

In the preceding theory, the concepts *transformation*, *group* and *invariant* lie hidden but they are not explicitly developed by the author. It is of course well known that these concepts hovered before the mind of the elder Grassmann

[1] Collected Works, vol. 1–2, pp. 256–257.

and vague references to them may be found in the Ausdehnungslehre of 1844. Indeed, Grassmann's geometrical algebra forms an admirable basis for the genetic motivation of the preceding concepts. A valuable contribution in this direction has recently been made by Dr. E. Carus in his Chicago dissertation.[1] Grassmann, Jr. also refers to the "mehrfaltige Produkte" of E. Müller and the "Faltungsprozess" of Gordan in a note on page 188 of the present work. A further application of certain concepts of the Ausdehnungslehre, with special reference to the theory of invariants, has recently been made by E. Study in his vectorial "Theorie der Invarianten linearer Transformationen." However, Study apparently interprets mathematics narrowly as a mere science of number; consequently he rejects the axiomatic foundation of geometry and would subordinate the Ausdehnungslehre to the theory of invariants. Such a position is clearly opposed by the elder Grassmann and no doubt also by the author of the present work.

Hermann Grassmann, Jr. did not live to see his splendid volume in print. The publication of this work was made possible through the financial support of a group of friends, notably Dr. E. Carus of La Salle, Ill. Hermann Grassmann, Jr.'s work represents mature scholarship and is indispensable to all students interested in the important field of vectorial geometry; it should also promote research in the Ausdehnungslehre itself, whose application to geometry has been judged "the most comprehensive and above all the most natural system of geometric analysis yet discovered."

ARTHUR R. SCHWEITZER

Neue Astronomie. By Johannes Kepler. Translated and introduced by Prof. Dr. Max Caspar. R. Oldenburg, Munich-Berlin, 1929. 66+416 pages, xiii+68 figures.

This translation of Kepler represents the very commendable purpose of the translator and publisher to make the German readers better acquainted with the forceful style and colorful personality of their great countryman. There is no attempt to abridge or elaborate any of the passages. So far as possible it is Kepler's writing unembellished and unadorned.

It is also hoped that there will result a greater appreciation of Kepler's accomplishment, comparable in many ways, as he thought, with that of Columbus, and a fulfillment of his expressed wish that many would enjoy the narrative of his celestial discoveries as they had those of the geographical ones of Columbus, Magellan, etc.

The book performs a long neglected duty of doing for Kepler what had long been done for Ptolemy, Copernicus, Newton, and Descartes, for strangely enough it seems to be the only complete German edition ever published.

Preceding the translation there is a preface, an introduction, a brief outline

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[1] *Invariants as products and a vector interpretation of the symbolic method*, The Open Court Publishing Co., Chicago and London, 1927.

of planetary theory before the time of Kepler, the origin and development of the *Neue Astronomie*; and at the end of the book there are comments by the publisher and a table of contents that groups the material into five parts according to the outstanding ideas.

The book is printed on firm unglazed paper with very legible typography and is altogether a very attractive volume. The price ranges from M38.50 for the linen to M100 for the pigskin binding.

F. E. CARR

Bibliography, Practical, Enumerative, Historical; An Introductory Manual. By H. B. Van Hoesen with the collaboration of F. K. Walter. Charles Scribner's Sons, New York, 1928. Price $7.50.

American students are frequently accused of lack of erudition, perhaps not without cause. Erudition is acquired, if at all, by extensive and selective reading. At most institutions of higher learning in this country the supply of useful books is large and the conditions for consulting them most favorable. If the accusation is well founded one may consequently be justified in concluding that the students do not know when, how, or what to read. At Princeton University the assistant librarian, Dr. H. B. Van Hoesen, has found it necessary to give courses for graduate students in the use of the library. The highly interesting and entertaining book under review is an outgrowth of these courses. The book starts out with practical, much needed advice to authors. Subject bibliography takes about 85 pages which includes two pages devoted to mathematics. About 120 pages are given to the use of the library in general: library science, general reference books, special, national, and universal bibliographies. The historical bibliography takes about 150 pages devoted to the history of writing, printing, books and libraries. A bibliographical appendix and a topical index, each with about 2000 entries, occupy the rest of the book, an impressive object lesson.

EINAR HILLE

MATHEMATICS CLUBS

EDITED BY H. J. ETTLINGER, University of Texas, Austin, Texas.

All reports of club activities should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES

The Euclidean Circle of the Illinois State Normal University, Normal, Illinois.

The Euclidean Circle was organized on Dec. 14, 1927, under the leadership of Mr. C. N. Mills, professor of mathematics, and Miss Edith Irene Atkin, assistant professor of mathematics.

The officers for the year 1928–29 were as follows: Elmer J. Graber ('29)—"Major Arc"—Presiding Officer; Pauline Whipple ('29)—"Minor Arc"—Vice-Presiding Officer; Willis T. Maas ('29)—"Inscribed Polygon"—Secretary; Dorothea Concklin ('31)—"Center"—Treasurer. Miss

Edith Irene Atkin—"Circumscribed Polygon"—Acting Head of the Mathematics Department, 1928–29, was Sponsor.

The program for the college year 1928–29 was as follows:

Sep. 27, 1928. Business. Homecoming Social Committee.
Oct. 11. "The theorem of Nicomachus with extensions," by Elbert C. Parker, '30; "Magic squares," by Blanche Hinthorne, '29.
Oct. 20. First Annual Homecoming Breakfast. 8:30 A.M. Toasts: "Limits," by Christian E. Harpster, '28; "Going off on a tangent," by Blanche Hinthorne, '29; "Signs of grouping," by Clara Whitfield, '28; "Talk," by President David Felmley.
Oct. 25. Proofs of the Pythagorean theorem: "*The Pythagorean Proposition*," by E. S. Loomis and Miss Atkin; "Geometric Proofs," by Ella Iliff, '31; "Geometric and algebraic proof," by Sue Szabo, '30; "Geometric and algebraic proof," by Ella Rosenthall, '31.
Nov. 22. "Flatland," by Verna Mae Thomassen, '31; "Fourth dimension," by Arthur R. Grismer, '29.
Dec. 13. Initiation of new members. Social Hour.
Jan. 31, 1929. "Old measuring instruments," by Dorothea Concklin, '31.
Feb. 28. "Mathematics in chemistry," by Pauline Whipple, '29.
Apr. 4. "Methods of drawing a straight line by linkages," by Clyde Kaiser, '30.
May 2. "A number system with a base twelve," by Agnes Hanson, '31; "How to trisect an angle with a straight edge containing two marked points," by Elmer J. Graber, '31.
May 30. A Play: "Mock trial of B versus A, or solving a personal equation by judicial process." Adapted by Kathryn McSorley, Hunter College. Social Hours.

The officers for the year 1929–30 are: Major Arc, Clyde Kaiser ('30); Minor Arc, Verna Mae Thomassen ('31); Inscribed Polygon, Dorothea Concklin ('31); Center, W. A. Fiske ('31).

(Report by Willis T. Maas)

The Junior Mathematics Club of the University of Chicago, Chicago, Ill.

The officers for the year 1927–1928 were: B. W. Jones, President; May M. Beenken, Chairman of Committee on Program, T. F. Cope, Secretary and Treasurer.

The program for the year 1927–1928 was as follows:

Oct. 21, 1927. "The history of the Junior Mathematics Club," by Professor H. E. Slaught; "Mathematical study in Italy," by Professor E. P. Lane.
Nov. 2. "Selected topics in the calculus of variations," by Mr. T. F. Cope.
Nov. 16. "Construction of higher plane curves," by Mr. R. S. Shaw.
Dec. 7. "The Borel theorem and its applications," by Mr. E. J. McShane.
Jan. 11, 1928. "The evolution and dissolution of matter," by Professor MacMillan.
Feb. 9. "Various map projections," by Mr. C. A. Sherer.
Feb. 21. "Hyperbolic functions," by Mr. G. D. Gore.
Mar. 14. "The Lorentz transformations," by Mr. R. H. Bardell.
Apr. 11. "Sturm's theorems of comparison and oscillation for solutions of differential equations," by Mr. F. R. Bamforth.
Apr. 25. "How can graduate mathematical instruction be improved?" by everyone, especially B. J. Jones, R. S. Shaw, A. Woods.
May 9. "Geometric representation of real and complex points of elementary curves," by Mr. G. W. Spenceley.
May 23. "Various types of coordinate systems in geometry," by Dr. Jesse Douglas.

(Report by T. F. Cope)

Zeta Mu Tau, University of Washington, Seattle, Wash.

The program of Zeta Mu Tau, the mathematics club of the University of Washington, for the year 1927–1928, was as follows:

Oct. 5, 1927. "The Path of light in the gravitational field," by John Hicks, '29.
Oct. 19. "The history of our number system," by Lucile Morry, '28.
Nov. 2. "Four times," by Larned Meacham, '29.
May 17, 1928. Initiation of pledges, followed by a banquet with addresses by Dr. R. E. Moritz and Dr. R. M. Winger.

Zeta Mu Tau is essentially an undergraduate society, with graduate students and members of the Mathematics Department taking little part in its activities. It was formed for the purpose of stimulating campus interest in a very little-appreciated branch of learning, and to provide a means for the exchange of ideas among the three groups interested in mathematics, viz., the mathematics majors, the engineers, and the students of the physical sciences.

The officers for the year 1927–28 were as follows: Louis Berger, President; Martha Hardy, Vice-President; Lucile Morry, Secretary; Lucile Anderson, Treasurer.

(Report by Martha Hardy)

The Mathematics Club of the University of Colorado, Boulder, Colorado.

The program for the year 1928–29 was as follows:
Oct. 11, 1928. "Man-made science," by S. Hacker.
Oct. 25. "Impressions of the 1928 International Congress of Mathematicians, Bologna, Italy," by Associate Professor C. Kendall.
Nov. 18. "Graphical representation of complex functions," by E. Rainville.
Nov. 25. "The place of mathematics in education," by R. B. Pinson.
Jan. 17, 1929. "Circles connected with the triangle," by H. Harms.
Jan. 31. "Magic squares," by P. Folk.
Feb. 21. "Interesting theorems in modern geometry," by H. A. Miley.
Feb. 28. "Mathematical geography," by Professor C. A. Hutchinson.
Mar. 28. "Calculation of the date of Easter," by R. Remke.
Apr. 11. "Mathematics, from the points of view of the mathematician and the physicist," by L. Strait.
Apr. 25. "Majoring in mathematics," by Professor A. J. Kempner.
May 16. "Conics in a spring suit: or, conic sections in terms of the length of arc and radius of curvature," by Professor G. H. Light.
May 24. A Fry in Gregory Canyon.

The officers for 1928–29 were: Sidney Hacker, President; Earl Rainville, Vice-President; Pauline Folk, Secretary. The officers for 1929–30 are: Earl Rainville, President; Louis Strait, Vice-President; Janet Hall, Secretary.

The Mathematics Club, Cornell College, Mt. Vernon, Iowa.

The regular meeting time is the third Thursday of every month. The permanent arrangements committee consists of Hazel Cory, Myrtle McIntosh, and Arnold Herkleman.

The programs for the year 1928–29 were as follows:
Nov. 1928. Dinner followed by an address by Prof. E. W. Chittenden. *Entertainment Committee*: Ethel Cain and Iva Shaffer.
Dec. Christmas party with recreational mathematics. *Committee*: Viola Smith, James Nauman, Irma Kaufman, Leona Barnes, and Albert Nelson. *Entertainment Committee*: Mary Schmeiser, Newell Lumsden, and Velda McCauley.
January, 1929. "The Calendar," by Professor E. E. Moots. *Entertainment Committee*: Pauline Davidson, Margaret Kopf, and Ernest Nielson.
February. (1) A Drama, "The evolution of numbers;" (2) Arnold Herkleman, "The history of $\sqrt{(-1)}$. *Committee*: Lilliam Frink, Beatrice Burge, Philip Switzer, Arthur Rouse. *Entertainment Committee*: Ruth Stewart, Grace Tielkemeir, William Shaw.
March. "The relation of mathematics to biology," by Professor H. M. Kelley. *Entertainment Committee*: Bernadine Burge, Florence Wentzel, Portia Tracy.

April. "Fundamental concepts," by Professor F. M. McGaw. *Entertainment Committee*: Ruth Ketzle, Gertrude Sayer, Elmer Meyers.

May. "Recent astronomical discoveries," by Leonard Hute; "Comets," by Albert Nelson. *Entertainment Committee*: Eleanor Martin, Gladys Berry, Evelyn McMeans.

There was also an afternoon meeting at which Professor Wiley, of Iowa City, spoke; and there was a picnic at the end of the year.

(Report by James L. Nauman)

The Delta Chi Fraternity of the University of New Hampshire, Durham, New Hampshire.

The officers for the year 1928–29 were: Real Dis Rochers, President; Louise Woodman, Vice-president; Florence M. Brown, Secretary; Philip Nudd, Treasurer.

The programs for the year 1928–29 were as follows:

January 19, 1928. "Diophantine analysis," by Jessie Daniels, '28.
February 9. Initiation banquet.
February 23. "The theory and uses of the planimeter," by Carroll Avery, '28.
March 8. "How trigonometric and logarithmic tables are made," by Malcolm Sargent, '28.
March 22. "Comparison of circular and hyperbolic functions," by Charles Morreels, '28.
April 5. "Partial differentiation theory and certain uses," by Priscilla Morris, '28.
April 19. "Mean value and probability theory and illustrations," by Lawrence Smith, '28.
May 3. Annual party.
May 17. Election of officers.
October 11. Election of committees.
October 25. "Hyperbolic functions," by Philip Nudd, '30.
November 8. Social.
November 22. "Vector analysis," by Marvin P. Salt.
January 17, 1929. "Mathematics and thermo-dynamics," by Assistant Professor Edward Donovan.
January 31. Annual Business Meeting.
February 14. Initiation Banquet.
February 28. "Mathematics and mechanics," by Assistant Professor Edward L. Getchell.
March 14. "Mathematics and electricity," by Mr. William B. Nulsen.

The Mathematics Club of George Washington University, Washington, D. C.

The officers for the year 1928–1929 were: Dr. F. E. Johnston, President; Mr. Michael Goldberg, Secretary.

The program for the year 1928–1929 was as follows:

October 15, 1928. "The calculus of tensors," by Professor Edgar W. Woolard.
October 29. "The complex quantity slide rule," by Mr. Michael Goldberg.
November 12. "The problem of Apollonius," by Dr. Paul Wernicke.
November 26. "Line values of the trigonometric functions and their use in constructing curves," by Dr. F. E. Johnston.
December 10. "Solution of equations by Graeffe's method," by Professor W. J. Berry.
January 7, 1929. "The story of numbers," by Dr. Tobias Dantzig.
February 18. "Short methods in arithmetic and algebra," by Mr. B. Z. McLeroy.
March 5. "Multiple points," by Mr. P. J. Federico.
March 18. "The three problems of antiquity," by Mr. D. B. Lloyd.
April 15. "Heaviside's operational calculus," by Dr. Louis Cohen.
April 29. "Special methods of integration," by Mr. S. J. Snyder.
June 8. Picnic at Chapel Point, Md.

(Report by Michael Goldberg)

The Mathematics Section of the Hanover College Science Club, Hanover, Indiana.

The officers for 1928–29 were: George N. Bishop, President; William Willis, Vice-President; Louisa Plummer, Secretary-Treasurer; Dr. Harvey A. Zinszer, Faculty Adviser.

The program for the year 1928–29 was as follows:

October 10, 1928. "History and evolution of the electric spark," by Dr. H. A. Zinszer; "Horner's synthetic division," by Mary E. Holderman, '29.

November 27. "Science and student life at Cambridge," by Dr. Mason Hufford, (Indiana University).

January 9, 1929. "Evolution of pi," by Jessie Hope Rankin, '29; A reel on the "Development of the telephone."

February 13. "Variation of volume with depth in horizontal tanks, by Thirza Kurtz, '29.

March 13. "Sturm's functions," by Cecil Collins, '29; "Curve tracing," by George Bishop, '29.

April 24. "Prime numbers," by Louisa Plummer, '29; A reel on "Revelations of the X-ray."

May 22. Picnic at Cedar Bluff; Election of officers; A reel on "Theory and operation of the radio."

The new officers for 1929–30 are: William Willis, President; Harry Francke, Vice-president; Helen Campbell, Secretary-Treasurer.

Three members of the club attended the state mathematical meeting at Culver Military Academy, where Margaret Darragh, '29, gave a paper on "Three methods for solving skew lines." Eight senior mathematics majors made Delta Epsilon, a national scientific honor fraternity.

(Report by Helen Campbell)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3394. *Proposed by Paul Wernicke, Washington, D. C.*

In space of three or more dimensions two n-gons A, B have the consecutive vertices $A_1, A_3, \cdots, A_{2n-1}$ and $B_2, B_4, \cdots, B_{2n}$. The feet of perpendiculars from A_1 to $B_{2n}B_2, \cdots$, from A_{2k-1} to $B_{2k-2}B_{2k}$ are $B_1 \cdots B_{2k-1}$. Those of perpendiculars from B_{2k} to $A_{2k-1}\,A_{2k+1}$ are A_{2k}. Subscripts are to be taken modulo $2n$, and $k=1, 2 \cdots n$.

(a) Show that

$$\sum_k (A_{2k-1}A_{2k})^2 - \sum (A_{2k}A_{2k+1})^2 = -\sum (B_{2k-2}B_{2k-1})^2 + \sum (B_{2k-1}B_{2k})^2$$

(b) What does it mean geometrically if the members of this equation both vanish?

3395. *Proposed by J. Rosenbaum, Milford, Conn.*

Given an n-gon, $A_1A_2 \cdots A_n$, show how to locate a point X such that the vectors XA_1, XA_2, $\cdots$, XA_n form a closed n-gon.

3396. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Let D, E, F be the feet of the altitudes, and A', B', C', the mid-points of the sides of a triangle ABC. Show that the double elements of the three involutions (BC, DA'), (CA, EB'), (AB, FC') are three pairs of opposite vertices of a complete quadrilateral.

3397. *Proposed by V. Rojansky, Washington University.*

Let $L_k(u) = e^u(d^k/du^k)(u^k e^{-u})$ be the Laguerre polynomial of the kth order. Let $L_k^m(u)$ be the mth derivative with respect to u of the Laguerre polynomial of the kth order.

Let n, l, q, and p be integers such that $n>0$, $q>0$, $0 \leqq l \leqq n-1$, and $0 \leqq p \leqq q-1$.

Then show that the algebraic sign of the definite integral

$$\int_0^\infty x^{l+p+3} e^{\mu} L_{n+l}^{2l+1}\left(\frac{x}{n}\right) L_{q+p}^{2p+1}\left(\frac{x}{q}\right) dx, \quad \text{where} \quad \mu = -\tfrac{1}{2}(n^{-1} + q^{-1})x,$$

is the same for $p = l+1$ as it is for $p = l-1$, when either value of p is permitted by the conditions given above.

The proposer infers the truth of this proposition from the physical considerations involved in the theory given in the Physical Review, vol. 33 (1929), p. 1. It is of interest to prove this on purely mathematical grounds. Integrals of this type were discussed by Schroedinger, Annalen der Physik, vol. 80 (1926), p. 437. It may be added that the Laguerre polynomials are closely related to Whittaker's W function. (See Whittaker and Watson's *Modern Analysis*.)

SOLUTIONS

3174 [3170; 1926, 104]. *Proposed by R. H. Sciobereti, University of California.*

Given the base BC of a spherical triangle in position and magnitude and given the magnitude of the angle A which is opposite to BC, find the locus of A.

Solution by Otto Dunkel, Washington University.

Let M be the middle point of the base, the extremities of which will be denoted by B, B', $B'M = MB = b$; and let the vertex P be located so that angle $B'PB = A$. Let also a set of rectangular axes be taken with the origin at O, the center of the sphere, so that OM is along the positive z-axis and the plane of OBB' is the yz-plane. Then the direction cosines of OB and OB' are respectively $(0, \sin b, \cos b)$, $(0, -\sin b, \cos b)$. Let α, β, γ be the direction angles of OP, and denote the angles $B'OP$ and BOP by p' and p. Then

$$\cos p = \cos\beta \sin b + \cos\gamma \cos b, \qquad \cos p' = -\cos\beta \sin b + \cos\gamma \cos b,$$

and from the triangle $B'BP$ we have

$$\cos 2b - \cos p \cos p' = \sin p \sin p' \cos A,$$

$$\cos 2b - \cos^2 \gamma \cos^2 b + \cos^2 \beta \sin^2 b = \sin p \sin p' \cos A.$$

If now we square both sides, replace $\sin^2 p$ and $\sin^2 p'$ by values obtained from the equations above, and then make some reductions, we obtain the equation of the cone passing through the desired locus,

$$(1)\quad [\cos^2 b(x^2+y^2) - \sin^2 b(x^2+z^2)]^2 \sin^2 A - x^2[x^2+y^2+z^2]\sin^2 2b \cos^2 A = 0.$$

If $A = 90°$ the equation reduces to one of the second degree (repeated),

$$(2)\qquad \cos^2 b(x^2+y^2) - \sin^2 b(x^2+z^2) = 0.$$

We have here the equation of a particular case of a spherical conic. We shall derive the focal property for such conics. Let F and F' be two points on the great circle of the sphere determined by the xz-plane at the distances c and $-c$ from M, and suppose that P is located on the sphere so that $FP+F'P=2a$. The direction cosines of OF, OF' and OP are $(\sin c, 0, \cos c)$, $(-\sin c, 0, \cos c)$, $(\cos\alpha, \cos\beta, \cos\gamma)$ respectively. If we set $FP=\rho$, $F'P=\rho'$, we have at once

$$\cos\rho = \cos\alpha \sin c + \cos\gamma \cos c,\ \cos\rho' = -\cos\alpha\sin c + \cos\gamma\cos c.$$

Hence

$$2\cos\gamma\cos c = \cos\rho + \cos\rho' = 2\cos\tfrac{1}{2}(\rho+\rho')\cos\tfrac{1}{2}(\rho-\rho') = 2\cos a\cos\tfrac{1}{2}(\rho-\rho'),$$

$$2\cos\alpha\sin c = \cos\rho - \cos\rho' = -2\sin\tfrac{1}{2}(\rho+\rho')\sin\tfrac{1}{2}(\rho-\rho')$$
$$= -2\sin a\sin\tfrac{1}{2}(\rho-\rho').$$

By squaring and combining these two equations we obtain

$$\cos^2\gamma\sin^2 a\cos^2 c + \cos^2\alpha\cos^2 a\sin^2 c - \cos^2 a\sin^2 a = 0.$$

The constant term may be multiplied by $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$ in order to make the equation homogeneous. It then becomes

$$(3)\quad \cos^2 a(\sin^2 a - \sin^2 c)x^2 + \sin^2 a\cos^2 a\, y^2 - \sin^2 a(\cos^2 c - \cos^2 a)z^2 = 0.$$

We shall now identify (3) with (2). If $b=45°$ it is easily seen geometrically that the locus of P for angle $B'PB=90°$ is two great circles passing through the pole of $B'B$ and its extremities. If $90°>b>45°$, extend the arc BB' to B_1 so that $BB_1=180°$; also extend $B'B$ to B_1' so that $B'B_1'=180°$. The arcs B_1B' and BB_1' are diametrically opposite, and the locus will be obtained by making B_1PB' and $BP'B_1'$ right angles. Hence we need to consider only the case where $b<45°$. The locus of P, angle $B'PB=90°$, has intercepts on the meridian circle at M which may be denoted by $a=MA$ and $-a=MA'$. In the right triangle AMB, angle $MAB=45°$ and hence $a>b$ and $\tan b=\sin a$. Locate the points F and F' on AA' so that $FB=F'B=a$, and set $MF=c$; then $\cos a=\cos b \cos c$, and from the equation above $\sin b=\cos b\sin a$. Hence we derive $\sin^2 a\ \cos^2 a=\cos^2 c-\cos^2 a=\sin^2 a-\sin^2 c$. After inserting these values in (3)

that equation reduces to $x^2 \cos^2 a + y^2 - z^2 \sin^2 a = 0$, and by use of $\sin a = \tan b$ this last equation reduces to (2).

This shows that, if in a spherical ellipse with major axis $A'A$ and minor axis $B'B$ the angle BAB' is a right angle, then if P is any other point on the ellipse BPB' is also a right angle.

3348 [1928, 446]. *Proposed by A. C. Aitken, University of Edinburgh.*

Show that

$$\frac{\pi}{6} = \sum_{1}^{\infty} \operatorname{arccot} 2u_r^2,$$

where $u_r = 4u_{r-1} - u_{r-2}$ with $u_1 = 1$, $u_2 = 3$.

Solution by Harry Langman, Arverne, L. I.

We have

$$\cot \frac{\pi}{6} = \sqrt{3} = 1 + \frac{1}{1+}\,\frac{1}{2+\cdots}, \tag{1}$$

the partial quotients 1, 2, repeating. Let p_t/q_t be the tth convergent. Then

$$p_{2r+1} = p_{2r+2} - p_{2r}, \quad p_{2r+3} = 2p_{2r+2} + p_{2r+1};$$

whence,

$$p_{2r+4} = p_{2r+3} + p_{2r+2} = 4p_{2r+2} - p_{2r}, \tag{2}$$

a recurring relation for the even convergents by themselves. It also follows that

$$q_{2r+4} = 4q_{2r+2} - q_{2r}. \tag{3}$$

We now set

$$a_r = \operatorname{arccot} \frac{p_{2r+2}}{q_{2r+2}} - \operatorname{arccot} \frac{p_{2r}}{q_{2r}}. \tag{4}$$

Here $a_r > 0$. We may write

$$\frac{\pi}{6} = \operatorname{arccot} \frac{p_{2t}}{q_{2t}} + \sum_{r=t}^{\infty} a_r. \tag{5}$$

From (4),

$$\cot a_r = \frac{p_{2r+2}p_{2r} + q_{2r+2}q_{2r}}{p_{2r}q_{2r+2} - q_{2r}p_{2r+2}}.$$

By the use of (2) and (3), the denominator reduces to $p_{2r-2}q_{2r} - q_{2r-2}p_{2r}$, hence to $p_2q_4 - q_2p_4$, or 1. Hence,

$$\cot a_r = p_{2r+2}p_{2r} + q_{2r+2}q_{2r}. \tag{6}$$

We now note that

$$p_{2r} = 2p_{2r-2} + 3q_{2r-2} \quad \text{and} \quad p_{2r-2} = 2p_{2r-4} + 3q_{2r-4} \tag{7}$$

hold for the initial values of r. Hence, by (2) and (3), $p_{2r+2}=2p_{2r}+3q_{2r}$, and (7) holds universally. In similar fashion we find that

$$q_{2r} = p_{2r-2} + 2q_{2r-2}. \tag{8}$$

From (6), (7), and (8), we have

$$\cot a_r = (2p_{2r} + 3q_{2r})p_{2r} + (p_{2r} + 2q_{2r})q_{2r} = 2(p_{2r} + q_{2r})^2. \tag{9}$$

Hence, (5) becomes

$$\frac{\pi}{6} = \operatorname{arccot}\frac{p_{2t}}{q_{2t}} + \sum_{r=t}^{\infty} \operatorname{arccot} 2(p_{2r} + q_{2r})^2. \tag{10}$$

If we set $u_{r+1}=p_{2r}+q_{2r}$, and choose $t=1$, (10) becomes

$$\frac{\pi}{6} = \operatorname{arccot} 2 + \sum_{r=2}^{\infty} \operatorname{arccot} 2u_r^2 = \sum_{r=1}^{\infty} \operatorname{arccot} 2u_r^2\,. \tag{11}$$

From (2) and (3) we obtain the recurrence relation $u_{r+1}=4u_r-u_{r-1}$.

If now we consider the sequence of odd convergents in (1), and set

$$b_r = \operatorname{arccot}\frac{p_{2r-1}}{q_{2r-1}} - \operatorname{arccot}\frac{p_{2r+1}}{q_{2r+1}}, \tag{4'}$$

then $b_r > 0$, and

$$\frac{\pi}{6} = \operatorname{arccot}\frac{p_{2t-1}}{q_{2t-1}} - \sum_{r=t}^{\infty} b_r. \tag{5'}$$

We find $\cot b_r=(p_{2r-1}+q_{2r-1})^2$, whence

$$\frac{\pi}{6} = \operatorname{arccot}\frac{p_{2t-1}}{q_{2t-1}} - \sum_{r=t}^{\infty} \operatorname{arccot}\,(p_{2r-1} + q_{2r-1})^2. \tag{10'}$$

If we set $v_r=p_{2r-1}+q_{2r-1}$ and choose $t=1$,

$$\frac{\pi}{6} = \frac{\pi}{4} - \sum_{r=1}^{\infty} \operatorname{arccot} v_r^2\,, \tag{11'}$$

with $v_{r+1}=4v_r-v_{r-1}$ and initial values $v_1=2$, $v_2=8$.

Note by the Editors. This problem is similar to problem 3051 [1924, 49] proposed by Norman Anning. A solution [1925, 386] was given by the proposer which, after a slight modification, shows that

$$\frac{\pi}{12} = \operatorname{arccot}\frac{u_2}{u_1} + \sum_{r=1}^{\infty} \operatorname{arccot}\left(\frac{4u_{r+1}^2}{D}\right),$$

where $u_{r+2}=4u_{r+1}-u_r$ and $D=u_2^2-u_1u_3$. This result assumes different forms according to the values assigned to u_1 and u_2. If $u_1=2$, $u_2=8$, then $D=4$ and we have the result given by Anning, which is also the second result given above.

If $u_1=1$, $u_2=1$, $u_3=3$, then $D=-2$ and we obtain the result in this problem after a slight modification.

This method may be modified by using other suitable difference equations. Thus the equation $x^2-2x-1=0$ has the root $1+\sqrt{2}=\cot(\pi/8)$, and this root is equal to Limit u_{r+1}/u_r $(r\to\infty)$, where $u_{r+2}=2u_{r+1}+u_r$. Hence by a similar analysis

$$\frac{\pi}{8}=\operatorname{arccot}\frac{u_2}{u_1}+\sum_{r=1}^{\infty}(-1)^{r+1}\operatorname{arccot}\left[\frac{2u_{r+1}(u_{r+1}+u_r)}{D}\right].$$

Thus, if $u_1=1$, $u_2=1$, then $D=-2$ and we find that

$$\frac{\pi}{8}=\sum_{r=1}^{\infty}(-1)^{r+1}\operatorname{arccot}\left[u_{r+1}(u_{r+1}+u_r)\right].$$

3331 [1928, 321]. *Proposed by Otto Dunkel, Washington University.*

In a spherical triangle ABC such that $AC<BC$ two points A', B', are taken on the side AB so that $\angle A'CA=\angle BCB'\leqq C/2$. Prove that AA' is greater than, equal to, or less than BB' according as $AC+CB$ is greater than, equal to, or less than 180°.

Solution by the Proposer.

Consider first the case $AC+CB<180°$. Since $BC>AC$ it follows that $AC<90°$ and $A>B$. Extend the arcs CA and CB until they meet again at C', and let L and N be the mid-points of CAC' and CBC', respectively. Then the arc LN is less than 180° since $C<180°$. Draw the arc CM_1C' bisecting LN at M_1 and cutting AB at M, and then produce AM_1 to meet CBC' in B_1. The two right triangles AM_1L and B_1M_1N are congruent, and hence $AC+CB_1=180°$, and also $AM_1=M_1B_1<90°$. Thus B lies within the segment B_1C. Since the triangle ACB lies within ACB_1 we have $A+B<180°$; thus $B<A$, $B<180°-A$, $B<90°$.

Now let CA' and CB' cut AB_1 in A'' and B'', then $CA'+CB'<CA''+CB''=180°$. Lay off on CB the arc $CA_1=CA$, and draw the arc A_1M cutting CB' in A_1'. Then A_1 falls within CN, $CA_1'=CA'<90°$, and $A_1A'_1=A'A$. Since $CA_1'+CB'=CA'+CB'<180°$, the mid-point of $A_1'B'$ lies between the arc LN and the point C, and hence B' is nearer LN than A_1'. If then $B'N_b$ and $A_1'N_a$ are the perpendiculars to CNC', $N_aA_1'<N_bB'$, since $\angle BCB'<90°$. The point N_a lies within NC as well as A_1, and hence $N_aA_1<90°$, also $N_aA_1'<NM_1<90°$; and it follows that $A_1A_1'<90°$. Now if $BB'\geqq 90°$, then $BB'>A_1A_1'=A'A$ and the theorem is true. Hence we need to consider only the case $BB'<90°$. In the right triangle N_bBB', $BB'<90°$, and $\angle N_bBB'=B<90°$; hence N_bB' and N_bB are each less than 90°. Lay off on the prolongation of BN_b the arc $N_bK=N_aA_1$ (in absolute value) and on N_bB', the arc $N_bH=N_aA_1'$. Draw the arcs KH and KB'. Then $\angle N_bKH=\angle N_aA_1A_1'$. If $A\leqq 90°$, $\angle N_aA_1A_1'=A$; if $A>90°$, $\angle N_aA_1A_1'=180°-A$. But in either case it follows that $B<\angle N_bKH$. Since $N_bH<N_bB'$ and these two arcs as well as N_bK are each less than 90°,

$B < \angle N_bKH < \angle N_bKB'$. Since $BK < 180°$, it now follows that $BB' > KB' > KH$, or $BB' > A'A$, and the proof is complete.

If $AC+CB=180°$, then $A' \equiv A''$, $B' \equiv B''$, $B \equiv B_1$, and $BB' = A'A$. If $AC+CB > 180°$ and $BC > AC$, then $AC'+C'B < 180°$ and $AC' > BC'$. Hence from the above proof $AA' > B'B$.

Note. In spherical geometry the condition $AC+CB < 180°$ insures the fact that the external angle at A of a triangle ABC is greater than the internal angle at B. In the geometries for which the sum of the angles of a triangle is less than or equal to 180° this inequality of angles is always true and the proof that $BB' > A'A$ is simpler. We draw as before CM bisecting the angle BCA and take on CB the length $CA_1 = CA$. Then draw A_1M cutting CB' in A_1'. The corresponding parts of the triangles CA_1A_1' and CAA' are equal, and $CB' > CA_1'$. Then if $B'N_b$ and $A_1'N_a$ are the perpendiculars to CB, $N_aA_1' < N_bB'$. Also $B < A$ and $B < 180° - A$, and the proof follows as before.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Dr. William E. Wickenden has assumed his work as president of the Case School of Applied Science. Fitting ceremonies commemorating the founding of the School in 1880 will be held sometime during the year, possibly in conjunction with the installation of Dr. Wickenden as third president of the School within the fifty year period.

Professor W. L. Crum, of Stanford University, has been appointed professor of economics at Harvard University.

Dr. H. B. Curry has been appointed assistant professor of mathematics at the Pennsylvania State College.

Assistant Professor C. C. MacDuffee, of the Ohio State University, has been promoted to an associate professorship.

Mr. E. D. McCarthy, of the Pennsylvania State College, has accepted a position at the University of Detroit, and Miss Frances Monteith of the same college has gone into statistical work.

Assistant Professor Florence M. Mears, of the Pennsylvania State College, has been appointed assistant professor at George Washington University.

Mr. Albert E. Meder, Jr. has been promoted to an assistant professorship at the New Jersey College for Women.

Assistant Professor Gaylord M. Merriman, of Grinnell College, has been appointed assistant professor of mathematics at the University of Cincinnati.

Dr. John A. Miller, of Swarthmore College, is retiring from teaching and administrative work to become research professor of astronomy. He will continue in charge of the Sproul Observatory.

Dr. Hillel Poritsky has accepted a position with the General Electric Company.

Professor O. H. Rechard, head of the Department of Mathematics in the University of Wyoming, is on leave of absence in the University of Wisconsin. Assistant Professor C. F. Barr is acting head of the Department.

Professor J. Shibli and Mr. W. O. Rogers, of the Pennsylvania State College are on leave of absence; Professor Shibli is studying at Teachers College, Columbia University, and Mr. Rogers, at the University of Chicago.

Professor Charles C. Wagner and Mr. O. J. Farrell, of the Pennsylvania State College remain on leave of absence; Professor Wagner will study at the University of Michigan, and Mr. Farrell at Harvard University.

Professor Norbert Weiner, of the Massachusetts Institute of Technology, is visiting professor of physics at Brown University.

Dr. Harvey A. Zinszer who for the past two years has been professor of physics and acting professor of mathematics at Hanover College has been appointed professor of physics and astronomy at the Kansas State Teachers' College.

The following appointments to instructorships are announced:

New Jersey College, Mr. Robert M. Walter.
Rutgers University, Mr. Hubert B. Huntly.
Pennsylvania State College, Mr. C. H. Graves, Mr. William Mann, Miss Gladys Quigg, Dr. Leo Zippin.
South Dakota State School of Mines, Mr. Harry Pool.
University of Wyoming, Mr. Richard W. Warner.

The Fourth Carus Mathematical Monograph

The Carus Monograph Committee is pleased to announce that the fourth number is now in process of publication and will be ready for distribution by the time of the annual meetings in Des Moines. The title of this Monograph is "Projective Geometry" by Professor JOHN W. YOUNG of Dartmouth College, now President of the Association. The preceding numbers are: (1) "Calculus of Variations" by Professor GILBERT A. BLISS; (2) "Analytic Functions of a Complex Variable" by Professor DAVID R. CURTISS, (3) "Mathematics of Statistics" by Professor HENRY L. RIETZ.

The price of these Monographs is $1.25 to institutional and individual members of the Association when ordered directly through the Secretary, one copy to each member; this is the bare cost of production. The price to all non-members of the Association and for all quantity orders for class use is $2.00 per copy, obtained only through the Open Court Publishing Company, 339 East Chicago Avenue, Chicago, Illinois, distributors to the general public of Association publications.

As heretofore, for the convenience of members, the forthcoming Monograph will be charged along with the bill for annual dues late in December. (This item may be cancelled in case it is not wanted.) New members and those who have neglected to subscribe for the previous numbers may still do so by ordering directly from the Secretary. As the series goes on, the complete list of Monographs will become more valuable if not indispensable to the individual library of an increasing number of members as well as to most college and all university libraries. It is gratifying to announce that the sales of the preceding numbers are continuing very favorably and that two of them have already gone to second editions. It would be still more gratifying if a larger proportion of members (now somewhat more than fifty per cent) should become regular subscribers to this Monograph series. Failure to do so is, doubtless, in many cases due to oversight or procrastination. Now is a good time to remedy such a condition.

Attention is called to the enlargement of the membership of the Monograph Committee by the addition of Professors AUBREY J. KEMPNER of the University of Colorado, and JOHN W. YOUNG, of Dartmouth College. The other members of the Committee are: Professor GILBERT AMES BLISS, of the University of Chicago; Professor DAVID RAYMOND CURTISS, of Northwestern University; and Professor HERBERT ELLSWORTH SLAUGHT, of the University of Chicago.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Thirteenth Summer Meeting of the Association, Boulder, Colorado, August 26-27, 1929.
Fourteenth Annual Meeting, Des Moines, Iowa, December 31, 1929, January 1, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1929.

ILLINOIS, Carthage, Ill., May 3-4.
INDIANA, Culver Military Academy, May 3-4.
IOWA, Fairfield, Iowa, April 26-27.
KANSAS, Topeka, Kansas, February 2.
KENTUCKY, Lexington, Ky., April 13.
LOUISIANA-MISSISSIPPI, Lafayette, La., April 12-13.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, May 4.
MICHIGAN, Ann Arbor, Mich., March 16.
MINNESOTA, St. Paul, Minn., May 11.
MISSOURI, Kansas City, Mo., November 16
NEBRASKA.
OHIO, Columbus, Ohio, April 4.
PHILADELPHIA, University of Pennsylvania, November 30.
ROCKY MOUNTAIN, Greeley, Colo., April 12-13.
SOUTHEASTERN, Macon, Ga., April 19-20.
SOUTHERN CALIFORNIA, University of Redlands, March 9.
TEXAS, Houston, Texas, Jan. 26.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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THE AMERICAN MATHEMATICAL MONTHLY

By H. S. EVERETT, The University of Chicago.

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ADDENDA AND CORRIGENDA

Volume XXXIV, 1927:

P. 446, 10th and 11th lines, replace "Twelfth Summer Meeting" by "Eleventh Summer Meeting."

Volume XXXV, 1928:

P. 451, 1st and 2nd lines, replace "Thirteenth Summer Meeting" by Twelfth Summer Meeting."
P. 560, 6th line, replace x_k by x_4.
P. 574, 5th line, replace "Bailieff" by "Ballieff."
P. 575, 7th line under Notes and News, for "508" read "503."
P. 577, insert after "Dunkel, O.," "Dunlap, L. T., 569."
P. 577, under "Fort" for "564" read "554."
P. 580, under "Poritsky" insert "568."
P. 582, in "Wheeler, A. H., " replace "n" by "h."

Volume XXXVI, 1929:

P. 84, 15th line from bottom, or "convergant" read "convergent."
P. 120, under LEPESHKIN for "L. A." read "S. A."
P. 175, 11th line from bottom, for "Miler" read "Miles."
P. 240, 3rd line, for "Kbhn" read "Kuhn."
P. 256, 19th line, for "s" read "is.".
P. 317, 4th line for "days'" read "day's."
P. 421, 18th line, for "form" read "from."
P. 422, 10th line from bottom, for "to" read "of."
P. 423, first line, for "to" read "of;" 8th line, for "to" read "for."
P. 436, second line, for "dring" read "during."
P. 453, 17th line, invert "te" in compteition;" 22nd line, for "Union" read "Electric."
P. 454, 20th line, for C. K. read C. R.
P. 490, 5th line, for "Wiley" read "Wylie."

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXVI, 1929

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DIFFERENTIAL EQUATIONS AS A FOUNDATION FOR ELECTRICAL CIRCUIT THEORY

By THORNTON C. FRY, Bell Telephone Laboratories

1. *Introductory Remarks.* The subject of electrical circuit theory is of fundamental importance, not only in the field of communication, but in power engineering as well. From the physical standpoint it is naturally a study in real quantities, since the physical forces and currents are themselves real, and the mathematical methods originally used in connection with it dealt almost exclusively with real variables. During the present century, however, a progressive transition has taken place, until now the use of complex quantities is by all odds the usual thing, and real numbers are rather rare. This transition has been brought about, not through the influence of mathematicians who have taught the engineers the advantage of complex quantities, but rather through the initiative of a few engineers who have preached the idea to their own kind. As a natural consequence the duty of explaining why *imaginary* numbers are so peculiarly appropriate in a problem in which the physical quantities are so obviously *real* has fallen to the professor of electrical engineering, though the reasons are essentially mathematical in character, and can be taught much more concisely and effectively in connection with a study of elementary differential equations. It is my purpose to emphasize this fact by outlining as briefly as possible the nature of the problem, and the ideas upon which its solution is based.

Perhaps the matter would be of little consequence if the results were satisfactory, but this is not the case. For, simple as we shall find the ideas to be when presented in a suitable setting, they are really so foreign to the subject of electricity itself that it is not easy to explain them effectively in a technical course. Hence, even if the instructor has the proper perspective, which is not always the case, he usually meets his immediate needs by some sort of vectorial analogy, and the student is left with a very hazy idea as to what it is all about. The accuracy of this statement will be readily attested by those mathematicians who have occasion to meet representative groups of such students after graduation.

2. *The Differential Equations of Circuit Theory.* The differential equations of elementary circuit theory—and the same is true of the simpler dynamical systems in general—are characterized by five principal characteristics:

1. They are linear.

Of course, any dynamical system can be overloaded to the point where it becomes non-linear. Very frequently, however, its practical usefulness ceases when non-linearity sets in, and because of this fact we are justified in thinking of it as a linear system. There is also another class of dynamical systems, of which vacuum-tube devices are excellent electrical illustrations, many of whose uses depend upon the fact that they are essentially non-linear in character. Such systems, however, require special methods of treatment, and are not usually regarded as part of the subject-matter of "elementary" circuit theory. More-

over, the physical peculiarities of the systems play such a fundamental part in their discussion that the mathematician is likely to treat them less effectively than the engineer.

2. The coefficients are constant.

This also is only true within certain limits; but when it is not, the problem is no longer "elementary" in the sense in which we are using the term.

3. There are usually several dependent variables, so that the problems formulate themselves as *systems* of differential equations, rather than single equations.

It is often true that all but one of these variables are without any technical interest. In such cases, of course, the problem could be expressed either as a single equation of high order or as a system of low order. But even when this is true, the nature of the boundary conditions is usually such that the use of the system is to be preferred.

4. Very frequently—so frequently in fact as to merit almost exclusive attention in an elementary course—the boundary conditions are given by one or the other of two physical statements: either the system is "at rest" at a certain instant, or else it is in a "steady state."

5. The driving force which actuates the system is usually of the simple harmonic (a.c.) type $E_0 \cos(pt-\epsilon)$, constant (d.c.) forces being included as the special case $p=0$.

Superficially, there are two reasons why this type of driving force should be so nearly universal. The first is the dynamical fact that rotating generators and oscillating circuits produce electromotive forces of substantially this type. The second is the fact that it is the basic function out of which Fourier series and Fourier integrals are built.

The first of these reasons is a fundamental one. It is inherent in the nature of the systems with which we deal. The second, however, is only superficially different, since the superiority of Fourier expansion to other forms, Hermite polynomials for instance, resides primarily in the fact that simple harmonic oscillations are characteristic of linear dynamical systems.

Reduced to mathematical terms, then, the problem of elementary circuit theory is that of solving the system of differential equations:

$$
\begin{array}{ccccccccc}
F_{11}(p)y_1 & + & F_{12}(p)y_2 & + & \cdots & + & F_{1r}(p)y_r & = & f_1(t), \\
F_{21}(p)y_1 & + & F_{22}(p)y_2 & + & \cdots & + & F_{2r}(p)y_r & = & f_2(t), \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
F_{r1}(p)y_1 & + & F_{r2}(p)y_2 & + & \cdots & + & F_{rr}(p)y_r & = & f_r(t),
\end{array}
\tag{1}
$$

in which the F's are linear differential operators with constant coefficients, the f's are either simple harmonic functions or sums of such functions, and the boundary conditions require that the system shall be either at rest or else in a steady state at the time zero.

I need scarcely remark that the study of such systems is a recognized part of first courses in differential equations. The suggestion that the foundation for circuit theory should be laid in such courses is therefore not a very revolutionary one: it requires at most only a slight change of emphasis to accomplish it. Perhaps this will become even clearer as we mention the four facts upon which the use of complex quantities rests, none of which is unusual or non-mathematical in character.

3. *The Principle of Superposition and the Principle of Decomposition.* Two of these four facts are properties of linear differential equations. They are generally known in applied mathematics as the principle of superposition and the principle of decomposition. In order to simplify their wording it is desirable to call any set of y's which satisfies (1) a solution "due to" the functions $f_j(t)$, the phrase being suggested by the fact that, in the electrical applications, the y's are currents and the f's are electromotive forces. If we adopt this form of speech, the principle of superposition becomes:

The sum of any solution of (1) *due to the set* f_j *and any solution due to another set* g_j *is a solution due to* f_j+g_j.

Of course, if this theorem is true for two sets of functions, it is true for any finite number.

The principle of superposition is of importance because it enables us to simplify our problem in three ways:

In the first place, we can consider the r sets of functions,

$$\begin{array}{ccccc} f_1, & 0, & 0, & \ldots, & 0, \\ 0, & f_2, & 0, & \ldots, & 0, \\ 0, & 0, & f_3, & \ldots, & 0, \\ \ldots & \ldots & \ldots & \ldots & \ldots, \\ 0, & 0, & 0, & \ldots, & f_r, \end{array}$$

separately, and by adding together the results thus obtained find a solution due to the set

$$f_1, f_2, f_3, \ldots, f_r.$$

In other words, in the development of the theory we may assume that all of the f's except one are zero; and since the order in which the equations are written is immaterial, we may always suppose that the one which does not vanish is the first.

In the second place, if the one driving force which remains is the sum of a number of simple harmonic terms (as it will be if it can be represented by a Fourier series or integral) we may consider each of these terms separately and add the results together. In other words, in the development of our theory we may assume that the one f which does not vanish is a single trignometric term.

The third simplification is, in reality, a sort of inverse use of the theorem. Of course, there is no *exact* converse of the principle of superposition, for even if we know a solution of (1) due to f_j+g_j, we cannot ordinarily separate the part which is due to f_j from that which is due to g_j. But there is one exceptional case

in which the component parts can be recognized. *It occurs when the functions* f_j *are real while the functions* g_j *are pure imaginary; for then the real part of a solution due to* f_j+g_j *is a solution due to* f_j *and the imaginary part is a solution due to* g_j.

This is the principle of decomposition.

4. *Two Peculiarities of Exponential Functions.* The other two facts upon which the use of complex quantities in electrical network theory is based are even more elementary. They are:

The derivatives of an exponential function are proportional to the function itself.

I need hardly remark that it is this property which enables us to reduce the system of *differential* equations (1) to a system of *algebraic* equations, and thus determine, in the traditional way, not only the complementary part of the solution due to any sort of function[1] $f(t)$, but also a particular solution as well when $f(t)$ happens to be exponential.

The other important property of the exponential function is Euler's equation, $e^{ix}=\cos x+i\sin x$.

5. *Complex Numbers in Electrical Circuit Theory.* With these four properties before us, the reason for using complex quantities is readily seen. For if it is true that the system (1) can be easily solved when $f(t)$ is an exponential, and if it is true that when $f(t)$ is complex we can separate out the solution due to its real part from the solution due to its imaginary part, it follows at once that we can easily obtain the solution due to a simple harmonic function $\cos(pt-\epsilon)$ by first finding the solution due to a complex exponential $e^{i(pt-\epsilon)}$, and then discarding its imaginary part.

This is the entire argument in favor of the use of complex numbers in electrical circuit theory. It is hard to imagine any explanation based upon vectorial ideas which would approach it either in conciseness, simplicity, or generality. Indeed, there is none *provided the matter is brought up during a discussion of differential equations.*

But suppose it is not. Suppose instead that it arises in a technical course and that the class either has not been exposed to the argument at all, or else that the form of presentation has not been so molded as to conform in some degree to their technical needs. It is obvious that the engineering instructor must then either present a connected account of part of the theory of linear differential equations, or else he must find some brief but plausible explanation of what he proposes to do. He may *suspect* that if he chooses the latter alternative his students may some day have to unlearn some crude ideas; but he *knows* that the other will sacrifice time which he needs for the presentation of technical ideas. If we remember where his interest lies, there is little doubt which alternative he will choose.

6. *The Steady-state Solution; Impedance; Natural Frequencies; Damping.* I

[1] Since we may now assume that all the f_j's except one are zero, we need no subscript to distinguish this one which remains.

think it must now be evident why I believe that this subject matter should be presented in the mathematical instead of the engineering classroom, and why I believe the subject of differential equations should be required of those students who are likely to be interested in circuit theory. So far, however, I have spoken only of its desirability from the standpoint of the engineer himself. I believe, however, that with comparatively little effort the subject may be made to yield a return to the mathematician also in increasing his fund of illustrative material, and in deepening the interest of his class.

I need hardly mention the fact that the imaginary parts of the roots of the auxiliary equation give us the "natural frequencies" of the system, and the real parts the "damping." Few instructors would fail to mention such obvious facts, at least when talking to a class with practical interests. But once we have introduced the electrical circuit as a sort of collateral objective, a number of ideas suggest themselves which might otherwise not be thought of. For instance, since the particular solution (which we obtain by algebraic means when we deal with exponential or sinusoidal forces) is independent of the boundary conditions, we at once suspect that it must correspond to the steady-state condition of the system. It is a simple matter to verify this assumption. Conversely, the solution of the complementary equation can be obtained once for all without any knowledge of the nature of the driving force, except in so far as that knowledge enters into the evaluation of the constants of integration. We would therefore suspect it to be the "transient" reaction of the circuit, and again it is a simple matter to prove that this is true.

We may also observe that the particular (or steady-state) solution is obtained by merely writing down a few determinants, whereas in order to find the complementary (transient) part it is necessary to locate the roots of an algebraic equation. As the first of these processes is very much simpler than the second, we conclude that it will in general be immensely simpler to find those properties of a system which are defined in terms of its steady-state reaction, than to find its transient reaction to impulses and the like. Chief among the ideas associated with the steady state is that of impedance, which is defined as the ratio of the driving force to the steady-state current which it produces. It follows at once that the impedance of the system is just the ratio of the two determinants by means of which our particular solution is derived. In this connection we may observe that in the days when electrical theory was a science of real numbers, it was the ratio of the *magnitudes* of these two quantities which was called impedance, but today the term is applied about equally often to this real quantity, and to the complex ratio itself.

Also, if we separate the driving force into two components, one in phase with the current and the other 90° out of phase, the ratios obtained when we divide these two components by the current are called the "resistance" and "reactance" of the circuit, respectively. It requires only a moment's inspection to show that these definitions merely state that the resistance of a circuit is the real part of its impedance, and the reactance the imaginary part.

How much of this sort of illustration should be used depends, of course, upon the interests of the instructor and the needs of his class. In some cases it might be unwise to use even as much as I have mentioned; in others it might be desirable to speak of such matters as Rayleigh's reciprocity theorem and Heaviside's expansion. Very often the presentation of Cauchy's solution, at least in the simplified form which corresponds to the case of a system originally at rest, will be of interest, not only because of its technical importance but also because it is a beautiful piece of mathematics in itself.[1]

I hold no brief as to just which of these things shall be done, or indeed as to how they shall be done, my only point being that in my opinion both the mathematician and the engineer will benefit from the doing of them: the engineer because of the more suitable foundation which will be laid for his technical work, and the mathematician through the fact that his subject will have a vitality and interest which is too often lacking when the study of the linear system is merely an exercise in p's, D's and θ's.

SOME GEOMETRICAL APPLICATIONS OF COMPLEX NUMBERS

By LLOYD L. SMAIL, Lehigh University

Many teachers of algebra have doubtless wished for some applications of complex numbers outside the field of algebra which would indicate their usefulness in a way to appeal to students. Some textbooks of college algebra do, indeed, remark that complex numbers find important applications in certain parts of physics and electrical engineering, but such application is too remote to appeal greatly to the student at this stage.[2] In the search for material which would illustrate the usefulness of complex numbers for use in his own classes, the writer developed some simple geometric applications of these numbers, a few of which are presented in this paper. These applications are simple enough to be used with an algebra class in connection with the study of complex numbers. The methods used are rather obvious adaptations of familiar applications of the methods of elementary analytic geometry and vector analysis to the proof of simple geometric propositions.

In all our diagrams we shall write in braces adjacent to the capital letter denoting each point the corresponding complex number which is represented by the point.

We shall use the following two formulas, which are easily proved:

The distance between two points $P_1\{z_1\}$ *and* $P_2\{z_2\}$ *is given by*

$$P_1P_2 = |z_1 - z_2| . \tag{1}$$

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[1] See F. D. Murnaghan, *The Cauchy-Heaviside expansion formula and the Boltzmann-Hopkinson principle of superposition*, Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 81–89.

[2] Similar remarks were made by G. A. Bingley in a note on *The complex variable in the solution of problems in elementary analytic geometry* in this Monthly, vol. 33 (1926), p. 418.

The point of division $P\{z\}$ *of a line-segment joining points* $P_1\{z_1\}$ *and* $P_2\{z_2\}$ *which divides the segment in the ratio* $P_1P/PP_2=\lambda$ *is given by*

$$z = (z_1 + \lambda z_2)/(1 + \lambda). \tag{2}$$

For the special cases of the mid-point, where $\lambda=1$, *and the trisection-point, where* $\lambda=2$, *we have:*

$$z = \tfrac{1}{2}(z_1 + z_2) \tag{2'}$$

and

$$z = \tfrac{1}{3}(z_1 + 2z_2)\ , \tag{2''}$$

respectively.

The geometric applications of complex numbers will now be illustrated by proving the following propositions.

Theorem 1: *The mid-point of the hypotenuse of a right triangle is equidistant from the three vertices.*

Proof: Place the triangle as shown in Fig. 1; let the numbers corresponding to the points A and B be a and bi; then the number for the point C is found by use of formula (2′) to be $\frac{1}{2}(a+bi)$. By formula (1),

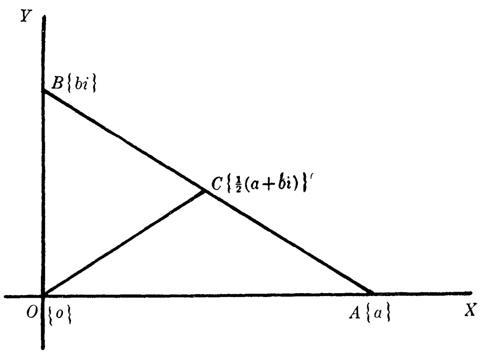

Fig. 1.

$$OC = \left|\tfrac{1}{2}(a + bi) - 0\right| = \tfrac{1}{2}\left|a + bi\right| = \tfrac{1}{2}(a^2 + b^2)^{1/2},$$

$$AB = \left|a - bi\right| = (a^2 + b^2)^{1/2}.$$

Hence,

$$OC = \tfrac{1}{2}AB = BC = CA.$$

For the next two propositions, take *any* triangle, place it as in Figs. 2 and 3, and let the numbers corresponding to the vertices A and B be a and $b+b'i$.

Theorem 2: *The line-segment joining the mid-points of two sides of a triangle is equal to half the third side and is parallel to it.*

Proof: The numbers for points C and D are written down on the figure (Fig. 2) by use of formula (2′). By formula (1),

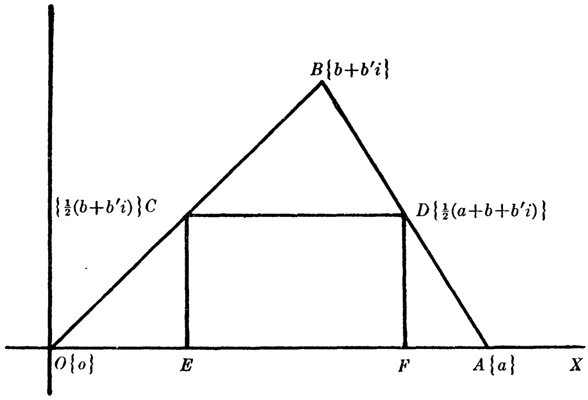

FIG. 2.

$$CD = \left| \tfrac{1}{2}(b + b'i) - \tfrac{1}{2}(a + b + b'i) \right| = \tfrac{1}{2}a = \tfrac{1}{2}OA.$$

If CE and DF are perpendicular to OA, then CE is equal to the coefficient of i in the number corresponding to the point C, and similarly for DF; hence $CE = \frac{1}{2}b' = DF$, so that CD is parallel to OA.

Theorem 3: *The medians of a triangle meet in a point which is two-thirds of the distance from each vertex to the opposite side.*

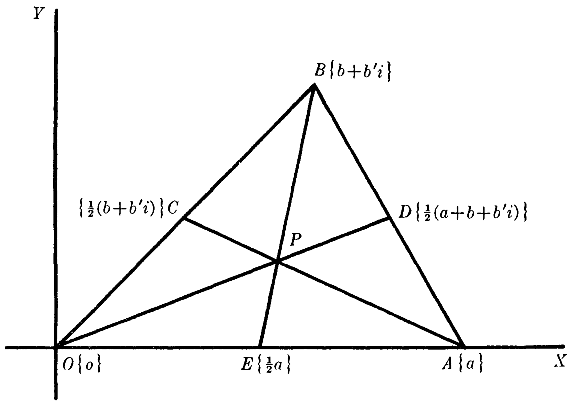

FIG. 3.

Proof: Let $P_1\{z_1\}$ be the point of trisection, two-thirds of the distance from O to D, in Fig. 3, and let $P_2\{z_2\}$ and $P_3\{z_3\}$ be the corresponding points on AC and BE. By formula (2″), we find

$$z_1 = \tfrac{1}{3}[0 + 2\cdot\tfrac{1}{2}(a + b + b'i)] = \tfrac{1}{3}(a + b + b'i),$$
$$z_2 = \tfrac{1}{3}[(b + b'i) + 2\cdot\tfrac{1}{2}a] = \tfrac{1}{3}(a + b + b'i),$$
$$z_3 = \tfrac{1}{3}[a + 2\cdot\tfrac{1}{2}(b + b'i)] = \tfrac{1}{3}(a + b + b'i).$$

Since $z_1 = z_2 = z_3$, the points P_1, P_2 and P_3 coincide, and the theorem is proved.

For the next four propositions, take *any* quadrilateral, place it as in Figs. 4–6, and let the numbers corresponding to the vertices A, B, C be a, $b+b'i$ and $c+c'i$. Numbers corresponding to mid-points in these figures are found by formula (2′).

Theorem 4: *The line-segments joining the mid-points of opposite sides of any quadrilateral bisect each other.*

Proof: Let $P_1\{z_1\}$ and $P_2\{z_2\}$ be the mid-points of DF and EG, respectively (Fig. 4); then

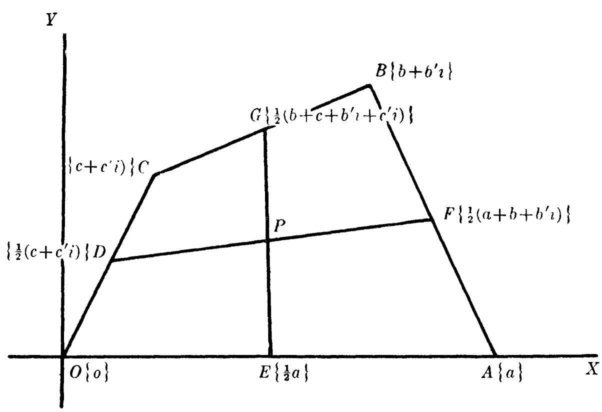

FIG. 4.

$$z_1 = \tfrac{1}{2}[\tfrac{1}{2}(c + c'i) + \tfrac{1}{2}(a + b + b'i)] = \tfrac{1}{4}(a + b + c + b'i + c'i),$$
$$z_2 = \tfrac{1}{2}[\tfrac{1}{2}(a) + \tfrac{1}{2}(b + c + b'i + c'i)] = \tfrac{1}{4}(a + b + c + b'i + c'i).$$

Since $z_1 = z_2$, the points P_1 and P_2 coincide, and the theorem is proved.

Theorem 5: *The mid-point of the line-segment joining the mid-points of the diagonals of a quadrilateral coincides with the point of intersection of the line-segments joining the mid-points of opposite sides of the quadrilateral.*

Proof: Let $R\{z_3\}$ be the mid-point of MN (Fig. 5), which joins the mid-points of the diagonals; then

$$z_3 = \tfrac{1}{2}[\tfrac{1}{2}(b + b'i) + \tfrac{1}{2}(a + c + c'i)] = \tfrac{1}{4}(a + b + c + b'i + c'i).$$

Since z_3 agrees with the values of z_1 and z_2 of the preceding proposition, the conclusion of the theorem follows.

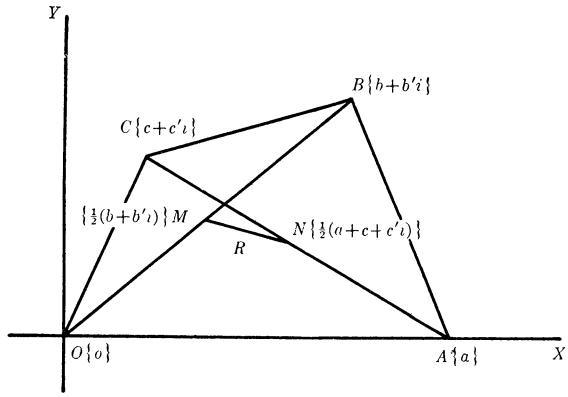

Fig. 5.

Theorem 6: *The lines joining mid-points of adjacent sides of a quadrilateral form a parallelogram.*

Proof: From Fig. 6, by formula (1),

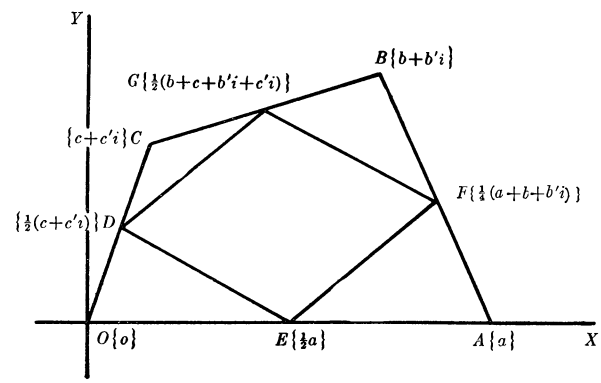

Fig. 6.

$$
\begin{aligned}
DE &= \left| \tfrac{1}{2}(c + c'i) - \tfrac{1}{2}a \right| = \tfrac{1}{2}\left| c - a + c'i \right| , \\
GF &= \left| \tfrac{1}{2}(b + c + b'i + c'i) - \tfrac{1}{2}(a + b + b'i) \right| = \tfrac{1}{2}\left| c - a + c'i \right| ; \\
EF &= \left| \tfrac{1}{2}(a + b + b'i) - \tfrac{1}{2}a \right| = \tfrac{1}{2}\left| b + b'i \right| , \\
DG &= \left| \tfrac{1}{2}(b + c + b'i + c'i) - \tfrac{1}{2}(c + c'i) \right| = \tfrac{1}{2}\left| b + b'i \right| .
\end{aligned}
$$

Hence, $DE = GF$ and $EF = DG$, and $DEFG$ is a parallelogram.

Theorem 7: *The sum of the squares of the sides of a quadrilateral is equal to the sum of the squares of its diagonals plus four times the square of the segment joining the mid-points of the diagonals.*

Proof: From Fig. 5, by formula (1), we have

$$\overline{OA}^2 + \overline{AB}^2 + \overline{BC}^2 + \overline{CO}^2$$

$$\begin{aligned} &= |0-a|^2 + |a-(b+b'i)|^2 + |(b+b'i)-(c+c'i)|^2 + |(c+c'i)-0|^2 \\ &= [a^2] + [(a-b)^2 + b'^2] + [(b-c)^2 + (b'-c')^2] + [c^2 + c'^2] \\ &= 2a^2 + 2b^2 + 2c^2 + 2b'^2 + 2c'^2 - 2ab - 2bc - 2b'c'. \end{aligned} \tag{3}$$

Similarly,

$$\overline{OB}^2 + \overline{AC}^2 + 4\overline{MN}^2$$

$$\begin{aligned} &= |0-(b+b'i)|^2 + |(c+c'i)-a|^2 + 4\left|\tfrac{1}{2}(b+b'i) - \tfrac{1}{2}(a+c+c'i)\right|^2 \\ &= [b^2+b'^2] + [(c-a)^2+c'^2] + 4[\tfrac{1}{4}(b-a-c)^2 + \tfrac{1}{4}(b'-c')^2] \\ &= 2a^2 + 2b^2 + 2c^2 + 2b'^2 + 2c'^2 - 2ab - 2bc - 2b'c'. \end{aligned} \tag{4}$$

Comparing (3) and (4) gives the theorem.

Theorem 8: *If in any triangle OAB, a line OC be drawn to the mid-point of AB and extended to any point as D, and the sides of the triangle be extended to meet AD and BD at E and F respectively, then EF will be parallel to AB.*

Proof: Place the triangle as in Fig. 7. Let

$$OF/OA = m, \quad OE/OB = n, \quad OD/OC = k, \quad AE/ED = \lambda, \quad BF/FD = \mu.$$

The numbers corresponding to points F, E, and D will then be ma, $n(b+b'i)$

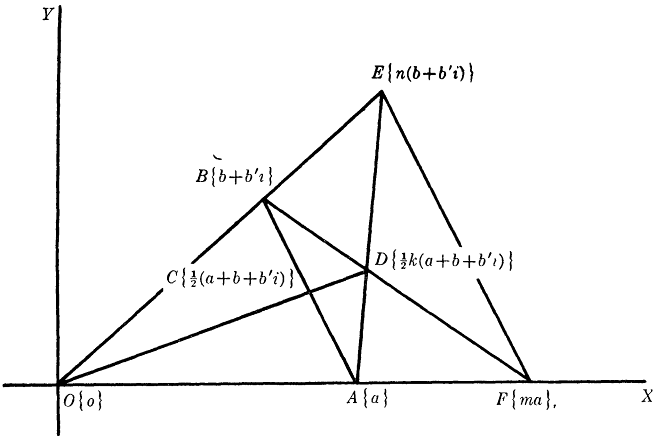

FIG. 7.

and $\frac{1}{2}k(a+b+b'i)$, as shown in Fig. 7. By use of formula (2), we find that E and F represent the complex numbers

$$\frac{a + \lambda\cdot\frac{1}{2}k(a+b+b'i)}{1+\lambda} \quad \text{and} \quad \frac{(b+b'i) + \mu\cdot\frac{1}{2}k(a+b+b'i)}{1+\mu},$$

respectively. In the resulting equations

$$\frac{a + \lambda \cdot \frac{1}{2}k(a + b + b'i)}{1 + \lambda} = n(b + b'i), \tag{5}$$

$$\frac{(b + b'i) + \mu \cdot \frac{1}{2}k(a + b + b'i)}{1 + \mu} = ma, \tag{6}$$

equate the real and imaginary parts separately, and we have

$$\frac{a + \lambda \cdot \frac{1}{2}k(a + b)}{1 + \lambda} = nb, \qquad \frac{\lambda \cdot \frac{1}{2}kb'}{1 + \lambda} = nb', \tag{7}$$

$$\frac{b + \mu \cdot \frac{1}{2}k(a + b)}{1 + \mu} = ma, \qquad \frac{b' + \mu \cdot \frac{1}{2}kb'}{1 + \mu} = 0. \tag{8}$$

Eliminating λ from equations (7) and solving for n, and similarly eliminating μ from equations (8) and solving for m, we find

$$n = k/(2 - k) \text{ and } m = k/(2 - k).$$

Hence, $m = n$ or $OF/OA = OE/OB$, so that EF is parallel to AB.

Theorem 9: *If OABC is any parallelogram, and DE is any line-segment parallel to OA, and if lines OD, AE, CD and BE are drawn, meeting at F and G respectively, then FG is parallel to OC.*

Proof: Place the parallelogram as in Fig. 8, and affix complex numbers to the figure as indicated. Let

$$OF/FD = m, \qquad CG/GD = n, \qquad AF/FE = \lambda, \qquad BG/GE = \mu.$$

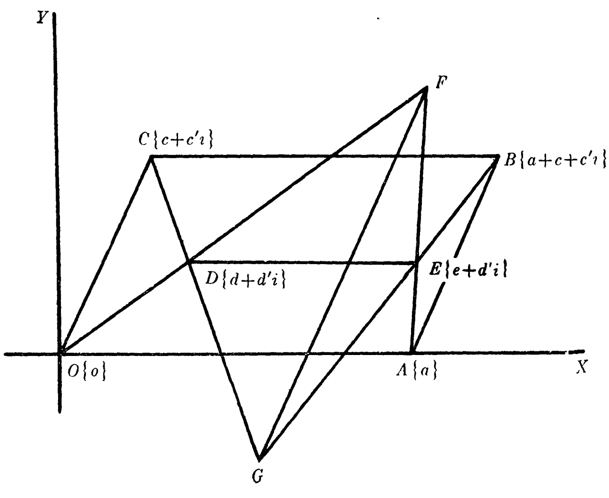

FIG. 8.

For the point F, lying on OF and AF, and for the point G, lying on CG and BG, by use of formula (2), we have

$$\frac{0 + m(d + d'i)}{1 + m} = \frac{a + \lambda(e + d'i)}{1 + \lambda}, \tag{9}$$

$$\frac{(c + c'i) + n(d + d'i)}{1 + n} = \frac{(a + c + c'i) + \mu(e + d'i)}{1 + \mu}. \tag{10}$$

Equating real and imaginary parts separately in these equations, we get

$$\frac{md}{1 + m} = \frac{a + \lambda e}{1 + \lambda}, \qquad \frac{md'}{1 + m} = \frac{\lambda d'}{1 + \lambda}, \tag{11}$$

$$\frac{c + nd}{1 + n} = \frac{a + c + \mu e}{1 + \mu}, \qquad \frac{c' + nd'}{1 + n} = \frac{c' + \mu d'}{1 + \mu}. \tag{12}$$

Eliminating λ from equations (11) and solving for m, and eliminating μ from equations (12) and solving for n, we find

$$m = a/(d - e) \quad \text{and} \quad n = a/(d - e).$$

Hence, $m = n$ or $OF/FD = CG/GD$, which shows that FG is parallel to OC.

Many more examples of this and other kinds might be considered, but enough has been presented to show the possibilities of this method of applying complex numbers to geometric proofs.

PLANE CONFIGURATIONS AND THEIR SPACE ANALOGUES IN THE CASE OF PINCH POINTS OF RATIONAL SURFACES

By A. R. WILLIAMS, University of California

Zeuthen in a well known memoir on reciprocal surfaces[1] studied extensively the properties of a surface at its various singularities. In the case of rational surfaces many of these can be inferred from the corresponding configurations in the plane; and it is interesting to trace this correspondence, even when the properties noted are true of surfaces in general. The following note illustrates this, particularly with respect to pinch points.[2]

The plane sections of a rational surface correspond point for point to ∞^3 plane curves ϕ. To a net of fundamental curves ϕ correspond a bundle of plane sections; *i.e.*, the ∞^2 planes through a point. If the net consists of the ∞^2 curves determined by a point of the plane, the center of the bundle is on the surface. To the ∞^1 curves of the net that have double points correspond the tangent planes of the tangent cone whose vertex is the center of the bundle. Two nets have a pencil in common. The number of curves of this pencil that have double points is the number of tangent planes to the surface that pass through an arbitrary line; *i.e.*, the class of the surface, or class of a general

[1] Mathematische Annalen, vol. 10 (1876), page 446.

[2] See Caporali in Collectanea Mathematica: *I Sistemi Lineari Triplimente Infiniti*, §§9 and 16 (Hoepli, Milan, 1881).

tangent cone. The locus of double points of the curves of a net is the Jacobian. The latter, therefore, is the image of the curve of contact of the corresponding tangent cone. To a general point P of the double curve correspond two points of the plane, P_1' and P_2', such that the net of fundamental curves that pass through one pass also through the other. To this net of curves corresponds the bundle of plane sections through P. A general section of the bundle has of course a node at P corresponding to the passages of the plane curve through P_1' and P_2'. To the pencil of plane curves tangent to the image of the double curve at P_1', and therefore at P_2', correspond the pencil of plane sections containing the tangent to the double curve at P. The section by a general plane of this pencil has therefore a tacnode at P, the two branches being tangent to the double curve. One plane curve osculates the image of the double curve at P_1' and P_2'. The corresponding plane is the osculating plane of the double curve at P, and the two branches of the corresponding section osculate there. One curve of the net has a double point at P_1' and is tangent to the image of the double curve at P_2'. The corresponding plane is one of the tangent planes to the surface at P, and the section by this plane has a triple point at P, one branch being tangent to the double curve.

In the above the elements of direction at P_1' and P_2' correspond to the elements of direction in the tangent planes at P. If, therefore, the two images of P coincide, P is a pinch point, or point of the double curve where the two tangent planes coincide. The corresponding point, P', lies on the Jacobians of all the ∞^3 nets of fundamental curves. For the net of fundamental curves through P' have a common tangent there, determined by two consecutive points of the image of the double curve. Therefore a pencil, g', of the net has double points at P', and there is one curve of this pencil in any net of fundamental curves. Thus every tangent cone includes one of the planes of this pencil among its tangent planes. This agrees with the fact that all first polar surfaces have the same tangent plane at a pinch point as the original surface, and that therefore a pinch point is on the curve of contact of any tangent cone. To find the number of such points P', *i.e.*, the number of pinch points of a rational surface, it is only necessary to deduct from the intersections of two general Jacobians the number of intersections accounted for by the base points and the number of curves of the pencil common to the two nets that have double points. Since two consecutive points of the common tangent at P' correspond to the same point of the surface it follows that the section of the surface by a general plane through a pinch point has there a stationary point or cusp. The pencil g' of fundamental curves that have double points at P' has been mentioned. Since two curves of this pencil have 4 intersections at P', the corresponding pencil of plane sections has for axis a line which meets the surface 4 times at the pinch point. This line, which Zeuthen calls "the particular tangent," will therefore meet the section by a general plane of the pencil 4 times at the pinch point. Since there is no reason to expect a cusp of higher order, we may infer that such a section has there a point of contact of two branches, or tacnode.

For example, in the case of the ruled cubic surface the "particular tangent" is the single generator from the pinch point, and the residual section is simply a conic tangent to it. If a direction at P' is given there is a curve of the pencil g' one of whose tangents at P' has this direction. Thus to any plane curve passing through P' corresponds on the surface a curve tangent to the "particular tangent" at P. In particular, since the Jacobians of all nets of fundamental curves pass through P', the curves of contact of all tangent cones have the same tangent at the pinch point.

One curve, ϕ_1, of the pencil g' is tangent to the image of the double curve at P'. The plane of the corresponding section contains the "particular tangent," and one other line which meets the surface 4 times at the pinch point. For any curve of the net which osculates ϕ_1 at P' on the branch tangent to the image of the double curve has 4 intersections with ϕ_1 at P' and the same is true of any two curves of the pencil determined by them. Hence the corresponding pencil of plane sections has for axis a line which meets the surface 4 times at the pinch point, and which must be in fact the tangent to the double curve. For that tangent evidently has the property just mentioned; and there is no other pencil of fundamental curves any two of which have 4 intersections at P', and therefore no third line meeting the surface 4 times at the pinch point. Since ϕ_1 has 3 intersections at P' with a general curve of the net, a general line through the pinch point in the corresponding plane meets the surface and the section corresponding to ϕ_1 three times there. That is, the section has a triple point whose tangents are the "particular tangent" and the tangent to the double curve. Such a triple point is composed of a cusp and two nodes. In this case the cuspidal tangent is the tangent to the double curve and the other tangent is the "particular tangent." For we have noted that a general curve, ϕ_0, of the net is simply tangent to the image of the double curve at P', while the corresponding section of the surface has an ordinary cusp at the pinch point whose cuspidal tangent lies in the plane of the section corresponding to ϕ_1. Therefore, as a point tracing ϕ_1 passes through P' on the branch that crosses the image of the double curve, the corresponding point on the surface passes through the pinch point tangent to the "particular tangent;" and as the point in the plane passes through P' on the branch tangent to the image of the double curve, the point on the surface becomes stationary, and a cusp results whose cuspidal tangent is the tangent to the double curve. It follows that the section of the surface by a general plane of the pencil whose axis is the tangent to the double curve has at the pinch point a cusp of higher order, node+cusp. The cuspidal tangent is the tangent to the double curve, and the 4 intersections at P of this tangent with the section are thus accounted for. One plane of the pencil is the osculating plane of the double curve. The branches of its section have higher contact at the pinch point and the corresponding plane curve osculates the image of the double curve at P'. Furthermore in the plane pencil g' are two curves, ϕ_2 and ϕ_3, that have cusps at P'. The section of the surface corresponding to one of them will therefore have at the pinch point instead of a tacnode a cusp of higher order,

node+cusp. The cuspidal tangent will be the "particular tangent." The cuspidal tangents to ϕ_2 and ϕ_3 at P' are the tangents to the locus of cusps of curves ϕ, which itself has a double point at P'. This locus is of course the image of the parabolic curve of the surface. Hence the planes of the sections corresponding to ϕ_2 and ϕ_3 are stationary tangent planes. Each section has three point contact with the corresponding branch of the parabolic curve.

Finally, the Jacobian J, of the net, that is, the image of the curve of contact of the tangent cone whose vertex is the pinch point, has a triple point at P'. One branch is tangent to the image of the double curve and osculates the corresponding branch of ϕ_1, and the other two touch the cuspidal tangents of ϕ_2 and ϕ_3. Therefore as compared with a general tangent cone the genus of the cone whose vertex is a pinch point is less by 3 on account of the triple point of the Jacobian. Its order is less by 4 on account of the 4 intersections of the Jacobian at P' with a general curve of the net. And its class is reduced by 2; for in an arbitrary pencil of the net the number of curves that have double points, other than P', is reduced by 2 by reason of the common tangent at P'. From these the modifications of the other Plücker numbers are easily determined. Thus the number of stationary tangent planes is less by 6. This follows also from the fact that the Jacobian is tangent to both branches of the cuspidal locus, and has 6 more intersections with that locus at P' than does the Jacobian of a general net. Thus the sections corresponding to ϕ_2 and ϕ_3 count each for 3 in making up the number of stationary tangent planes through the pinch point. The number of double edges is increased by 1. This follows also from the fact that a general curve of the pencil g' meets J 6 times at P' instead of 4 times, as is the case for a general curve of the net. Thus the additional double edge is the "particular tangent." Therefore the number of tangent planes to the cone which are planes of the pencil g should be less by 6 than the class of an ordinary tangent cone, that is, the number of curves ϕ of an ordinary pencil that have an additional double point. The number of curves of the pencil g' that have another double point, additional to P', is less by 7 than for an ordinary pencil. But ϕ_1 has 8 intersections with J at P'; and the corresponding plane, which contains the "particular tangent" and the tangent to the double curve, is therefore the remaining tangent plane to the cone that belongs to the pencil g. In the same way the characteristics of the tangent cone whose vertex is a point on the "particular tangent" may be obtained. For the corresponding net of plane curves is determined by ϕ_2 and ϕ_3 and a general curve ϕ. Therefore the Jacobian of that net will have a double point at P' whose tangents are the cuspidal tangents of ϕ_2 and ϕ_3. From the Plücker equations the number of double tangent planes of the cone whose vertex is the pinch point itself is less by $2(n-7)$, where n is the class of the surface, or class of a general tangent cone. And for a cone whose vertex is an arbitrary point of the "particular tangent" the reduction in the number of double tangent planes is $n-7$. Now the curve of contact of a tangent cone meets 2 times the locus of points of contact of double tangent planes for every such plane belonging to the cone. $n-7$ is

just the number of curves of the pencil g' that have a double point, additional to P', and which will therefore be double tangent planes to the surface, though not proper double tangent planes of the cone whose vertex is the pinch point or an arbitrary point of the "particular tangent." Now the image of the curve of contact of double tangent planes passes $n-7$ times[1] through P'. The Jacobian of a general net has a simple point at P'. The Jacobian, J, of the net determined by P' has a triple point at P', and the Jacobian of the net corresponding to a cone whose vertex is on the "particular tangent" has a double point at P' with two of the same tangents as J. The pencil g' belongs to both nets. Thus it appears that J is tangent to the image of the curve of contact of double tangent planes at the $n-7$ points which are double points for curves of the pencil g'. But the Jacobian of the net corresponding to a cone whose vertex is a general point of the "particular tangent" has simple intersections with that locus at the same points. A corresponding state of affairs holds on the surface.

If we reciprocate the above configuration we get another rational surface having a pinch plane; and we see that the section by this plane consists of a line corresponding to the pencil g, and a residual portion which is of order less by 2 than a general plane section since the class of the tangent cone whose vertex is the pinch point is two less than the class of a general tangent cone. Hence the pinch plane is tangent to the surface along a "singular line." The residual section is tangent to the "singular line" at the two points which are the reciprocals of the planes of g that correspond to ϕ_2 and ϕ_3, and meets it again at a third point which is the reciprocal of the plane of g corresponding to ϕ_1. Most of the other properties of pinch planes given by Zeuthen,[2] and true for surfaces in general, may be inferred in this way for rational surfaces.

In conclusion a little may be said on the question of finding the number of pinch points of a given surface whether it is rational or not. In case the double curve is the complete intersection of two surfaces, so that the equation of the given surface may be put in the form $Au^2+2Buv+Cv^2=0$, the pinch points are given by the intersections of the surface $B^2-AC=0$ with the curve $uv=0$. This method has been extended by Cayley[3] to the case when the double curve is the partial intersection of three surfaces given by

$$\left\| \begin{matrix} u & v & w \\ u' & v' & w' \end{matrix} \right\| = 0.$$

Here u, v, w are functions of the coördinates of degree μ and u', v', w' of degree ν. The common curve is therefore of degree $(\mu+\nu)^2-\mu\nu$. Salmon[4] has given a method for finding the reduction in the class of a surface due to the double curve which incidentally determines the number of pinch points. In this connection a statement is made without proof which is not obvious. The intersection of

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[1] See Caporali, loc. cit., §29.

[2] Loc. cit., §21.

[3] Quarterly Journal of Mathematics, vol. 9 (1868), p. 332.

[4] Solid Geometry, vol. 2, article, 616 (Longmans, Green and Co., 1915).

two general first polar surfaces consists of the double curve, which is simple on each of them, and a residual portion which meets the double curve where the two first polar surfaces are tangent to each other. In the absence of isolated singularities the class of the surface is the number of intersections of this residual portion with the original surface that do not occur on the double curve. At a pinch point any two first polar surfaces have the same tangent plane, i.e., the plane of the "particular tangent" and the tangent to the double curve. Their residual intersection meets, therefore, the original surface three times at a pinch point. Now the statement just mentioned as requiring proof is that in addition to the pinch points the two first polar surfaces are tangent to each other, but not to the original surface, at a number of points on the double curve equal to the rank of the latter; that is, the number of tangents that meet a given line. This follows from the fact, which is easily proved, that the first polar surfaces of all points of a plane tangent to the double curve at any point S have themselves the same tangent plane at S. Thus let u and v be the first polar surfaces of two general points P and Q, and let S be a point on the double curve at which the tangent intersects PQ. Then u and v will be tangent at S (though not to the plane PQS); and their residual intersection will meet the double curve, and therefore the original surface twice, at S. The number of points S is the rank of the double curve.

If it is possible to obtain independently the sum of the rank of the double curve and the number of pinch points, one can be found when the other is known. Salmon accomplishes this as follows. Consider a point whose polar planes with respect to the first polar surfaces u and v intersect in a line that meets a given line, say

$$ax + by + cz + d = 0, \qquad a'x + b'y + c'z + d' = 0.$$

The locus of these points is evidently the surface,

$$\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ u_x & u_y & u_z & u_w \\ v_x & v_y & v_z & v_w \end{vmatrix} = 0.$$

This surface is of order $2n-4$ where n is the order of the original surface. Any point R on the double curve at which the tangent meets the line determined by the first two rows of the determinant is a point on the surface. And so is any point T where the double curve is met by the residual intersection of u and v. For at such a point corresponding elements of the 3rd and 4th rows are proportional. Therefore, if b is the order of the double curve, we have $2b(n-2) = r+t$, where r is the number of points R (or rank of the double curve), and t is the number of points T. That is, $2b(n-2) - r = t$; and t is composed of the pinch points and the points, called S above, where the two first polar surfaces are tangent to each other but not to the original surface. But we have seen that the number of the latter is also r. Hence the number of pinch points is $2b(n-2) - 2r$. This result,

which is given by Salmon, requires of course modification when the double curve has triple points. Such points are triple points of the surface and hence double points of all first polar surfaces. All the elements of the 3rd and 4th rows of the above determinant vanish at such a point, and we see immediately that the surface represented by the determinant has itself a double point there. Hence each triple point of the double curve absorbs 6 intersections of the latter with the auxiliary surface and reduces the number of pinch points by 6.

THE TRIGONOMETRY OF HYPERSPACE[1]

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1. *Introduction.* In recent years many references have been made to the problem of the extension of the formulas of spherical trigonometry to space of higher dimensions. In particular, Karl Pearson,[2] emphasized the importance of such an extension for the sake of its application to the theory of multiple correlation. E. V. Huntington[3] revoiced the sentiment of Pearson in urging the importance of mathematics in modern statistics. Later, Dunham Jackson[4] gave a trigonometric representation of correlation. The problem was considered in more detail by James McMahon[5] in the paper, *Hyperspherical goniometry and its appliction to correlation theory for n variables.* In this article he generalized the formulas of spherical trigonometry for the hypersphere in n dimensions and then applied these formulas to correlation for n variables. McMahon considered only Euclidean spaces.

It is the object of the present paper to show that the desired trigonometric formulas are essentially contained in well known relations connecting certain invariants of space. Some of the more important of these date back to Grassmann.[6] Formulas which may be regarded as the extension of the Grassmann formulas to curved or Riemannian space were established by Maschke.

2. *Notations and their geometric interpretation.* In the n-dimensional space S_n, let $f(u^1, \cdots, u^n)$ denote a vector from an arbitrary origin to the point having the coordinates $u^1, u^2, \cdots, u^n$. Let f_i denote the partial derivative of the function f with respect to u^i, i.e., $f_1 = \partial f/\partial u^1$, $f_2 = \partial f/\partial u^2$, $\cdots$, $f_n = \partial f/\partial u^n$. These are considered as vectors tangent to the parameter curves of parameters

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[1] Presented to the Missouri Section of the Mathematical Association of America, November 26, 1927.

[2] Biometrica, vol. 11 (1916), p. 237.

[3] This Monthly, vol. 26 (1919), p. 422.

[4] This Monthly, vol. 31 (1924), pp. 275–280.

[5] Biometrica, vol. 15 (1923), pp. 173–208.

[6] Grassmann's formulas contain implicitly the fundamental formulas for the trigonometry of Euclidean n-dimensional space. See, for example, section 175 of the *Ausdehnungslehre* of 1862. Certain explicit trigonometric formulas also occur in sections 195–215. The formulas of Grassmann are special cases of the formulas of this paper.

$u^1, \cdots, u^n$, respectively. The scalar products, $f_i f_j$, will be denoted by g_{ij}, which are the coefficients of the differential form,

$$ds^2 = f_i f_j du^i du^j = \sum g_{ij} du^i du^j,$$

where ds^2 is the square of the differential of arc length along any curve of S_n.

It is often convenient to introduce additional symbols, ϕ_i, ψ_i, which are equivalent notations for the tangent vectors, f_i. (i.e., $\phi_i\phi_j = \psi_i\psi_j = g_{ij}$.)

Notations or symbols which occur twice in any product will always be combined as scalar products, and it will be understood that the same symbol will not be used more than twice as a factor. If only one symbol occurs just once, the expression represents a one-dimensional vector. We frequently use an expression containing r distinct symbols each occurring just once as a factor, which has been arranged such that it represents an r-dimensional vector.

Throughout this paper, the different notations for the vector f have the properties of the Maschke symbols in his symbolic treatment of the invariants of a differential form.

Maschke showed that if $A^1, A^2, \cdots, A^n$ are n invariants of the quadratic differential form, $G = \sum g_{ij} du^i du^j$ in n variables, then $1/\sqrt{|g_{ij}|}$ times the Jacobian of $A^1, \cdots, A^n$ is also an invariant. This he denoted by $(A^1, \cdots, A^n)$. Maschke's result really holds if some or all of the A's are vectors. Hence any of the A's of this invariant may be replaced by arbitrary functions of the coordinates or by the vector functions f, ϕ, etc., since f is a function of the coordinates and is an invariant. Consequently, such expressions as $(f^1 \cdots f^{n-k} a^1 \cdots a^k)$ are invariants, where $f^1, \cdots, f^{n-k}$ are $n-k$ equivalent notations for the vector f. Furthermore, the product of any number of invariants is an invariant.

When n hypersurfaces, $a^1 = \text{const.}$, $a^2 = \text{const.}, \cdots, a^n = \text{const.}$ are given, the invariant $(fa^1 \cdots (a^i) \cdots a^n)$, where the notation indicates that a^i is omitted, represents a vector tangent to the intersection of all the hypersurfaces except $a^i = \text{const.}$ Similarly, if f and ϕ are equivalent symbols, $(f\phi a^1 \cdots (a^i a^j) \cdots a^n)$ represents a two-dimensional vector tangent to the intersection of all except $a^i = \text{const.}$ and $a^j = \text{const.}$ In general, $(f^1 \cdots f^{n-k}\, a^1 \cdots a^k)$ represents a vector of dimensions $(n-k)$.

For example, in four-dimensional space, let $w^1 = \text{const.}$; $w^2 = \text{const.}$, $w^3 = \text{const.}$, $w^4 = \text{const.}$ be four hypersurfaces. Then $(f\phi w^1 w^2)$, $(f\phi w^2 w^4), \cdots$, are two dimensional vectors; $(f\phi\psi w^1)$ is a three-dimensional vector, etc.

The angle between the three-dimensional vectors $(f\phi\psi w^1)$ and $(f\phi\psi w^2)$ will be denoted by $[w^1; w^2]$; the angle between the two-dimensional vectors $(f\phi w^1 w^2)$ and $(f\phi w^2 w^3)$ by $[w^1 w^2; w^2 w^3]$, or on occasion by $[1, 2; 2, 3]$; similarly, the angle between the one-dimensional vectors $(fw^1 w^2 w^3)$ and $(fw^1 w^2 w^4)$ will be denoted by $[w^1 w^2 w^3; w^1 w^2 w^4]$. Thus, it will be understood that $[\alpha_1 \cdots \alpha_k; \beta_1 \cdots \beta_k]$ denotes the angle between the two $(n-k)$-dimensional vectors $(f^1 \cdots f^{n-k} a^{\alpha_1} \cdots a^{\alpha_k})$ and $(f^1 \cdots f^{n-k} a^{\beta_1} \cdots a^{\beta_k})$. This is of course equal to one of the angles between the complementary k-dimensional vectors orthogonal to $(f^1 \cdots$

$f^{n-k}\, a^{\alpha_1} \cdots a^{\alpha_k})$ and $(f^1 \cdots f^{n-k} a^{\beta_1} \cdots a^{\beta_k})$. Here each of the two sets of functions $a^{\alpha_1} \cdots a^{\alpha_k}$ and $a^{\beta_1} \cdots a^{\beta_k}$ is a set of any k of the functions $a^1 \cdots a^n$. The magnitude of the vector $(f^1 \cdots f^{n-k}\, a^1 \cdots a^k)$ is by definition $\sqrt{(f^1 \cdots f^{n-k}\, a^1 \cdots a^k)^2}$.

In general, we define the angle $[\alpha_1 \cdots \alpha_k; \beta_1 \cdots \beta_k]$ by the formula

$$(1)\quad \cos\,[\alpha_1 \cdots \alpha_k\,;\beta_1 \cdots \beta_k] = \frac{(f^1 \cdots f^{n-k} a^{\alpha_1} \cdots a^{\alpha_k}) \quad (f^1 \cdots f^{n-k} a^{\beta_1} \cdots a^{\beta_k})}{\sqrt{(f^1 \cdots f^{n-k} a^{\alpha_1} \cdots a^{\alpha_k})^2}\sqrt{(f^1 \cdots f^{n-k} a^{\beta_1} \cdots a^{\beta_k})^2}}\,;$$

that is, the cosine of the angle between two r-dimensional vectors is equal to the scalar product of the vectors divided by the product of their magnitudes.

Thus in four dimensions, $\sqrt{(f\phi w^1 w^2)^2}$ is the magnitude of the two-dimensional vector $(f\phi w^1 w^2)$, and the cosine of the angle between $(f\phi w^1 w^2)$ and $(f\phi w^3 w^4)$ is defined by

$$\cos\,[w^1 w^2\,;\, w^3 w^4] = \frac{(f\phi w^1 w^2) \quad (f\phi w^3 w^4)}{\sqrt{(f\phi w^1 w^2)^2}\sqrt{(f\phi w^3 w^4)^2}}\,.$$

3. *Trigonometric formulas in n-dimensional space.* Maschke has given the following general formula:[1]

$$(2)\quad (f^1 \cdots f^{n-k} a^{\alpha_1} \cdots a^{\alpha_k})(f^1 \cdots f^{n-k} a^{\beta_1} \cdots a^{\beta_k}) = \frac{(n-k)!}{\{(n-1)!\}^k} \begin{vmatrix} (fa^{\alpha_1})(fa^{\beta_1}), & \cdots, & (fa^{\alpha_1})(fa^{\beta_k}) \\ (fa^{\alpha_2})(fa^{\beta_1}), & \cdots, & (fa^{\alpha_2})(fa^{\beta_k}) \\ \vdots & & \vdots \\ (fa^{\alpha_k})(fa^{\beta_1}), & \cdots, & (fa^{\alpha_k})(fa^{\beta_k}) \end{vmatrix}$$

where each f is used for a set of $n-1$ symbols, $f^1, \cdots, f^{n-1}$, and the sets are supposed to be the same in the two factors of any given product but are different in the different products.

When the definition (1) for the scalar product of two $(n-1)$-dimensional vectors is applied to each term of the determinant in the right hand member of (2), the equation becomes

$$(3)\quad (f^1 \cdots f^{n-k} a^{\alpha_1} \cdots a^{\alpha_k})(f^1 \cdots f^{n-k} a^{\beta_1} \cdots a^{\beta_k})$$

$$= \frac{(n-k)!}{\{(n-1)!\}^k} \begin{vmatrix} \sqrt{(fa^{\alpha_1})^2}\sqrt{(fa^{\beta_1})^2}\cos\,[\alpha_1\,;\beta_1], & \cdots, & \sqrt{(fa^{\alpha_1})^2}\sqrt{(fa^{\beta_k})^2}\cos\,[\alpha_1;\beta_k] \\ \vdots & \cdots & \vdots \\ \sqrt{(fa^{\alpha_k})^2}\sqrt{(fa^{\beta_1})^2}\cos\,[\alpha_k\,;\beta_1], & \cdots, & \sqrt{(fa^{\alpha_k})^2}\sqrt{(fa^{\beta_k})^2}\cos\,[\alpha_k\,;\beta_k] \end{vmatrix}$$

$$= \frac{(n-k)!}{\{(n-1)!\}^k} L_{\alpha_1} L_{\beta_1} \cdots L_{\alpha_k} L_{\beta_k} \begin{vmatrix} \cos\,[\alpha_1\,;\beta_1], & \cdots, & \cos\,[\alpha_1, \beta_k] \\ \vdots & & \vdots \\ \cos\,[\alpha_k\,;\beta_1], & \cdots, & \cos\,[\alpha_k\,;\beta_k] \end{vmatrix}$$

where L_{α_1} denotes $\sqrt{(fa^{\alpha_1})^2}$, etc.

[1] See the paper, *Differential parameters of the first order*, Transactions of the American Mathematical Society, vol. 7 (1906), pp. 69–80.

Now by definition (1),

$$(f^1 \cdots f^{n-k}a^{\alpha_1} \cdots a^{\alpha_k})(f^1 \cdots f^{n-k}a^{\beta_1} \cdots a^{\beta_k})$$
$$= \sqrt{(f^1 \cdots f^{n-k}a^{\alpha_1} \cdots a^{\alpha_k})^2}\sqrt{(f^1 \cdots f^{n-k}a^{\beta_1} \cdots a^{\beta_k})^2} \cos [\alpha_1 \cdots \alpha_k ; \beta_1 \cdots \beta_k].$$

Formula (2) may be applied to the expressions under each of the radicals on the right, and the elements of the resulting determinants may be expressed in terms of the quantities L and cosines as was done above; thus we obtain finally,

$$(4) \qquad (f^1 \cdots f^{n-k}a^{\alpha_1} \cdots a^{\alpha_k})(f^1 \cdots f^{n-k}a^{\beta_1} \cdots a^{\beta_k}) =$$
$$R\sqrt{\begin{vmatrix} 1 & , \cdots , & \cos [\alpha_1 ; \alpha_k] \\ \vdots & & \vdots \\ \cos [\alpha_k ; \alpha_1], & \cdots , & 1 \end{vmatrix}}$$
$$\times\sqrt{\begin{vmatrix} 1 & , \cdots , & \cos [\beta_1 ; \beta_k] \\ \vdots & & \vdots \\ \cos [\beta_k ; \beta_1], & \cdots , & 1 \end{vmatrix}} \cos [\alpha_1 \cdots \alpha_k ; \beta_1 \cdots \beta_k],$$

where

$$R = \frac{(n-k)!}{\{(n-1)!\}^k} L_{\alpha_1} L_{\beta_1} \cdots L_{\alpha_k} L_{\beta_k}.$$

Therefore, by equating (3) and (4),

$$(5) \quad \cos [\alpha_1 \cdots \alpha_k ; \beta_1 \cdots \beta_k] =$$
$$\frac{\begin{vmatrix} \cos [\alpha_1 ; \beta_1], & \cdots , & \cos [\alpha_1 ; \beta_k] \\ \vdots & & \vdots \\ \cos [\alpha_k ; \beta_1], & \cdots , & \cos [\alpha_k ; \beta_k] \end{vmatrix}}{\sqrt{\begin{vmatrix} 1 & , \cdots , & \cos [\alpha_1 ; \alpha_k] \\ \vdots & & \vdots \\ \cos [\alpha_k ; \alpha_1], & \cdots , & 1 \end{vmatrix}}\sqrt{\begin{vmatrix} 1 & , \cdots , & \cos [\beta_1 ; \beta_k] \\ \vdots & & \vdots \\ \cos [\beta_k ; \beta_1], & \cdots , & 1 \end{vmatrix}}}.$$

4. *Formulas of spherical trigonometry in three dimensions.* In ordinary space let $w^1 = \text{const.}$, $w^2 = \text{const.}$, and $w^3 = \text{const.}$ be three arbitrary functions of the coordinates u^1, u^2, u^3.

According to formula (2),

$$(fw^1w^2)(fw^2w^3) = \frac{1}{4}\begin{vmatrix} (f\phi w^1)(f\phi w^2), & (f\phi w^1)(f\phi w^3) \\ (f\phi w^2)(f\phi w^2), & (f\phi w^2)(f\phi w^3) \end{vmatrix}.$$

When the definition (1) for the scalar product of two vectors is applied to the right member of this equation, (fw^1w^2) $(fw^2w^3) =$

$$\frac{1}{4}\begin{vmatrix} \sqrt{(f\phi w^1)^2}\sqrt{(f\phi w^2)^2}\ \cos[w^1\,;\,w^2], & \sqrt{(f\phi w^1)^2}\sqrt{(f\phi w^3)^2}\ \cos[w^1\,;\,w^3] \\ \sqrt{(f\phi w^2)^2}\sqrt{(f\phi w^2)^2}\ \cos[w^2\,;\,w^2], & \sqrt{(f\phi w^2)^2}\sqrt{(f\phi w^3)^2}\ \cos[w^2\,;\,w^3] \end{vmatrix}$$

$$= \frac{1}{4}\sqrt{(f\phi w^1)^2}\left(\sqrt{(f\phi w^2)^2}\right)^2\sqrt{(f\phi w^3)^2}\begin{vmatrix} \cos[w^1\,;\,w^2], & \cos[w^1\,;\,w^3] \\ 1 & , \cos[w^2\,;w^3] \end{vmatrix}.$$

However, if the scalar product, (fw^1w^2) (fw^2w^3) be determined by use of (1),

$$(fw^1w^2)(fw^2w^3) = \sqrt{(fw^1w^2)^2}\sqrt{(fw^2w^3)^2}\cos[w^1w^2\,;\,w^2w^3].$$

Expanding the expressions under the radicals by formula (2),

$$(fw^1w^2)(fw^2w^3)$$
$$= \sqrt{\frac{1}{4}\begin{vmatrix} (f\phi w^1)(f\phi w^1), & (f\phi w^1)(f\phi w^2) \\ (f\phi w^2)(f\phi w^1), & (f\phi w^2)(f\phi w^2) \end{vmatrix}} \times \sqrt{\frac{1}{4}\begin{vmatrix} (f\phi w^2)(f\phi w^2), & (f\phi w^2)(f\phi w^3) \\ (f\phi w^3)(f\phi w^2), & (f\phi w^3)(f\phi w^3) \end{vmatrix}}$$
$$\times \cos[w^1w^2\,;\,w^2w^3].$$

Hence,

$$\cos[w^1w^2;\,w^2w^3] = \frac{\cos[w^1\,;\,w^2]\cos[w^2\,;\,w^3] - \cos[w^1\,;\,w^3]}{\sin[w^1\,;\,w^2]\sin[w^2\,;\,w^3]}, \tag{6}$$

which is a familiar equation in spherical trigonometry, where the angles are considered as exterior angles.

5. *Application to four dimensions.* Let $w^1 = \text{const.}$, $w^2 = \text{const.}$, $w^3 = \text{const.}$, and $w^4 = \text{const.}$ be four arbitrary functions of the coordinates u^1, u^2, u^3, u^4.

Formula (2) in this case becomes

$$(f\phi w^1w^2)(f\phi w^3w^4) = \frac{2!}{(3!)^2}\begin{vmatrix} (f\phi\psi w^1)(f\phi\psi w^3), & (f\phi\psi w^1)(f\phi\psi w^4) \\ (f\phi\psi w^2)(f\phi\psi w^3), & (f\phi\psi w^2)(f\phi\psi w^4) \end{vmatrix},$$

which reduces in the usual way to

$$(f\phi w^1w^2)(f\phi w^3w^4) = \frac{2!}{(3!)^2}L_1L_2L_3L_4\begin{vmatrix} \cos[w^1\,;\,w^3], & \cos[w^1\,;\,w^4] \\ \cos[w^2\,;\,w^3], & \cos[w^2\,;\,w^4] \end{vmatrix},$$

where $L_1 = \sqrt{(f\phi\psi w^1)^2}$, etc.

On the other hand, if the scalar product of $(f\phi w^1w^2)$ $(f\phi w^3w^4)$ is obtained by (1) and then (2) is applied to the expressions under the radical, we have

$$(f\phi w^1w^2)(f\phi w^3w^4) =$$
$$\frac{2!}{(3!)^2}L_1L_2L_3L_4\sqrt{\begin{vmatrix} 1 & , \cos[w^1\,;\,w^2] \\ \cos[w^1\,;\,w^2], & 1 \end{vmatrix}}$$
$$\times\sqrt{\begin{vmatrix} 1 & , \cos[w^3;\,w^4] \\ \cos[w^3\,;\,w^4], & 1 \end{vmatrix}}\ \cos[w^1w^2;\,w^3w^4].$$

Therefore,

$$(7)\quad \cos\left[w^1w^2\,;w^3w^4\right] = \frac{\cos\left[w^1\,;w^3\right]\cos\left[w^2\,;w^4\right] - \cos\left[w^1\,;w^4\right]\cos\left[w^2\,;w^3\right]}{\sin\left[w^1\,;w^2\right]\sin\left[w^3\,;w^4\right]}.$$

6. *Application to statistics.* For applications to statistics it is convenient to regard the space in which the sets of variables are represented as Euclidean though it is not really necessary to restrict the number of dimensions. In fact, all the points may, if desired, be regarded as points of a function space where instead of considering a point as represented by a set of coordinates x_i, we suppose a point to be represented by a function $x(t)$, where t is a continuous variable on a given interval.[1]

Let there be given n points $X, Y, Z \cdots$ in an n-dimensional Euclidean space, with coordinates $x_1 \cdots x_n$, $y_1 \cdots y_n$, $z_1 \cdots z_n$, etc. referred to an origin O. As usually defined, the coefficient of correlation between any two of these sets of coordinates, as the x's and the y's, is simply the cosine of the angle between the lines OX and OY. Thus, we have

$$r_{12} = \cos\left[OX\,;OY\right]\,;\quad r_{23} = \cos\left[OY\,;OZ\right]\,;\ \text{etc.}$$

In Euclidean space it can be supposed that the products $f_if_j = 1$ or 0 depending on whether $i=j$ or $i\neq j$. This will be the case if the axes form a rectangular cartesian system.

Let $w^1=0$, $w^2=0, \cdots, w^n=0$ be the equations of n hyperplanes through the origin orthogonal to OX, OY, OZ, etc., respectively; evidently we then have

$$r_{12} = \cos\left[OX\,;OY\right] = \cos\left[w^1\,;w^2\right],\ \text{etc.}$$

All of the formulas of this paper may now be interpreted in this Euclidean space.

In the general formula (5) the determinants on the right are minors of the determinant,

$$\begin{vmatrix} 1 & r_{12} \cdots r_{1n} \\ r_{21} & r_{22} \cdots r_{2n} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ r_{n1} & r_{n2} \cdots r_{nn} \end{vmatrix}$$

The minor of second order in which the p^{th} and q^{th} rows and r^{th} and s^{th} columns are deleted may be conveniently denoted by $\Delta_{pq,rs}$ with similar notations for minors of higher order; then

$$\cos\left[w^1w^4 \cdots w^n\,;w^3w^4 \cdots w^n\right] = \frac{\Delta_{12,23}}{\sqrt{\Delta_{12,12}}\sqrt{\Delta_{23,23}}}.$$

[1] The reader may consult the paper by Dunham Jackson, *The elementary geometry of function space,* in this Monthly, vol. 31 (1924), pp. 461–471, for a clear and simple treatment of these ideas.

This gives a geometrical interpretation of the partial correlation coefficient[1] $r_{13.45\ldots n}$.

It may be seen in a similar manner that in general

$$\cos\left[w^1 w^{i+2}\cdots w^n; w^{i+1}w^{i+2}\cdots w^n\right] = r_{1,\,i+1;\,i+2,\,\cdots,\,n-1,\,n},$$

which is a partial correlation coefficient of order $n-i$.

ON AN ALGORITHM AND ITS USE IN APPROXIMATING ROOTS OF ALGEBRAIC EQUATIONS

By T. A. PIERCE, University of Nebraska

A continued fraction is the result of eliminating $x_2, x_3, \cdots$, successively in the generating equations

$$(1) \qquad x_1 = q_1 + 1/x_2, \quad x_2 = q_2 + 1/x_3, \cdots.$$

The algorithm which we propose to treat in this paper is defined by the series of generating equations

$$(2) \qquad x_1 = q_1(1 - x_2), \quad x_2 = q_2(1 - x_3), \quad x_3 = q_3(1 - x_4), \cdots.$$

In these equations it will be noticed that multiplication takes the place of addition in (1) and that subtraction from unity takes the place of the division into unity of (1).

On eliminating $x_2, x_3, \cdots$, in (2) we have

$$(3) \qquad x_1 = q_1(1 - q_2(1 - q_3(\cdots))).$$

The expansion of this is

$$(4) \qquad x_1 = q_1 - q_1q_2 + q_1q_2q_3 - q_1q_2q_3q_4 + \cdots.$$

We specify that the q's, which will be precisely defined presently, are all positive. If the terms of the right member of (4) form a monotonically decreasing sequence whose terms ultimately approach zero we know from the theory of alternating series that the process represented by the algorithm (2) is convergent. We will be justified therefore in defining the successive convergents of x_1 to be

$$q_1, \quad q_1 - q_1q_2, \quad q_1 - q_1q_2 + q_1q_2q_3, \cdots.$$

The nth convergent is obtained from (3) or (4) by setting q_{n+1} equal to zero. We know also that the nth convergent will differ from the true value of the expansion by at most the absolute value of the $(n+1)$st term and therefore certainly by at most the absolute value of the nth term which is the last component term of the nth convergent itself.

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[1] See, for example, Kelley, *Statistical Method*, pp. 298–299.

When expanding a number greater than unity by this algorithm we take q_1 to be the least positive integer such that $x_2=1-(x_1/q_1)$ is positive; thereafter for $n=2, 3, \cdots$, we take $q_n=1/p_n$ where p_n is the greatest positive integer such that $x_{n+1}=1-p_nx_n$ is positive. If the number to be expanded is less than unity we take $q_1=1/p_1$ where p_1 is the greatest positive integer such that $x_2=1-p_1x_1$ is positive, and then choose $q_2, q_3, \cdots$, as before. Note that $x_2, x_3, \cdots$, are each less than unity.

By a periodic or recurring expansion is meant the expression (3) wherein the q's recur periodically from some point on.

Rational numbers except 1 and $\frac{1}{2}$ expand by (2) into periodic expansions with an infinite number of terms. For if the rational number to be expanded be greater than unity the choice of q_1 will give x_2 as a rational number less than unity. We consider therefore only rational numbers less than unity. Let $x_1=r/s$, where $r<s$, be the number to be expanded. From the very manner in which the succeeding x's are formed in (2) these x's will be rational numbers having the same denominator as x_1 since it is never necessary to cancel a factor. As it is impossible to have an infinite number of different rational fractions each less than unity and each having the same denominator we see that a repetition must exist in the numerators of the x's and this entails a recurrence among the q's.

Conversely every periodic expansion represents a rational number. For if

$$x_i = q_i(1 - x_{i+1}), \cdots, x_{i+k-1} = q_{i+k-1}(1 - x_{i+k})$$

are the equations of (2) which contain the first period of the recurrence then $x_{i+k}=x_i$. Upon elimination of the x's intervening between x_{i+k} and x_i and solving for x_i we obtain a rational value of x_i. Substituting this value of x_i in the $(i-1)$st equation of (2) and eliminating $x_2, \cdots, x_{i-1}$ we obtain a rational value of x_1.

It follows that every non-periodic expansion represents an irrational number. Since $\frac{1}{2}(1-\frac{1}{3}(1-\frac{1}{4}(\cdots)))$, which equals $1-1+(1/2!)-(1/3!)+\cdots=1/e$, is a non-periodic expansion we have proved very simply that the base of the natural system of logarithms is an irrational number.

Irrational roots of algebraic equations may be approximated by the algorithm represented in (2). By preliminary transformations of the equation its roots are separated and the one x_1 to be approximated is brought into the interval from 0 to 1. This root may be assumed to be the only positive root of the equation which is less than unity. The process then is to multiply x_1 by the greatest positive integer p_1 which will make their product less than unity, change sign of the product, and increase by unity in accordance with the equation $x_2=-p_1x_1+1$. The original equation in x_1 is subjected to this transformation. The root x_2 of the transformed equation corresponding to x_1 is positive and less than unity. Next the same process that was applied to x_1 is applied to x_2 to obtain x_3 as the root of the second transformed equation and such that $0<x_3<1$. The process is continued, giving

$$x_1 = \frac{1}{p_1}\left(1 - \frac{1}{p_2}\left(1 - \frac{1}{p_3}(\cdots))\right)\right).$$

Since the p's are positive integers the condition of convergence is satisfied. It is important to notice that since the process is convergent x_n approaches zero as n grows large. This after the first few stages of the process allows us to compute p_{n+1} by solving the first degree equation obtained from the transformed equation in x_n by dropping all terms except the last two. Also if we put $x_n = 0$ we obtain the nth convergent and we know that this differs from the true value of x_1 by less than $1/(p_1 \cdots p_{n+1})$. Thus the degree of approximation is known at each stage of the process.

As a numerical illustration let us approximate the root of the equation $x^3 - 5x + 2 = 0$ which lies between 0 and 1. We give the successive transformed equations and the corresponding transformations.

$$\begin{array}{rl}
x_1^3 - 5x_1 + 2 = 0 & x_1 = \dfrac{1}{2}(1 - x_2) \\[2ex]
-x_2^3 + 3x_2^2 + 17x_2 - 3 = 0 & x_2 = \dfrac{1}{5}(1 - x_3) \\[2ex]
x_3^3 + 12x_3^2 - 452x_3 + 64 = 0 & x_3 = \dfrac{1}{7}(1 - x_4) \\[2ex]
-x_4^3 + 87x_4^2 + 21977x_4 - 111 = 0 & x_4 = \dfrac{1}{197}(1 - x_5) \\
\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots
\end{array}$$

The third convergent is

$$\frac{1}{2} - \frac{1}{10} + \frac{1}{70} = .41428$$

which may be in error as much as $1/13790 = .00007$. The fourth convergent is

$$\frac{1}{2} - \frac{1}{10} + \frac{1}{70} - \frac{1}{13790} = .414213564$$

which is correct to within 2 units in the 9th decimal place.

The transformation of the equations in accordance with $x_{n+1} = -p_n x_n + 1$ is performed in two operations, first multiplying the roots of the equation by p_n and changing their sign, next increasing the roots by unity. The latter operation is particularly simple if carried out synthetically.

It will be noted that the above method of approximating roots bears the same relation to Lagrange's continued fraction method that the algorithm (2) bears to the continued fraction algorithm (1).

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

DISCUSSION

A GRAPHICAL METHOD FOR THE SOLUTION OF CERTAIN TYPES OF EQUATIONS

By T. R. C. WILSON, Madison, Wisconsin

A labor saving graphical method for solving certain types of equations is illustrated by the following example.

In connection with a series of tests on wooden columns it was desired to find values of e and E from equations of the type

$$y + e = e \cdot \sec \left(\frac{Pl^2}{4EI} \right)^{1/2} \tag{1}$$

or

$$y = e \cdot \operatorname{exsec} \left(\frac{Pl^2}{4EI} \right)^{1/2}, \tag{1a}$$

where y is the deflection of an Euler column under the load P which was placed as centrally as possible but found to have a slight eccentricity e whose value was sought as well as the value of E, the modulus of elasticity of the column, where l is the length of the column; and where I is the moment of inertia of the column section.

Simultaneous observations had been taken of loads and deflections and a curve had been plotted. From this curve two values of P, designated P_1 and P_2, were taken together with the corresponding deflections, y_1 and y_2. Putting these two sets of values in equation (1a) and transposing and dividing resulted in

$$\frac{y_2}{y_1} = \frac{\operatorname{exsec} (P_2 l^2/4EI)^{1/2}}{\operatorname{exsec} (P_1 l^2/4EI)^{1/2}}. \tag{2}$$

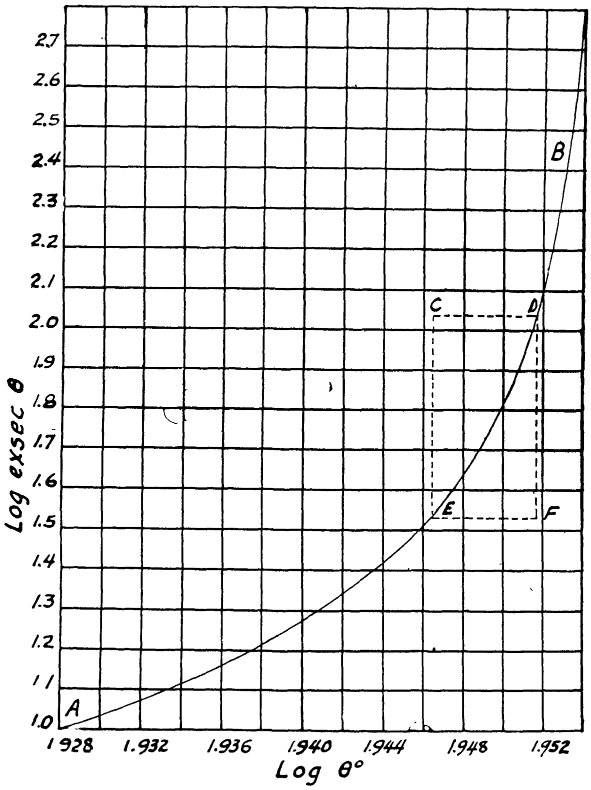
Here e has been eliminated and E is to be solved for. Attempts were made to find the value of E by expanding the secant into a series by MacLaurin's theorem and solving the resulting equation. It was found, however, that the use of the number of terms of the series which could be handled by ordinary algebraic methods did not give sufficient accuracy, and a different method was sought.

Equation (2) may be written:

$$Y = \frac{\operatorname{exsec} \theta_2}{\operatorname{exsec} \theta_1},$$

where $Y = y_2/y_1$ and θ_2 and θ_1 represent the values of the radicals occurring in the numerator and denominator, respectively, of the right-hand term of equation (2). Since it is evident that $\theta_2/\theta_1 = (P_2/P_1)^{1/2}$, a constant, let $(P_2/P_1)^{1/2} = K$, whence $\theta_2/\theta_1 = K$.

By cut and try methods two angles may be found whose ratio is K and the ratios of whose exsecants is Y. The cut and try is eliminated by the procedure illustrated in the diagram. AB is the curve resulting from plotting values of log exsecant θ as ordinates against corresponding values of log θ^0 as abscissae.



A rectangle ($CDFE$ in the diagram) whose width equals log K to the same scale as log θ in the plot and whose height equals log Y to the same scale as log exsecant θ is drawn on a piece of tracing paper. While keeping the sides of the rectangle parallel to the axes of the plot this tracing paper is shifted until two diagonally opposite corners of the rectangle are on the curve AB. The vertical sides of the rectangle are then over the values of log θ_1 and log θ_2 and the horizontal sides over the values of log exsec θ_1 and log exsec θ_2.

The diagram represents a solution from the following data: $P_1 = 330000\#$; $P_2 = 338000\#$; $K = 1.012$; $y_1 = .65''$; $y_2 = 2.10''$; $Y = 3.23$; $I = 1620$ ins.[4]:

$L=288.75''$. From the diagram, log $\theta_2^0=1.9516$, whence $\theta_2=89.45°=1.558$ radians; and log exsec $\theta_2=2.040$, whence exsec $\theta_2=109.6$.

By substituting the values of P_2 and θ_2 in the equation

$$\theta_2 = \left(\frac{P_2 l^2}{4EI}\right)^{1/2},$$

E is found to be 1,784,000 pounds per square inch; and putting values of exsec θ_2 and y_2 in equation $(1a)$ we get $e=.0187''$.

By plotting to natural as well as to logarithmic scales and by reflecting the plotted curves in the x-axis, or y-axis, or both, as may be required, the method illustrated above can be adapted to the solution of any pair of simultaneous equations formed by choosing any equation from the first and any from the second line below:

$$x_1+x_2=A;\quad x_1-x_2=A;\quad x_1\,x_2=A;\quad x_1\div x_2=A,$$

$$f(x_1)+f(x_2)=B;\quad f(x_1)-f(x_2)=B;\quad f(x_1)f(x_2)=B;\quad f(x_1)\div f(x_2)=B.$$

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College of the City of New York.

All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Leçons sur la Géomêtrie des Espaces de Riemann. By E. Cartan. Gauthier-Villars, Paris, 1928. vi+273 pages.

This book begins with very simple and familiar ideas of vectors in Euclidean space in rectangular Cartesian coordinates and gradually arrives at the notion of a tensor and the algebraic and differential operations with tensors. In fact the study of the differential geometry of a Riemann space does not really begin until the fourth chapter. The method of study consists of associating with each point of the Riemann space an osculating Euclidean space; it then follows that the properties (at the point) of the Euclidean space which depend on the fundamental tensor and its first partial derivatives are valid for the Riemann space.

In order to study the properties of a curve in a Rieman space, a "levelling" or "flattening" space is introduced; that is, a Euclidean space which osculates the Riemann space along the curve. Its existance is established by proving Fermi's theorem from which a number of interesting geometric conclusions are drawn.

Although a great many theorems of classical differential geometry have been generalized or extended to any Riemann space, the author usually confines his attention to two and three dimensional Riemann spaces with positive definite metrics. Some of the proofs are given for spaces of n dimensions, others for spaces of two or three dimensions; often it is not clear to what kind of space the proof does apply. Thus in §87 by "any Riemann space" the author means one of three dimensions whose metric is positive definite and whose fundamental tensor admits continuous partial derivatives of at least the second order. The same defect exists in chapter V where a surface is under discussion; there one has to read about half a page further where "the" unit normal vector is mentioned, so that one concludes that the surface is immersed in a three dimensional Riemann space since in an n-dimensional space there would be $n-2$ linearly independent normals.

In this chapter there is a very interesting discussion of the "plane axiom" and the "axiom of free motion," which analytically amount to Beltrami's theorem that the only spaces in geodesic correspondence with a space of constant Riemannian curvature are spaces of constant Riemannian curvature.

Nothing is said about motion in a space which does not satisfy these two axioms although motion may take place in a space of this sort. Throughout the text one will find many ideas of analysis situs; and many interesting topological properties of Riemann spaces homeomorphic to the surface of a sphere, torus, etc. are obtained.

There is also a brief exposition of exterior multiplication and exterior differentiation of scalar differential forms which is extended to vector and tensor forms, the whole being given a clear geometrical significance. The last chapter deals with Riemann's normal coordinates which could be introduced advantageously much earlier in the text as it would simplify some of the proofs.

There are very few misprints and those are obvious (p. 190, l. 2 and p. 206, l.17). One theorem (§172) is improperly stated; it should read as follows: "The Riemannian curvature being known for every orientation at a given point *and the fundamental tensor being known at this point*, the values of the components of the Riemann-Christoffel tensor at this point are uniquely determined."

With this change, the proof as given in the text is valid but is somewhat long and fails to give the values of R_{ijkl}. These may be obtained as follows: Since

$$R_{ijkl}\lambda^i\lambda^k\mu^j\mu^l = K(\lambda, \mu)(g_{ik}g_{jl} - g_{jk}g_{il})\lambda^i\lambda^k\mu^j\mu^l) = \Phi(\lambda, \mu, g),$$

we have

$$R_{ijkl} + R_{ilkj} = \tfrac{1}{2}\partial^4\Phi/\partial\lambda^i\partial\lambda^k\partial\mu^j\partial\mu^l.$$

Interchanging j and k and adding the result to twice the above equation we get

$$R_{ijkl} = \tfrac{1}{3}\partial^4\Phi/\partial\lambda^i\partial\lambda^k\partial\mu^j\partial\mu^l + \tfrac{1}{6}\partial^4\Phi/\partial\lambda^i\partial\lambda^j\partial\mu^k\partial\mu^l.$$

On the whole, one leaves the book with a feeling of having read about something very concrete, the language throughout being vividly geometrical; yet there is no lack of analytical rigor.

A great many interesting and important problems are not mentioned at all in this book but, as the author states in the preface, they will probably form the subject matter of another volume which will no doubt be welcomed by every student of geometry.

M. S. KNEBELMAN

Six Lectures on Recent Researches in the Theory of Fourier Series. By Ganesh Prasad. The University Press, Calcutta, 1928, xiv+139 pp.

The book reproduces a set of lectures given by the author at the University of Calcutta. The plan of the book is easily read off from the detailed list of contents covering eight pages. The first of the six lectures is introductory and gives in outline all the material presented in the later lectures. These deal with convergence criteria, convergence defects, Cesàro summability, strong summability, properties of Fourier coefficients, etc. There are in addition two appendices; one containing part of the Riemannian theory, the other, corrections and additions.

The program of the book is obviously an interesting one. There is moreover a decided need of a clear, detailed, and accurate treatise on the modern theory of Fourier series. So far Hobson's *Theory of Functions of a Real Variable* stands unrivalled in this field. The excellent book by Schlesinger and Plessner unfortunately does not go beyond the convergence theory in the narrow sense. Accordingly Professor Prasad has had a first rate opportunity of producing a useful book.

Let us grant that the book has some very good points. The author's collection of convergence criteria seems to be complete though he restricts himself to regular points. He devotes much space to the important phenomena which pass under the name of convergence defects; the Fourier series of a continuous function may not converge uniformly in any interval, it may diverge at a point or at a set of points everywhere dense having the power of the continuum. We also note the account of the theory of strong summability. The book is obviously based on much reading.

The book is unfortunately marred by many defects. It is not easily read, a fact which may be due partly to the poor printing and the numerous misprints which are very annoying. When the lectures were given it may have been necessary to suppress many steps in the proofs or to employ heuristic arguments; such points should have been elaborated in the printed version. The reader receives his first shock at the bottom of p. 18 where a quantity a which must be kept fixed in the proof is equated to a quantity tending to zero. The discussion of de la Vallée-Poussin's criterion on pp. 23-24 is rather muddled. The formulation of Young's criterion on p. 24 differs considerably from that of p. 4; the author does not show that the two formulations are equivalent. The correction on p. 137 of the formulation on p. 4 is a good example of the confusion in the notation for the symbols "o" and "O" which prevails in the book.

The proof in the foot-note on pp. 113–114 is a mystery in its present form. Other points could be mentioned.

The author does not seem to care much for the Lebesgue integral and avoids it as much as possible. This is unfortunate since the modern theory of Fourier series is based on this instrument and owes its present high status to this fact. The convergence theory, in particular, is chiefly concerned with results which are true almost everywhere; local results are usually of secondary interest. This point does not seem to have occured to the author when he labelled the results of his own investigations, in themselves interesting, as "criticism" of various criteria of summability due to Lebesgue, Hardy, Carleman and others. The interest in these criteria lies in the fact that they are satisfied almost everywhere by integrable functions and their "failure" at a point may be startling but is not a tragedy. The sense of the word "failure" in the author's investigations will be discussed later.

Professor Prasad's recent investigations[1] play a conspicuous rôle in his book. Consider a function having a discontinuity of the second kind at $x=x_0$ such that $f(x_0+t)+f(x_0-t)=g(t)\cos h(t)$ where $g(t)$ and $h(t)$ are monotone for small values of t and at least $g(t)$ is unbounded. P. du Bois-Reymond has discussed the convergence of the Fourier series of $f(x)$ at the point $x=x_0$. Professor Prasad considers summability by arithmetic means or strong summability instead of convergence. He is able to prove that the series is summable by such means at $x=x_0$ for a suitable choice of $g(t)$ and $h(t)$ though the known sufficient conditions for the type of summability in question may fail in one sense or another.

It is desirable to consider the case of summability (C1) in more detail. Lebesgue has shown that if there exists a quantity S such that

$$t^{-1}\int_0^t \left| f(x_0+2u)+f(x_0-2u)-2S\right| du \to 0 \tag{1}$$

when $t\to 0$, then the Fourier series of $f(x)$ is summable (C1) to the sum S for $x=x_0$. It should be noticed that this theorem does not admit of a converse. If the series is summable (C1) to the sum T for $x=x_0$ it does not follow that the left hand side tends to 0 or to any limit at all if we give S the value T. It is obvious that it cannot tend to 0 for any other value of S. Professor Prasad has found that this case presents itself in the case of the functions which he investigated; he gets T equal to zero and the left hand side of (1) does not tend to zero if it tends to any limit at all. This is the sense in which Lebesgue's criterion fails: since the converse of Lebesgue's theorem is not true, there is no contradiction with known results. That this phenomenon may occur is a priori obvious; it is of course interesting to know that it really presents itself in simple cases and Professor Prasad deserves credit for having called attention to it. The person who reads the top of p. 10 of his book cannot escape the conclusion,

[1] Bulletin of the Calcutta Mathematical Society, vol. 18 (1927), two papers, and vol. 19 (1928), five papers. These papers are largely based on earlier papers of the author in the same journal dealing with the differentiability of integrals with respect to the upper limit.

however, that Professor Prasad thinks Lebesgue was not justified in giving his theorem because it does not apply to the case considered by the author.

It is clear that the behavior of the functions $g(t)$, $h(t)$ and their derivatives near $t=0$ is decisive in these investigations. To express this behavior Professor Prasad uses a modification of the notation due to du Bois-Reymond explained on p. 137. He writes

$$f_1(x) \succ f_2(x) , \tag{2}$$

meaning that the two functions are monotone near some point x_0 and that $f_2/f_1 \to 0$ when $x \to x_0$. From this relation the author concludes

$$f_1'(x) \succ f_2'(x) \tag{3}$$

and similar relations for the higher derivatives. All that can be concluded from (2), however, is that $f_1'(x)$ and $f_2'(x)$ exist almost everywhere; (2) does not imply (3) except for very limited classes of functions. The author usually assumes that one of his functions in (2) is a simple logarithmico-exponential function; this is not sufficient, however. It is necessary to impose severe restrictions on both functions in (2) in order that (3) shall follow, e.g. that they are both logarithmico-exponential functions. There is no trace of any such restrictions in the book or any where in the recent publications of the author.

EINAR HILLE

Analytic Geometry,. By A. M. Harding and G. W. Mullins. The Macmillan Company, New Yorks, 1929. viii+312 pages+14 pages of answers.

In some respects this textbook does not follow the usual tradition of most texts in its field. We find more space devoted to the locus of an equation than is usually given. In fact it seems to the writer that a little too much space is given to this. The article on building up a locus is somewhat of an improvement in that the student solves two or more simple equations to locate points on the locus instead of using a more laborious method of solving the equation as a whole. The ingenious method of solving higher equations, using a graphical method that is essentially Horner's method, gives the student an unusual insight into this rather complicated subject.

The article on a circle determined by three conditions is clear and timely as is the one on whether a point lies on, within, or outside a circle.

The treatment of conics is very brief, much more so than usual, and might be expanded without detriment, though the simultaneous treatment of the ellipse, hyperbola, and parabola saves much explanation on the part of the instructor.

The use of derivatives in finding the equations of tangents to algebraic curves is very appealing. Indeed it is a source of wonder that more texts have not made use of this terminology instead of the cumbersome method of finding the secant through the given point and then making the second point of intersection approach coincidence with this given point.

More use might have been made of Boscovich's definition of conics, especially bringing out the idea that the e used in the derivation of the equation of the conic is the eccentricity, also that the circle is a conic where e equals 0.

That part of the book devoted to solid analytics is brief, as it should be. Few teachers will find it possible to take up even this much in a single semester course of three hours per week.

Taking the book as a whole it seems to be very teachable but rather long for a one semester course.

J. A. CRAGWALL

Elementare Geometrie, von W. Schwan, Studienrat am Prinzgeorggymnasium zu Düsseldorf. Erster Band: Die Ebene, xviii+402 pp. Leipzig 1929, Akad. Verlagsges. M. B. H.

The first volume of a treatise on elementary geometry belongs to the series of monographs and text books on mathematics and its applications edited by Professor E. Hilb of the University of Würzburg. In the preface Professor Dehn gives an outline of the character and scope of the book, which in the introduction is supplemented by the author himself.

He promises to continue in a second volume in a similar manner the treatment of elementary and affine geometry of space, and finally, in a third volume he intends to draw the conclusions which will lead from affine to projective geometry.

The book is divided into 5 parts with 14 chapters. First under the title of "figure and transformation," Schwan writes about the elements of space, symmetry, congruence, and similitude, and introduces the group concept which in connection with the axiomatic bases dominates the whole structure. In the next two parts he establishes a sect and vector calculus with its implications and connection with coordinate and complex geometry. The fourth part deals with the transition from elementary to affine geometry, and in conclusion it is shown how plane geometry is derived from spatial considerations.

Taken as a whole, the treatise is a happy combination of the customary elementary type of geometric textbooks and recent pedagogical methods of presentation with the purely axiomatic standpoint and the modern geometric ideas which, principally under the influence of Klein, have been of the utmost importance in the development of geometry.

In this respect the book will be of great value to the teacher in secondary schools in so far as it will show him what is essential in elementary geometry and what its dominating features are when considered from a more advanced standpoint.

The author presumably did not intend the book to be used as a text in elementary instruction, since some of the axioms would appear as clever puzzles to the pupil. Consider for example the three axioms on page 52 upon which plane movements are based. The third states that there is just one

movement which will move a half-ray S into some other half-ray S' in the same plane. This is not at all obvious and not freely chosen; but it is known beforehand by the author that it will work, because it can be proved by elementary geometry that it is true in the Euclidean plane.

I add a simple proof for the amusement of the reader: Let A and A' be the finite ends of S and S'. At these points erect perpendiculars to S and S' respectively, which will intersect in a point B. Draw the perpendicular bisector of AA' which will cut the circle on $AA'B$ in two points one of which will be the center of rotation of S into S'.

The typography and general make-up of the book is excellent. But some of the figures, especially those showing architectural examples of symmetry are very poorly drawn, although the names of the draftsmen appear on the title-page.

ARNOLD EMCH

Theoretical Mechanics. By Joseph Sweetman Ames and Francis D. Murnaghan. Ginn and Company, Boston, 1929. ix+462 pages. $5.00.

This text book on advanced mechanics aims to meet the needs of the present day student of mathematical physics who will proceed from the study of classical mechanics to the recently developed theories of relativity and quantum mechanics. The choice of topics discussed, as well as the relative emphasis and point of view are all well suited to this end. Furthermore the authors have attempted to write so that the book "may be read by any competent student without the aid of an instructor," and have been almost more successful than could be hoped when one considers the large amount of material treated.

We shall briefly indicate the contents. This book begins with a lengthy chapter on vector analysis. Of particular interest to the mathematical reader, we may note the complete manner in which the derivation of the physical interpretation of divergence and curl, as well as the transformations of these expressions to orthogonal curvilinear coördinates are carried out. Vector methods and notation are freely used throughout the book, though not to the exclusion of other notations. After a chapter on kinematics, dynamics is introduced with a statement of Newton's laws of motion. These laws are followed by a very readable discussion of their precise content, and an outline of an equivalent set of postulates. Then follow four chapters on the detailed application to particle dynamics, a very brief one on impulsive forces and two on the rigid body. The second of these, which discusses the gyroscope, is particularly well written, presenting the fundamental principles in a form available for engineering applications as well as the integration of the differential equations of motion, The next chapter deals with Lagrangian and Hamiltonian coördinates, with application to impulsive forces. The derivation of Lagrange's equations is possibly too brief, as the proof is only sketched for two or more degrees of freedom, reference being made to the treatment of the one dimensional case.

Furthermore, the fact that the q_i and $\dot{q}_i$ are taken as independent in T, which is a stumbling block for students meeting this derivation for the first time, is never explicitly mentioned. After a chapter consisting of an excellent treatment of small oscillations, there follows one in which the Hamilton-Jacobi equation is introduced and interpreted. A chapter is then devoted to forms of the differential equations of motion applicable to non-holonomic systems, and the discussion of general dynamical theory ends with a chapter on methods of integration, which includes the application of integral invariants and Poisson brackets.

The four chapters which conclude the book deal with rather isolated topics. One deals with the Newtonian potential function, and handles it briefly but fairly completely. The chapter entitled "wave motion" presents the derivation and integration of the equation of the vibrating string and the telegraph equation, and ends with a discussion of the conditions which determine a solution of the general wave equation. The next one is devoted to the special theory of relativity, and takes the concept of a four-vector as an excuse for introducing the definitions and notations of tensors. The last chapter gives a short account of dimensional analysis.

Throughout the book, the presentation is unusually clear, and while at times the discussions are closely written, the conscientious student should be able to follow the argument without outside help. Particular pains have been taken to give an independent account of such mathematical facts as are needed, and are not ordinarily treated in our undergraduate mathematics courses. For example, the method of solving an ordinary linear differential equation with constant coefficients, the definition and elementary properties of the elliptic integrals of Legendre and the p-function of Weierstrass, and the deduction of a pair of quadratic forms to normal form are all included in the text. The standard of mathematical rigor is quite high, and, with the possible exception of a few places where infinitesimals are used, will satisfy the most fastidious. This is the more commendable in view of the too common attitude that all is fair in mathematical physics, and that an ordinarily careful mathematician has no need for his mathematical conscience when writing on physics.

At the end of each chapter a list of references to additional treatises and articles for further reading is given. The book is well supplied with problems, for the most part from traditional sources, which are well selected. A curious error occurs in problem 4, p. 327, which is presumably based on problem 1 at the end of chapter XI in Whittaker's *Analytical Dynamics*. The transformation as given by Whittaker is not (as stated) a contact transformation but (except for a minus sign) has the properties indicated. As modified in the present text, the transformation is not a contact transformation, and has not the required properties.

A possible correct statement of the problem is: Show that the transformation defined by the equations

$$Q_1 = \tfrac{1}{2}(p_1^2 - p_2^2 + \lambda^{-2}q_1^2 - \lambda^{-2}q_2^2),$$
$$Q_2 = \tfrac{1}{2}(p_1^2 + p_2^2 + \lambda^{-2}q_1^2 + \lambda^{-2}q_2^2),$$
$$P_1 = -\lambda/2 \text{ arc tan } (q_1/\lambda p_1) + \lambda/2 \text{ arc tan } (q_2/\lambda p_2),$$
$$P_2 = -\lambda/2 \text{ arc tan } (q_1/\lambda p_1) - \lambda/2 \text{ arc tan } (q_2/\lambda p_2),$$

is a contact transformation, and that it reduces the dynamical system whose Hamiltonian function is $\frac{1}{2}(p_1^2+p_2^2+\lambda^{-2}q_1^2+\lambda^{-2}q_2^2)$ to the dynamical system whose Hamiltonian function is Q_2.

The notation for equations is at first sight confusing to one familiar with the system of Peano, since, for example, the equation numbered "(3.2)" in the present text is not the same as that numbered "(3.20)" but rather the earlier equation which would be designated by (3.02) in the Peano system.

In conclusion, it is hoped that the few criticisms made on minor points will not be given undue weight; particularly as in most respects the reviewer considers the text under discussion better suited to the average American student of advanced mechanics than any other with which he is acquainted.

PHILIP FRANKLIN

An Editor's Note on a Review of the Emery and Jeffs "Algebra for Secondary Schools."

Mr Stephen Emery, one of the authors of Emery and Jeffs' Algebra for Secondary Schools, has raised the question whether the review[1] of that book in the March issue of the Monthly did full justice to the book. The department editor has undertaken to look into the matter and to arrive at a reasonable compromise between the points of view of the authors and of the reviewer.

The statement made in the review, that "logarithms and numerical trigonometry are not made integral parts of the book but are detached in the appendix" seems to have been a slip. The book has no appendix; the chapters in question are at the end of the book, but are not separated from the rest of the text in any way. It also appears that the full Euclidean method for the highest common factor is not, as was stated in the review, to be found in the book.

The review stated that "the book departs drastically from the recommendation of the National Committee that the function concept be made the keynote of mathematics." In the opinion of Mr. Emery, " it is true that the words function and functionality are not used with extreme frequency, but the concept is an essential and prominent part of the warp and woof of a large part of the book." It is the view of the present writer that the function concept is the head of King Charles which cannot possible be kept out of any mathematics text; but that perhaps we have had a tendency to too extensive harping on the word function. Whether the book under consideration presents adequately the spirit of the function concept is a question of opinion and judgment on which our author and our reviewer differ honestly.

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[1] This Monthly, vol. 36 (1929), p. 164.

Several criticisms by the reviewer as to the selection of material are questioned by the author. In the judgment of the editor, these are matters of opinion to a considerable degree, and subject to the principle "de gustibus non disputandum." Most of the criticisms indicated the inclusion of too much material, and omission is therefore always possible. The treatment of variation, found rather slight by the reviewer, is considered by the author as "condensed but entirely adequate." One wonders how long it will be before the treatment of variation as a separate topic in our algebras will have gone the way of cube root, alligation alternate and medial, and other useless lumber.

The editors of the Monthly do not wish to foment acrimonious debate concerning reviews, but are very desirous that full justice be done to every book reviewed and that every viewpoint be fully recognized. In the present instance it is clear that the reviewer was in error as to one or two matters of fact, and that her judgment differed from that of the authors at several points. As it has been said that difference of opinion is the motivating factor in the sport of horse-racing, so these differences of opinion will stimulate interested readers to examine the book for themselves and form their own conclusions.

R. A. J.

MATHEMATICS CLUBS

EDITED BY H. J. ETTLINGER, University of Texas, Austin, Texas.

All reports of club activities should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES

The Napierian Club, DePauw University, Greencastle, Indiana.

The officers for the year 1928–1929 were: Alfred Vaughan ('29), President; Louise F. Davis ('29), Vice-President; Geneva Annis ('29), Secretary; J. Merle Harris ('29), Treasurer.

The following programs were given at the regular monthly meetings:

October 18, 1928. Election of new members. "Solution of equations" by Paul Sharp.

November 8. "The history of the Napierian Club," by Mary E. Cline, "The life and works of John Napier," by Geneva Annis.

December 13. "History of the magic square," by J. Merle Harris; "Construction of the magic square," by Edward Sights.

January 10, 1929. "History of the decimal point," by Robert Hixson; "Squaring of a circle," by Edgar Young.

February 14. "Graphic method of solving quadratics," by Paul Godwin; "Calculating prodigies," by Russell Nichols.

March 14. "Proofs for the Pythagorean proposition," by Russell Rosenkrans; "Proofs for the impossibility of trisecting an angle," by Karen Ita Cooper.

April 11. "Mathematics in the development of the music scale," by Professor H. E. H. Greenleaf; "Continuous functions that do not have derivatives at some point," by Professor W. C. Arnold.

May 9, 1929. "Four-fours problem," by Charles Stunkel. Election of officers.

(Report by Miss Annis)

The Mathematics Club of the North Carolina College for Women, Greensboro, N. C.

The officers of the club for the year 1928–1929 were: Sallie Spratt ('29), President; Louise Leary ('30), Vice-President; Olive Renfroe ('30), Secretary-Treasurer; Edith Allie ('29), Chairman of the Program Committee; Professor C. Strong, Faculty Adviser.

The club held monthly meetings with the programs as follows:

October. Business meeting: drawing up and adopting a new constitution.

November. "Famous theorems of geometry," by Virginia Tucker, Margaret Redwine, Alma Smith.

December. "Geometrical symmetry," by Mary Kapp, and "Algebraic symmetry," by Roxana Yancy.

February. Initiation of new members. Social.

March. "Einstein's theory of relativity," by Elizabeth Hall.

April. "History of Women in mathematics," by Mary Dranghan; "Mathematics as a vocation for women," by Mrs. C. G. Woodhouse, Vocational Director.

May. Election of officers. The officers for 1929–30 are: Mary Kapp, President ('30); Margaret Redwine ('30), Vice-president; Mary Welch Parker ('31) secretary-treasurer.

(Report by Olive Renfroe)

Tuftconic Club, Tufts College, Mass.

The officers for the year 1927–1928 were: Richard Tousey ('28), President; Cora B. Harlow ('29), Vice-President; Helen G. Murray ('28), Secretary; Sumner Harwood ('28), Treasurer.

The program for 1928–1929 included the following topics:

February 16, 1928. "Maps," by Professor William R. Ransom.

March 22. "The function of a complex number," by Merrill Orswell, '26.

April 24. "Mechanically-drawn curves," by Richard Tousey, '23.

May 15. Picnic at Nahant Beach. The following officers were elected for the year 1928–1929; Burnham L. Paige ('29); Cora B. Harlow ('29), Vice-President; Eleanora L. Czerniewska ('31), Secretary; Dorothy L. Stone ('29), Treasurer.

May 31. "Euclid's beginning," by Professor William R. Ransom.

October 25. "Beginnings of non-Euclidean geometry," by Dorothy L. Stone, '29.

November 13. "Projective geometry and Pascal's hexagon," by Dr. William Fitch Cheney.

December 5. "The trisection of an angle," by Dean Frank G. Wren.

January 9, 1929. "Empirical equations," by Professor Titus E. Mergendahl.

February 7. "Methods of factoring large numbers," by Dr. William Fitch Cheney.

The club holds its meetings once a month during the academic year, the last meeting being in the form of a picnic at which officers for the coming year are elected.

(Report by Eleonora L. Czerniewska)

The Alabama Chapter of Pi Mu Epsilon.

Officers for the chapter for 1927–28 were elected as follows: Professor Fred A. Lewis, Director; William F. Adams ('27), Vice-Director; Margaret Perry ('28), Secretary; Vernon Leftwich ('28), Treasurer; Sarah E. Haughton ('29), Librarian.

Meetings were held as follows:

October 31, 1927. "Simpson's rule and its application," by William Adams, '27.

December 18. Initiation of new members followed by a party at the home of Director and Mrs. Lewis.

February 6, 1928. "The Jacobian," by Margaret Perry, '28.

February 27. "Inertial integrals," by Mrs. J. C. Nixon.

March 26. "Trilinear coordinates," by Jean Lang Kitchell, '28.

April 30. Picnic and initiation of new members.

(Report by Margaret Perry)

The Missouri Beta Chapter of Pi Mu Epsilon, Washington University, St. Louis, Mo.

Nine regular meetings were held during the year as follows:

Oct. 25. "Map projections," by Mr. H. R. Grummann.

Nov. 25. Reception for visiting members of the Southwestern Section of the American Mathematical Society and the Missouri Section of the Mathematical Association of America. Professor E. B. Stouffer, of Kansas University, gave an exceptionally interesting address on the study of mathematics in contemporary Italian universities.

Dec. 13. "Algebra for blind students," by Mr. Harry Bauer.

Jan. 23. "Numerical differentiation and integration," by Professor Jessica Young.

Feb. 23. "A defective proof in our geometries" by Professor Otto Dunkel.

Mar. 21. Business Meeting. Election of new members.

Apr. 13. Annual banquet and initiation of twenty-two new members. On this occasion Mr. W. O. Pennell, Chief Engineer of the Southwestern Bell Telephone Company, gave an address, illustrated by lantern slides, on "Telephoto." Each member was presented with an excellent photograph of Col. Chas. Lindbergh which had been sent to St. Louis over the lines of the Bell system.

Apr. 27. "The relation of geometry to physics," by Professor Frank Bubb.

May 18. On this occasion the members were guests of Dr. Young, the Director of the chapter. Mr. George Harvey spoke on "The theory of linear dependence." The following officers for the year 1928–29 were elected: Mr. W. O. Pennell, Director; Miss Elizabeth L. Harris, Vice-Director; Dr. Jessica M. Young, Secretary; Miss Amy R. Claus, Assistant Secretary; Mr. H. R. Grummann, Treasurer; Mr. George A. Harvey, Librarian; Miss Bernice Hosch, Miss Pearl Schukor, Mr. Harold J. Miller; Mr. Wm. L. Knaus, Members of the Executive Committee.

(Report by Professor Jessica M. Young)

The Hunter College Chapter of Pi Mu Epsilon.

The year of 1927–1928 was a most profitable one for the Hunter College Chapter of Pi Mu Epsilon. Four program meetings were held each semester. In commemoration of the two hundredth anniversary of Newton's death the members undertook to study during the first semester Newton's contributions to Science. The second semester was devoted to a study of the Theory of Numbers. Each student member presented a short report of about ten minutes duration. The papers thus presented were not independent of one another, but were designed so that those presented at each meeting formed a connected unit. The interest of the members and the success of the meetings were clearly demonstrated by the enthusiasm with which the reports were prepared and presented and the large attendance at each meeting.

Following an established precedent, a prize was awarded to the best speaker of each semester. A prize of a book, *Sir Isaac Newton*, was presented to Lillian Abramowitz, who was judged to be the best speaker of the Fall semester. The best speaker of the Spring semester, Natalie Birnkrant, was awarded a copy of Goursat-Hedrick's *Mathematical Analysis*.

Four social meetings were held during the year. On October 15th, an initiation dinner was held at Standish Hall, at which the members had the pleasure of hearing addresses by Professor Fort and Professor Fite. A less formal initiation was held February 18th. On April 22nd the Faculty invited the student members to tea. A bridge party for graduating members was held June 11th at the Hotel Woodward.

There were 46 active members during the Fall semester and 50 during the Spring semester. The officers for the year were:

Director: Professor Walker (*Fall*), Dr. Weisner (*Spring*); Vice-director: Jean Hutchinson (*Fall*), Bertha Boschwitz (*Spring*); Recording Secretary: Helen Dauenhauer (*Fall*), Anna Kenny (*Spring*); Corresponding Secretary: Frances Rice (*Fall*), Lillian Abramowitz (*Spring*); Treasurer: Marcella Votava (*Fall*), Frieda Talmey (*Spring*).

Pi Mu Epsilon and the White Mathematics Club of the University of Kentucky, Lexington, Ky.

The meetings of Pi Mu Epsilon and the White Mathematics Club alternated during the year. The combined program was as follows:

October 27, 1927. "The number system," by Professor Claiborne G. Latimer.

November 10. "A sufficiency condition for lower semicontinuity of $S_cF(x, y, x', y')ds$," by Professor H. H. Downing.

December 1. "Mathematical considerations in Spengler's *Decline of the West,*" by Professor J. M. Davis.

December 15. A report on the Convention of Pi Mu Epsilon at Nashville, by Dean P. P. Boyd; "Some applications of differential equations," by Professor D. E. South.

January 15, 1928. "On sketching certain curves," by Professor E. L. Rees.

February 9. "Absolute or infinite region of geometry," by Mr. Marion Brown. "Present Day Mathematics in Italy," by Professor H. H. Downing.

March 15. "On cubic congruences," by Professor C. G. Latimer.

April 12. "Cryptography," by Miss Sallie Pence; "Some Russian mathematicians," by Professor C. G. Latimer.

April 26. "Higher singularities," by Mr. M. B. Tolar.

April 26. "Integration by parts," by Miss Ethel Botts; "Chinese mathematics," by Mr. W. W. Chambers.

(Report by Elizabeth LeStourgeon)

The Mathematics Club of Hunter College of the City of New York.

The year just closing has been an active and interesting one for the Mathematics Club. Fortnightly meetings were held at which a series of topics too long to be printed in full were discussed. Outstanding among the papers presented were: Infinite series, and the properties of the trigonometric, exponential and logarithmic functions of a complex variable; Newton's interpolation polynomial, and several types of practical application; The normal frequency-distribution function for a single variate, and the adjustment of precision observations; Curve fitting by the method of averages, and by the method of least squares; Elements of the algebraic theory of quaternions; Mathematics and music. Mr. Arne Fisher, statistician of the Western Union, lectured to the club on "Certain topics of probability theory." There were two important social events—a reception and tea in the fall term, and a bridge party in the spring term.

The officers for the year 1928–1929 were: Lillian Glass, President; Miriam Fassler, Vice-President; Miriam Schechter, Secretary; Margaret Hosey, Treasurer; Muriel Rosner, Publicity Manager; Prof. Lester S. Hill, Faculty Adviser.

The three Annexes to the Main Building also did active club work. The lively interest in these various centers insures a successful club when the students become juniors and seniors. The 85th Street Mathematics Club consists entirely of lower freshmen. These are prospective majors. They have had talks from their own number and from members of the staff on such topics as the transit, physics or mathematics, mathematical paradoxes; they have presented a play, "The Mock Trial of A versus B," a dramatization by a former Hunter Student of Stephen Leacock's story "A, B, C;" they were entertained by the mathematics instructors toward the end of the term at a tea.

At the Annex in Brooklyn, one member of the club prepared her own slides for a talk on the history of numbers. The wide area of the mathematical field is shown by the list of papers presented here: Function concepts of mathematics; Fourth dimensional space; Fabre, entomologist and mathematician; Algebra in the American Colonies; History and transcendence of π; Extension of the number system; Mathematics and music; Nature of mathematics; Some unsolved problems; Four dimensional geometry. With the exception of three papers, the topics were presented by students.

A similar diversity of interest is shown in the work at the third Annex. The Department is confident that the Club work contributes materially to the success of its students in the class room.

(Report by Professor L. G. Simon)

The Mathematics Club of Wellesley College, Wellesley, Mass.

The Program of the Mathematics Club of Wellesley College for the year 1928–29 was as follows:

May, 1928. Election of Officers: Esther Neubrand ('29), President; Mildred Shineman ('29), Vice-President; Celia Russel ('29), Treasurer; Elsie Franck ('30), Secretary; Evelyn Bristol ('30), Junior Executive.

October 12, 1928. An account of the meeting of the Mathematical Association of America at Amherst, by Professor Lennie P. Copeland and Professor Clara Smith of Wellesley College. "Mathematics at the Sorbonne, France," by Eleanor Moise, '29.

November 23. "Chance in a game of dice," by Mildred Shineman, '29; "The nine point circle," by Celia Russel, '29, "Zeno's paradoxes," by Elsie Franck, '30; "Rule of single and double false position," by Esther Neubrand, '29.

January 25, 1929. Informal meeting in the Treasure Room of the library to look at the many 16th, 17th, and 18th century mathematic books.

February 15. "Some elementary properties of integers," by Professor Brinkman of Harvard.

April 5. "Four systems of numbers," by Professor Clara Smith of Wellesley College. "The straight edge," by Professor Marion Stark of Wellesley College.

May 17. Election of Officers: Elsie Franck ('30), President; Muriel Fuller ('30), Vice-President; Frances Kauffman ('30), Treasurer; Melita Holly ('31), Secretary; Adelaide Newman ('31), Junior Executive.

(Report by Elsie Franck)

Phi Chi Mu Honorary Science Club of Washington & Jefferson College, Washington, Pa.

As Washington & Jefferson College is a small "Arts College," limited to 500 men, all in regular course, it has seemed best to have one science club, including all honor men of the two upper years in mathematics, physics, chemistry, and biology; although most of the members are majoring in mathematics.

The officers for the year were: A. B. Bowden, President, Paul Jose, Secretary-Treasurer.

The program for the year was as follows:

November 6, 1928. "Color photography," by Dr. Malecot.

December 11. "Discussion of Einstein's work," by George Schweigert.

January 15, 1929. "Fourth dimension," by A. B. Bowden.

February 5. "Map projection," by Paul D. Jose.

March 5. "Gels," by John Dorn.

April 9. "The mathematics of chemistry," by F. B. Durigg.

May 7. "Duality in mathematics," by J. A. John.

(Report by Professor Atchison)

The Denison Mathematics Club, Denison University, Granville, O.

September 27, 1927. The duo-decimal system, by Miss Tippett; Magic squares, by Al Bakeman; Large numbers, by Dr. Wiley.

October 11, 1927. Charts, by Dr. Wiley.

October 25. The nine point circle, by Miss Peekham.

November 12. Galois number fields, by Mr. C. C. MacDuffee. (Joint meeting with Engineering Society.)

November 29. Napier's bones, by Donald Fitch.

December 13. Christmas Party.
January 10, 1928. The wonders of the heavens, by Dr. Biefeld.
January 31. Arithmetical progressions, by Lillian Dallman.
February 21. Units and dimensions as used in physics, by Dr. Coons.
March 6. Mathematics in chemistry, by Dr. Ebaugh.
March 20. Regular business meeting.
April 10. Units and dimensions as concerned in engineering, by Mr. Greenshields.
April 24. Complex numbers, by Mr. Kato.
May 8. Tests and measurements in educational computations, by Miss Wood.
May 22. Business meeting.
June 1. Club banquet.

(Report by Miss Deeds)

The Undergraduate Mathematics Club of the University of Iowa, Iowa City, Iowa.

The officers for the year were: Prof. L. E. Ward, Faculty Adviser; Mr. Neal H. McCoy, President; Mabel Huber, Secretary-Treasurer.

The program for the year was as follows:
October 4, 1928. Mr. Fred Reusser talked on "Euler and his works."
October 18. "The life and works of Gauss," by Mr. C. C. Sherman.
November 14. "Contemporary American Mathematicians," by Professor H. L. Rietz.
December 13. "The life and works of Laplace," by Mr. Milton J. Goldberg.
February 7, 1929. "The graphical solution of quadratic, cubic, and bi-quadratic equations," by Professor L. E. Ward.
February 21. "Some geometrical transformations," by Mr. Paul Trump.
March 7. "Some elementary invariants," by Mr. Allen T. Craig.
March 21. "A comparison of coordinate systems," by Mr. Wilbur B. Cliff.

Each talk was preceded by a short social meeting. The following officers were elected for 1929–1930: Professor L. E. Ward, Faculty Adviser; Mr. Wilbur B. Eliff, President; Miss Fern Barr, Secretary-Treasurer.

(Report by Mabel Huber)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3398. *Proposed by V. Ivanoff, San Francisco, California.*

Prove that

$$\frac{1}{R^2} = \frac{\sin^6 M}{R_x^2} + \frac{\sin^6 N}{R_y^2} + \frac{\sin^6 P}{R_z^2},$$

where R is the radius of curvature of a given curve in the point $(x_1y_1z_1)$; R_x, R_y, R_z are the radii of curvature of the projections of this curve in the co-ordinate planes, YOZ, XOZ, and XOY in the points (y_1, z_1), (x_1, z_1), and (x_1, y_1), respectively; M, N, P are the angles between the tangent to the curve in the point (x_1, y_1, z_1) and OX, OY, and OZ, respectively.

3399. *Proposed by B. C. Wong, Berkeley, California.*

Prove or disprove

$$\sum_{i=0}^{t}(-1)^i\binom{r+1}{i}\binom{2r-2i}{r} = r+1,$$

where $t=r/2$ if r is even and $t=(r-1)/2$ if r is odd.

3400. *Proposed by W. O. Pennell, St. Louis, Mo.*

Find a function of x, $f(x)$, such that

$$f(x) = xf(x-r) = x(x-r)f(x-2r) = x(x-r)(x-2r)f(x-3r), \quad \text{etc.}$$

and $f(r)=r$, where r is a given real quantity >0.

3401. *Proposed by Paul Wernicke, Washington, D. C.*

Let a, b, c, d, be four lines in a plane no three of which are concurrent. Let f be the join of the intersections ac and bd and g the join of the intersections of ad and bc. Prove that $(\sin af/\sin fc)/(\sin ag/\sin gd) = (\sin bg/\sin gc)/(\sin bf/\sin fd)$.

3402. *Proposed by S. A. Corey, Des Moines, Iowa.*

Let $f(y)=0$ be a numerical equation, algebraic or transcendental, a solution of which is sought. If $x=1$ this equation becomes identical with

$$f(y) + g(y) - xg(y) = 0, \tag{1}$$

where $g(y)$ may be any function of y whatsoever. If y_0 be so chosen that $f(y_0) + g(y_0)=0$, y_0 being preferably an approximate value of the root of $f(y)=0$ sought, y may be developed by (1) by infinite series in terms of x by Maclaurin's formula (Lambert's method). But the form of $g(y)$ is a matter of choice. Give a rule for finding such a form of $g(y)$ as to render the development of y in terms of x by Maclaurin's formula the most rapidly convergent.

NOTE: A prize of \$10 is offered by the proposer to the person sending in a solution or comment most enlightening, in the judgment of the editors.

3403. *Proposed by E. A. Whitman, Carnegie Institute of Technology.*

If in problem 3379, proposed in May, 1929, \$12 is replaced by "a" dollars, for what values of "a" is there a solution?

Problem 3379 reads: Two men own jointly x cows which they sell for x dollars per head, and with the returns buy sheep at \$12 per head. As their income from the cows is not divisible by 12 they purchase a lamb with the

remainder. Later they divided the flock so that each had the same number of animals. How much money was due the man with the lamb by the other man?

SOLUTIONS

3334[1928, 377]. *Proposed by James Singer, Graduate College, Princeton, N. J.*
Evaluate

$$\operatorname*{limit}_{n\to\infty}\left[\frac{1}{2^n}\left(\sum\frac{e_1}{2^{2\cdot1-e_1}}+\frac{e_2}{2^{2\cdot2-(e_1+e_2)}}+\cdots+\frac{e_n}{2^{2n-(e_1+e_2+\cdots+e_n)}}\right)\right],$$

where $e_i=0$ or 1, and the sum is extended over all possible choices of the e's; i.e., the sum of the n terms where all the e's except one are zero, plus the $n(n-1)/2$ terms where all the e's except two are zero, $\cdots$, plus the single term where all the e's are unity.

Solution by Schieffelin Claytor, Washington D. C.

It will be convenient to find first the sum of all the rth terms for a given n, where $e_r=1$, i.e., the sum of all the terms $1/2^p$, where $p=2r-(e_1+e_2+\cdots+e_{r-1}+1)$. For a given set of values for $e_1, e_2, \cdots, e_{r-1}$ this term will occur 2^{n-r} times. Also if K is an integer, $0\leqq K\leqq r-1$, there are ${}_{r-1}C_K$ ways of choosing unit values for the e's so that $e_1+e_2+\cdots+e_{r-1}=K$. Hence the sum of all the rth terms is

$$\sum_{K=0}^{r-1}\frac{2^{n-r}{}_{r-1}C_K}{2^{2r-K-1}}=2^{n-3r+1}\sum_{K=0}^{r-1}2^K{}_{r-1}C_K=2^{n-2}\left(\frac{3}{8}\right)^{r-1}.$$

Taking the sum of the above results for $r=1, 2, \cdots, n$ and dividing this sum by 2^n, we have

$$\frac{1}{4}\,\frac{1-\left(\dfrac{3}{8}\right)^n}{1-\dfrac{3}{8}}=\frac{2}{5}\left[1-\left(\frac{3}{8}\right)^n\right].$$

and hence the desired limit is 2/5.

3351 [1929, 494]. *Proposed by C. D. Smith, Louisiana College.*

Assume $AOBC$ to be an octant of a sphere of radius R and center O. Find the radius of the inscribed sphere. Also find the radius of a sphere which is tangent to the two spheres and to two planes of the octant.

Solution by J. H. Neelley, Carnegie Institute of Technology.

If r is the radius of the sphere inscribed in the octant, the coördinates of the center must be r, r, r. Since this sphere touches the sphere of radius R, we must have $r/(R-r)=3^{-1/2}$ or $r=(1+3^{1/2})^{-1}R$.

The second sphere will be assumed to be tangent to $y=0$ and $z=0$; and, if its radius is k, the coördinates of its center are h, k, k. Since it is tangent to

the sphere R, we have $(R-K)^2=h^2+2k^2$. Since it is tangent to the sphere r, we have $(h-k)^2+2(k-r)^2=(k+r)^2$. This pair of equations gives the desired value $k=(1+3^{3/2})^{-1}R$.

3356 [1928, 564]. *Proposed by L. S. Johnston, Pennsylvania State College.*

Consider a triangle ABC, with angles at A and B acute and the angle at A greater than the angle at B. Let A', B', and C' be the feet of the respective perpendiculars from A, B, and C to the opposite sides. Let the perpendiculars from A to the lines $A'B'$ and $A'C'$ meet those lines at R_1 and R_2, respectively; let the perpendiculars from B to the lines $B'C'$ and $B'A'$ meet those lines at S_1 and S_2, respectively; let the perpendiculars from C to the lines $C'A'$ and $C'B'$ meet those lines at T_1 and T_2 respectively. Show that the conic with A' and B' as foci and S_2 and R_1 as vertices is tangent to AB at C'; that the conic with B' and C' as foci and T_2 and S_1 as vertices is tangent to BC at A'; that the conic with C' and A' as foci and R_2 and T_1 as vertices is tangent to CA at B'.

Solution by Rufus Crane, Ohio Wesleyan University.

In any triangle, the vertices and the orthocenter are the excenters and incenter of the orthic triangle; if the given triangle is acute angled, its orthocenter is the incenter of the orthic triangle, but if the given triangle is obtuse angled, the vertex of the obtuse angle is the incenter of the orthic triangle.[1] Hence, a circle with center at B, and tangent to $B'C'$ at S_1 is also tangent to $B'A'$ at S_2. Thus $B'S_1=B'S_2$. Similarly, $C'T_1=C'T_2$ and $A'R_1=A'R_2$. Also, if $A'B'$ be denoted by c', $B'C'$ by a', $C'A'$ by b', and their sum by s', then $A'R_2=s'-c'=B'S_1$ and $C'S_1+C'R_2=c'$. Thus, if the angles A and B are both acute, we have

$$B'C' + A'C' = B'S_2 + A'S_2 = B'R_1 + A'R_1.$$

But if one of these angles, say A, is obtuse, we have

$$A'C' - B'C' = B'S_2 - A'S_2 = A'R_1 - B'R_1.$$

Also, BA is the bisector of the angle external to $B'C'A'$ if both A and B are acute, of the internal angle if either B or A is obtuse. Hence, a conic with A' and B' as foci and R_1 and S_2 as vertices will pass through C' and have for tangent at C' the line BA, this conic being an ellipse if both B and A are acute, and a hyperbola if either B or A is obtuse.

Thus, in an acute angled triangle, all three of the conics described in this problem are ellipses; in an obtuse angled triangle, that one which touches the side opposite the obtuse angle is an ellipse, while the other two are hyperbolas.

Also solved by Nathan Altshiller-Court, C. A. Rupp, and A. Pelletier.

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[1] Durell's *Modern Geometry*, Theorem 21 is a special case of this theorem, but is not so stated. In Altshiller-Court's *College Geometry*, section 147 is a similar treatment of a related theorem. Richardson and Ramsey, in *Modern Geometry*, page 27, give the theorem correctly stated.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Rutgers University conferred the honorary degree of Doctor of Science upon Professor Richard Morris in June, 1929.

Associate Professor A. E. Cooper, of the University of Texas, has been promoted to a professorship of applied mathematics.

Professor W. V. N. Garretson has been appointed associate professor of mathematics at Oklahoma Agricultural and Mechanical College.

Mr. Charles Hatfield has been elected professor of mathematics at Georgetown College, Georgetown, Ky.

Associate Professor W. R. Hutcherson, of Berea College, has been promoted to the rank of professor of the department of mathematics.

Dr. Oystein Ore, of Yale University, has been promoted to a professorship of mathematics.

Associate Professor T. A. Pierce, of the University of Nebraska, has been promoted to a professorship of mathematics.

Associate Professor J. F. Reilly, of the University of Iowa, has been promoted to a professorship of mathematics.

Professor W. T. Stratton, of Kansas State Agricultural College, is spending a year of sabbatical leave in graduate work at the University of Washington.

Professor E. B. Wilson, of Harvard University, has been elected President of the Social Science Research Council. Professor Wilson has been granted leave of absence for the present academic year.

Dr. Charles Ranold MacInnes, associate professor of mathematics at Princeton University, died on September 29, 1929, at the age of 53.

Dr. R. M. Mathews, associate professor of mathematics at West Virginia University, died on October 20, 1929, after suffering for several years from a weakened heart. He was a charter member of the Mathematical Association.

Professor Fred Reusser, of Buena Vista College, Storm Lake, Iowa, died October 11, 1929.

The Fourth
Carus Mathematical Monograph

The Carus Monograph Committee is pleased to announce that the fourth number is now in process of publication and will soon be ready for distribution. The title of this Monograph is "Projective Geometry" by Professor JOHN W. YOUNG of Dartmouth College, now President of the Association. The preceding numbers are: (1) "Calculus of Variations" by Professor GILBERT A. BLISS; (2) "Analytic Functions of a Complex Variable" by Professor DAVID R. CURTISS, (3) "Mathematics of Statistics" by Professor HENRY L. RIETZ.

The price of these Monographs is $1.25 to institutional and individual members of the Association when ordered directly through the Secretary, one copy to each member; this is the bare cost of production. The price to all non-members of the Association and for all quantity orders for class use is $2.00 per copy, obtained only through the Open Court Publishing Company, 339 East Chicago Avenue, Chicago, Illinois, distributors to the general public of Association publications.

As heretofore, for the convenience of members, the forthcoming Monograph will be charged along with the bill for annual dues late in December. (This item may be cancelled in case it is not wanted.) New members and those who have neglected to subscribe for the previous numbers may still do so by ordering directly from the Secretary. As the series goes on, the complete list of Monographs will become more valuable if not indispensable to the individual library of an increasing number of members as well as to most college and all university libraries. It is gratifying to announce that the sales of the preceding numbers are continuing very favorably and that two of them have already gone to second editions. It would be still more gratifying if a larger proportion of members (now somewhat more than fifty per cent) should become regular subscribers to this Monograph series. Failure to do so is, doubtless, in many cases due to oversight or procrastination. Now is a good time to remedy such a condition.

Attention is called to the enlargement of the membership of the Monograph Committee by the addition of Professors AUBREY J. KEMPNER of the University of Colorado, and JOHN W. YOUNG, of Dartmouth College. The other members of the Committee are: Professor GILBERT AMES BLISS, of the University of Chicago; Professor DAVID RAYMOND CURTISS, of Northwestern University; and Professor HERBERT ELLSWORTH SLAUGHT, of the University of Chicago.

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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Thirteenth Summer Meeting of the Association, Boulder, Colorado, August 26-27, 1929.

Fourteenth Annual Meeting, Des Moines, Iowa, December 31, 1929, January 1, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for **1929**.

ILLINOIS, Carthage, Ill., May 3-4.

INDIANA, Culver Military Academy, May 3-4.

IOWA, Fairfield, Iowa, April 26-27.

KANSAS, Topeka, Kansas, February 2.

KENTUCKY, Lexington, Ky., April 13.

LOUISIANA-MISSISSIPPI, Lafayette, La., April 12-13.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, May 4.

MICHIGAN, Ann Arbor, Mich., March 16.

MINNESOTA, St. Paul, Minn., May 11.

MISSOURI, Kansas City, Mo., November 16

NEBRASKA.

OHIO, Columbus, Ohio, April 4.

PHILADELPHIA, University of Pennsylvania, November 30.

ROCKY MOUNTAIN, Greeley, Colo., April 12-13.

SOUTHEASTERN, Macon, Ga., April 19-20.

SOUTHERN CALIFORNIA, University of Redlands, March 9.

TEXAS, Houston, Texas, Jan. 26.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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The Chauvenet Prize

The ASSOCIATION in 1926 established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the ASSOCIATION. In 1928 Professor W. B. Ford, then president of the ASSOCIATION, gave $500 the income from which was to supplement the original fund so that the prize could be awarded every three years.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a liberal cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

The first award, covering the five years preceding 1925, was made to GILBERT AMES BLISS for his paper on "Algebraic Functions and their Divisors" published in the *Annals of Mathematics.* The next award will be for the four years preceding 1929, and thereafter the prize will be awarded every three years.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION,—one more of the many good reasons for membership.

INDIVIDUAL MEMBERS OF THE MATHEMATICAL ASSOCIATION OF AMERICA

Closed for printing November 20, 1929.

Note. This list gives data for the members of the association as taken from the registei issued during December 1927, unless changes in position, mailing address, etc., have been reported to the Secretary. Records may be inexact or incomplete through the failure of members to return fresh information to the Secretary, but great efforts have been put forth to be full and correct. When members are known to be occupied in other lines than mathematics, this fact is indicated.

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COLLEGE PARK. Alrich, Spann, Taliaferro.
EMITTSBURG. Burke.
FREDERICK. Arnold, Brown.
PORT DEPOSIT. Haviland.
ROLAND PARK. Morrow.
WESTMINSTER. Hart.

MASSACHUSETTS. (98)

AMHERST. Esty, Moore, Olds, Porter.
ANDOVER. Newton.
BELMONT. Douglass, Rutledge.

BOSTON. Bruce, Downey, Gould, Holbrook, Laurentine, Leavens, Littauer, Marsh, Mode, M. M. Smith, Spear, Wilson.
BROOKLINE. Miller.
CAMBRIDGE. Bailey, Beatley, Birkhoff, Bradley, Brown, Coolidge, Cope, Crum, Franklin, Gaylord, Graustein, Hollis, Huntington, Kellogg, Kennelly, Moore, Morse, Osgood, Passano, Peterson, Phillips, Rice, Robinson, Sauté, Sherman, Tyler, Walsh, Wilson, Woods, Wright, Zeldin.
DANVERS. Majella.
DORCHESTER. Davis, Quigley.
EVERETT. Bryant.
FRAMINGHAM. Bickford.
GLOUCESTER. MacNutt.
HOLYOKE. Moriarty.
JAMAICA PLAIN. Andrew, Schroeder.
LYNN. Evans.
MEDFORD. Cheney.
NATICK. Nutt, Willis.
NORTHAMPTON. Benedict, Munroe, Rambo, Wood.
PITTSFIELD. Washburne.
SOUTH HADLEY. Anderton, Doak, Martin, S. E. Smith.
SPRINGFIELD. Cook.
TUFTS COLLEGE. Mergendahl, Ransom.
VINEYARD HAVEN. Manning.
WELLESLEY. Copeland, Doughty, Merrill, C. E. Smith, Stark, Young.
WESTON. O'Donnell.
WILLIAMSTOWN. Agard, Dorwart, Hardy, Wells.
WOLLASTON. Dennison, Gardner.
WORCESTER. Brown, Gay, Lepeshkin, Melville, Morley, Rice, Wheeler, Williams.

MICHIGAN. (73)

ALBION. Davis, Evers, Field, Sleight.
ALMA. Clack.
ANN ARBOR. Anning, Ayers, Baten, Bolks, Bradshaw, Coe, Denton, P. Field, S. E. Field, Ford, Glover, Grant, Hildebrandt, Hopkins, Hughes, Karpinski, Kazarinoff, Love, Markley, Nyswander, Raiford, Rainich, Rood, Rouse, Running, Schorling, Selheimer, Wagner, Wilder.
BATTLE CREEK. Van Deusen.
BAY CITY. Shellenberger.
DETROIT. Baldwin, Borgman, Chalmers, Darnell, Folley, Frumveller, Johnston, McCarthy, Mullen, Nelson, Paula, Schoonover, Thome.
EAST LANSING. Crowe, Emmons, Grove, Olson, Plant, Powell, Speeker.
FLINT. Stout, Swanson.
HILLSDALE. Herron, Penrod.
HOLLAND. Lampen.
JACKSON. Handy.
KALAMAZOO. Blair, Everett, Walton.
MARQUETTE. Spooner.
MOUNT PLEASANT, Richtmeyer.
SPRING ARBOR. M. G. Smith.
STURGIS. Steirnagle.
YPSILANTI. Barnhill, Erikson, Lindquist, Lyman, Matteson.

MINNESOTA. (43)

COLLEGEVILLE. Winklemann.
MANKATO. Robbins.
MINNEAPOLIS. Beal, Brink, Brooke, Bussey, Carlson, Constantine, Dalaker, Gibbens, Guttman, Hart, Hartig, Jackson, Jensen, Kirchner, McEwen Ness, Priester, Rosskopf, Saibel, Scammon, Schnell, Shuman, Shumway, Thorp, Underhill.
MOORHEAD. Leonard.
NORTHFIELD. Gingrich, Nordgaard, Solum, White.
ST. PAUL. Alice Irene, Kingery, Moench, Morgan, Reuterdahl, Taylor, Wood, Yvonne.
ST. PETER. Rundstrom.
VIRGINIA. Hancock.
WINONA. Bogard.

MISSISSIPPI. (18)

AGRICULTURAL COLLEGE. Drummond, Fox, Walker.
BLUE MOUNTAIN. Hutchins.
CLEVELAND. Dale.
CLINTON. Price.
GRENADA. Harris.
HATTIESBURG. Byrne, Scott.
JACKSON. Babbitt, Mitchell, Smylie.
MERIDIAN. I. K. Smith.
STARKVILLE. Edmondson.
UNIVERSITY. Hume, Rees, Wunder.
VICKSBURG. Newton.

MISSOURI. (45)

CANTON. Ingold.
CAPE GIRARDEAU. Johnson, Knepper.
CARTHAGE. Murto, Robinson.
CLAYTON. Haertter.
COLUMBIA. Callaway, Ingold, Wahlin, Westfall, Wyant.
FAYETTE. Fleet.
FULTON. Christian, Sweazey.
KANSAS CITY. Briggs, Cutting, Luby, Pierson.
KIRKSVILLE. Cosby, Jamison.
MAYSVILLE. Saunders.
MEXICO. McCoy.
PARKSVILLE. Wells.
ROLLA. Hinsch.
SPRINGFIELD. Finkel, Gibson.
ST. CHARLES. Karr.
ST. JOSEPH. Burney.
ST. LOUIS. Dunford, Dunkel, Gerst, Huntington, King, Middlemiss, Muehlman, Nauer, Osborn, Rider, Roever, Shannon, Stephens, Young.
WARRENSBURG. Scarborough.
WEBSTER GROVES. Clarke, Pennell.

MONTANA. (7)

BUTTE. Bowersox, Snodgrass.
HELENA. Canning, Wible.
MISSOULA. Carey, Lennes, Merrill.

NEBRASKA. (21)

BURWELL. Opp.
CERESCO. Walker.
GRAND ISLAND. H. Anderson.
HASTINGS. McDill.
KEARNEY. Hanthorn.
LINCOLN. Brenke, Camp, Candy, Congdon, Flood, Gaba, Gossard, Howie, Jackson, Pierce, Runge.
OMAHA. Bettinger, Campbell, Frankish.
PERU. Hill.
YORK. Feemster.

NEVADA. (1)

RENO. Haseman.

NEW HAMPSHIRE. (16)

CONCORD. Conwell.
DURHAM. Bauer, Kimball, Slobin, Wilbur.
EXETER. Barber, Sweet.
HANOVER. Beetle, Bill, Brown, Forsyth, Mathewson, Perkins, Silverman, Wilder, Young.

NEW JERSEY. (44)

BELLEPLAIN. Durell.
EAST ORANGE. Robinson, Stanwick.
JERSEY CITY. J. P. Smith.
LAWRENCEVILLE. Kimball, Mikesh.
LEONIA. Gafafer.
MONTCLAIR. Mallory.
MORRESTOWN. Wood.
MORRIS PLAINS. Johnson.
NEWARK. Conkling.
NEW BRUNSWICK. Carlson, Hall, Huber, Morris, Nelson, Schoonmaker, Wilson.
PATERSON. Caster, Okean.
POINT PLEASANT. Palmié.
PRINCETON. Adams, Alexander, Craig, Eisenhart, Garabedian, Gillespie, Hille, Knebelman, Lefschetz, Merrill, Moore, Morse, Thomas, Veblen, Wedderburn, Wheeler, Wilson.
RUTHERFORD. McMackin.
SOMERVILLE. Moyle.
TRENTON. Colliton.
WESTFIELD. Meder.
WEST ORANGE. Roman.
WORTENDYKE. F. E. Smith.

NEW MEXICO. (7)

ALBUQUERQUE. Barnhart, Newson, Smysor.
EAST LAS VEGAS. Rodgers.
SILVER CITY. Hunter, Mickelson.
SOCORRO. Reece.

NEW YORK. (242)

ALBANY. Beaver, Birchenough, Do Bell, Lester, Maynard.
ALFRED. Seidlin, Starr, Titsworth.
ALLEGANY. McLaughlin.
ANNANDALE-ON-HUDSON. Garabedian, Phalen.
AURORA. Barbour, Carroll, Hollcroft.
BALDWIN. Grove.
BELLPORT. Walsh.
BINGHAMTON. Patten.
BRONX. Tanzola.
BROOKLYN. Angelica, Bergstresser, Berry, Bowden, Emery, Fleisher, Griffin, Koch, Kreines, Kullback, Langman, Lieber, Locke, Schuyler, Shorr, Sinkov, Thecla, Thompson, Whitford.
BUFFALO. Archer, Cusick, Gehman, Harrington, Montague, Pound.
CLINTON. Brown, Carruth, Ferry, Fitch, Patterson.
CORONA. Hanson.
DALE. Whaley.
EDGEMERE. Poritsky.
ELMIRA. Suffa, Wright.
FLUSHING. Lehmann, Oglesby, Pruslin.
FRIENDSHIP. O'Kean.
GARDEN CITY. Rice.
GENEVA. W. H. Durfee, W. P. Durfee, Hubbs.
HAMILTON. Aude, A. W. Smith.
ITHACA. Agnew, Baker, Boothroyd, Calkins, Carver, Dye, Gillespie, Hadlock, Horsfall, Hurwitz, Karapetoff, Lowenstein, Paradiso, Ranum, Roos, Snyder, Torrance, Trevor, Williams.
JAMAICA. Barrett.
KINDERHOOK. Magee.
MOUNT VERNON. Breckenridge.
NEWBURGH. Miller.
NEW YORK. Allen, Allison, Berger, Berkeley, Blair, Bradley, Brewster, Burdick, Burgess, Campbell, J. R. Clark, R. L. F. Clark, Cooley, Darkow, Dauenhauer, Doermann, Eckersley, Edmonson, Eisele, Everett, Farnum, Fiske, Fite, Flanders, Foster, Frankel, Frary, Fry, Gentzler, Gill, Graham, Harper, Hawkes, Henderson, Hill, Himwich, Hirsch, Hoyt, Hutchinson, Jablonower, Jackson, Joffe, E. H. Johnson, M. I. Johnson, R. A. Johnson, P. C. Jones, Kasner, Kryder, Kunte, Larkin, Linehan, Lutz, MacGregor, Maiden, Miller, Mirick, Molina, Mullins, Murray, Paaswell, Packer, Payne, Pedersen, Penn, Phillips, Plimpton, Pooler, Post, Pride, Quilty, Raudenbush, Reddick, Rees, Reeve, Ritt, Saurel, Schelkunoff, H. M. Schlauch, W. S. Schlauch, Schmall, Schub, Seely, Shaw, Shewhart, Siceloff, Simons, D. E. Smith, R. F. Smith, R. R. Smith, Spies, Thorne, Tilley, Turner, Upton, E. Walker, H. M. Walker, Webster, Wechsler, Weisner, Werner, Whitford, Wilder, Woodyard, Young.
NISKAYUNA. Male.
OLEAN. Lowry.
ONEONTA. Haseman, Vivian.
PARISH. Church.
POTSDAM. Rowe, Waltz.
POUGHKEEPSIE. Cowley, Cummings, Wells.
ROCHESTER. Betz, Gale, Harding, Holt, Long, Silberstein, Watkeys, Welton.
SCARSDALE. Mac Neish, Thiesmeyer.
SCHENECTADY. Burkett, Caruthers, Fox, Hussey, Lerch, Libman, Morse, Oergel, Snyder, Stokes, Ulrich, Vedder.

SPUYTEN DUYVIL. Dean.
SYRACUSE. Campbell, Carroll, Decker, Harwood, Lindsey, Secy. Pi Mu Epsilon Frat., Randolph, Roe, Sperry, Taylor.
TARRYTOWN. Putnam.
TROY. Crockett.
VALLEY STREAM. Henry.
WELLSVILLE. Lish.
WEST POINT. Echols.
YONKERS. Hubert, John, Yanosik.

NORTH CAROLINA. (27)

CHAPEL HILL. Browne, Cain, Henderson, Hill, Lasley, Linker, Mackie.
CHARLOTTE. O. M. Jones, Woodson.
DAVIDSON. Douglas, Mebane.
DURHAM. Elliott, Hickson, Rankin, Robison.
ELON COLLEGE. Amick.
GREENSBORO. Barton, Pegram, Ragsdale, Strong, Watkins.
GREENVILLE. Graham.
HOT SPRINGS. Meyer.
MARS HILL. Robinson.
RED SPRINGS. Shuler.
SEABOARD. Harris.
WINGATE. Hendricks.

NORTH DAKOTA. (5)

FARGO. Householder, I. W. Smith.
GRAND FORKS. Staley.
UNIVERSITY. Hitchcock.
VALLEY CITY. Meyer.

OHIO. (115)

ADA. Fairchild, Whitted.
AKRON. Bender, Leigh, Silberfarb.
ASHLAND. Black.
ATHENS. Borger, Reed.
BEREA. Baur, Dustheimer.
BOWLING GREEN. Kelly, Overman.
BLUFFTON. Hirschler.
CANAL WINCHESTER. Bareis.
CHILLICOTHE. Mathias.
CINCINNATI. Barnett, Brand, Hancock, Justice, Kersten, Kindle, Lubin, Merriman, Moore, Mullings, Rhodes, Salkover, E. S. Smith, Wilczewski, Yowell.
CLEVELAND. Boyce, Burington, Focke, Freas, Getchell, Johnson, Jonah, Justin, Morris, Musselman, Nassau, Sanford, Simon, Thomas, Trofimov.
COLUMBUS. Arnold, Beatty, Blumberg, Harmount, Hobensack, Horn, M. E. Jones, Kuhn, MacDuffee, Manson, Morris, Peterson, Preston, Rasor, Rickard, Sanders, Singer, Wildermuth.
DAYTON. Hartwich.
DEFIANCE. Caris.
DELAWARE. Crane, Newlin, Rowland.
GAMBIER. Allen, Redditt.
GRANVILLE. Bumer, Ladner, Peckham, Sheets, Wiley.
HILLIARD. J. H. Weaver.
HIRAM. Clarke, Jerome.
KENT. Freeman, Manchester.
MACEDONIA. Burwell.
MARIETTA. Coar, Rea.
MILFORD. Butler.
MOUNT ST. JOSEPH. Corona.
NEW CONCORD. White.
NORWOOD. Wishard.
OBERLIN. Cairns, Carr, Johnson, Sinclair, Smyth, Yeaton.
OXFORD. Albert, Anderson, Dewey, Erickson, Pollard, Spenceley, Tappan.
PAINESVILLE. Lewis.
ROSS. Haldeman.
SPRINGFIELD. Tripp.
TIFFIN. Pierce.
TOLEDO. Brandeberry, Dancer, Mercedes, Winslow.
WESTERVILLE. Glover.
WILMINGTON. Spinks.
WOOSTER. Knight, Williamson, Yanney.
YELLOW SPRINGS. Dwyer.
ZANESVILLE. Riesbeck.

OKLAHOMA. (24)

ALVA. Hall.
BETHANY. Mooré.
CHICKASHA. J. J. Miller.
MIAMI. Whitney.
NORMAN. Brixey, Court, Hassler, D. McFarland, E. McFarland, Meacham, Raynor, Reaves, Stehn.
SHAWNEE. Short.
STILLWATER. Barnett, Flanders, Garretson, Gundersen, H. W. Smith.
TULSA. Byrd, Howard, Mitchell, West.
WEATHERFORD. McCormick.

OREGON. (12)

CORVALLIS. Beaty, Johnson, Kirkham, Williams.
EUGENE. Davis, De Cou, McAlister, Milne.
McMINNVILLE. Ramsey.
PORTLAND. Griffin, Merriss, Short.

PENNSYLVANIA. (118)

ANNVILLE. Wagner.
ARDMORE. Macdonald.
BALA. Gummere.
BEAVER FALLS. Cleland, McCormick.
BETHLEHEM. Barnes, Fort, Keeler, Lamson, Rau, Reynolds, Smail, Van Arnam, Weida.
BRYN ATHYN. Allen.
BRYN MAWR. Anderson.
CALIFORNIA. Foberg.
CARLISLE. Ayers, Landis.
COLLEGEVILLE. Clawson, Veatch.
CYNWYD. Sensenig.
DEVON. Clarke.
EASTON. Hall, Hatch, W. M. Smith.
ELIZABETHTOWN. Myers.
ERIE. Benedicta, Ewing, Wells.
GROVE CITY. Ramsey.
HARRISBURG. Whited.
HAVERFORD. Reid, Wilson.
KUTZTOWN. Bordner.
LANCASTER. Charles, Long.
LATROBE. Seubert.

LEWISBURG. Gold, Lindemann, MacCreadie, Richardson, Youtz.
LINCOLN UNIVERSITY. Wright.
MEADVILLE. Akers.
MILLERSVILLE. Seiverling.
NORRISTOWN. Roulton.
PENN. Brezler.
PHILADELPHIA. Adkisson, Caris, Chambers, Crawley, Davis, Eshleman, Evans, Jackson, Kevles, Kline, Knedler, Linton, Lufkin, Mitchell, Partridge, J. H. Roberts, W. Roberts, Rosengarten, Rothermel, Safford.
PINE GROVE. McDonough.
PITTSBURGH. Alden, Baird, Bushyager, Geckeler, Hoover, Johnson, Kaltenborn, Mathews, Neelley, Olds, Riggs, Rosenbach, Spencer, Swartzel, Taber, Taylor, Walker, Whitman.
SCRANTON. Bertrand.
SELINSGROVE. Williams.
SEWICKLEY. Miller.
SHIPPENSBURG. Kieffer.
STATE COLLEGE. Cohen, Dunlap, Gravatt, Hamilton, Kunkel, F. W. Owens, H. B. Owens, Peterson, Rupp, Shibli, Wagner, West.
SWARTHMORE. Dresden, Marriott, Miller, Pitman.
SWISSVALE. Foraker.
UNIONTOWN. Beisel.
UPPER DARBY. McDonough.
WASHINGTON. Atchison, Bert, Rasel, Shaub, Thomas.
WAYNESBURG. Bond.
WEST PHILADELPHIA. Latshaw.
WILLIAMSPORT. Hoshauer, Smink.

PANAMA. (3)

PANAMA CANAL. Judd.
PANAMA CITY. Linares, Lyons.

PHILIPPINE ISLANDS. (6)

LAGUNA. Salvosa.
LEYTE. Icamen.
MANILA. Jimenez, Mills, Tan, Tienzo.

PORTO RICO. (2)

MAYAGUEZ. Sanchez.
RIO PEDRAS. Horne.

RHODE ISLAND. (17)

NEWPORT. Chase, Kimball.
PROVIDENCE. Adams, Archibald, Bennett, Carlen, Chace, Currier, Fuller, Gilman, Lehmer, Manning, Moskovitz, Richardson, Suesman, Tamarkin, Watt.

SOUTH CAROLINA. (16)

CHARLESTON. Bond, Coleman.
COLUMBIA. Coleman, Jackson, Moorefield, Williams.
GREENVILLE. Bowen, Earle, ReBarker, Wood.
GREENWOOD. Weber.
HARTSVILLE. Reaves.
ROCK HILL. Grant, Pugh, Wood.
SALUDA. Ramage.

SOUTH DAKOTA. (11)

BROOKINGS. Fuller, Miller.
GLENHAM. Stagner.
HURON. Titt.
MITCHELL. Knox.
RAPID CITY. Bowles, March.
SIOUX FALLS. Day.
SPEARFISH. Hesseltine.
SPRINGFIELD. Hoopes.
YANKTON. Faught.

TENNESSEE. (14)

CHATTANOOGA. Perry.
CLEVELAND. Ryno.
JACKSON. Carr, Walden.
JEFFERSON CITY. White.
KNOXVILLE. J. D. Bond, Ghormley.
MARYVILLE. Knapp.
MEMPHIS. Armstrong.
NASHVILLE. Blair, S. I. Jones, N. P. Miser, W. L. Miser, Wren.

TEXAS. (81)

ABILENE. Burnam, Tate.
ALPINE. Gilley.
AMARILLO. Barrick.
AUSTIN. Barnes, Batchelder, Benedict, Cooper, Decherd, Dodd, Dorroh, Ettlinger, Feenberg, Holmes, Horton, Lubben, Mitchell, Moore, Muller, Quinn, Vandiver, Whyburn.
BENAVIDES. Pickett.
BOERNE. Hathaway.
BROWNSVILLE. de la Garza.
CAMERON. Newton.
CANYON. Murray.
COLLEGE STATION. Binney, Blumberg, Camp, Finlay, Hall, Halperin, D. C. Jones, McKee, Porter.
DALLAS. Dice, Jones, Mahoney, Reinsch.
DENTON. M. C. Brown, Duncan.
EL PASO. Kennedy.
FORT WORTH. Estes, Howard.
GALVESTON. Underwood, Van Fleet.
GEORGETOWN. Wapple.
HOUSTON. Blau, Blumenthal, Bray, Dean, Evans, Ford, Hickey, Lovett, Rees.
LUBBOCK. Hicks, Michie, Sparks, Thompson, Underwood.
MARSHALL. Tinner.
NACOGDOCHES. Cross, Ferguson, Oxsheer.
PRAIRIE VIEW. Randall.
SAN ANTONIO. Klipple, McNelly, Nelson, Udinski.
SHERMAN. May.
STEPHENVILLE, Jones, McSweeny, Marrs, Redden.
TYLER. Holmes, Nelson.
WACO. Harrell.
WICHITA FALLS. Adams, Shirley.

UTAH. (5)

SALT LAKE CITY. Axup, Gibson, Pehrson, Stevenson, Unseld.

VERMONT. (11)

BURLINGTON. Bullard, Butterfield, Donahue, Millington, Swift, Thomas.
MIDDLEBURY. Hazeltine, Perkins, Wiley.
WINOOSKI. Alliot.
WINDSOR. Tracy.

VIRGINIA. (42)

ABINGDON. Wright.
ASHLAND. Blincoe, Simpson.
BLACKSBURG. Brodie, Hatcher, O'Shaughnessy, Williams.
BRIDGEWATER. Shull.
BRISTOL. Mize.
CHARLOTTESVILLE. Stone, Wells.
EAST RADFORD. Bowers, McCain.
EMORY. Miller.
ETTRICK. Stokes.
FARMVILLE. Taliaferro.
HAMPTON INSTITUTE. Perkins.
HOLLINS. Dickinson.
LANGLEY FIELD. Pinkerton.
LEXINGTON. Funkhouser, Paxton, Purdie, L. W. Smith, Watts, Witt.
LYNCHBURG. Berry, Larew, Pattillo.
MONTEREY. Colaw.
RICHMOND. Gaines, Harris, M. L. Smith, Wheeler.
SALEM. Carpenter.
SWEET BRIAR. Morenus.
UNIVERSITY. Echols, Linfield, Luck, Sparrow, Thornton.
WILLIAMSBURG. Russell, Stetson.

WASHINGTON. (14)

PULLMAN. Butler, Isaacs.
SEATTLE. Ballantine, Carlson, Cramlet, Jerbert, Moritz, Mullemeister, Neikirk, Winger.
SPOKANE. Buxton.
TACOMA. Hanawalt.
WALLA WALLA. Bratton.
YAKIMA. Whitney.

WEST VIRGINIA. (11)

FAIRMONT. McCarty.
HARPERS FERRY. Drew.
HUNTINGTON. Hackney.
INSTITUTE. McFall.
MORGANTOWN. Colwell, Davis, Eiesland, Reynolds, Turner, Vehse.
WHEELING. Bagby.

WISCONSIN. (39)

BELOIT. Conwell, Huffer.
LA CROSSE. Adkins.
MADISON. Allen, Austin, Bennett, Evans, Hart, Ingraham, Langer, Lowney, Skinner, Slichter, Sromovsky, Stafford, Van Vleck, Vass, Weaver, Wilson.
MILTON. Whitford.
MILWAUKEE. Beckwith, Bunyan, Ericson, Evans, Knight, Lewandowski, Miller, Parkinson, Pettit, Quarles, Roth, Simpson.
OSHKOSH. Beenken.
PLATTEVILLE. Warner.
RIPON. Woodmansee.
RIVER FALLS. Eide.
SUPERIOR. C. W. Smith.
WAUKESHA. Hopkins.
WEST DE PERE. DeCleene.
WISCONSIN RAPIDS. McMillan.

WYOMING. (5)

LARAMIE. Barr, Bellamy, Neubauer, Rechard, Thom.

FOREIGN MEMBERS. (Other than Canada.)

ARGENTINE. (1)

BUENOS AIRES, Baidaff.

BELGIUM. (1)

ANTWERP. Van Hee.

BURMA. (1)

RANGOON. Campbell.

CHINA. (3)

CANTON. MacDonald.
NANSIANG. Loh.
PEKING. Konantz.

FRANCE. (3)

PARIS. Borel, Fréchet, Hadamard.

GERMANY. (3)

GÖTTINGEN. Boeder, Bond.
MUNICH. Wieleitner.

GREAT BRITAIN. (5)

BRISTOL. Chepmell.
CAMBRIDGE. Wood.
EDINBURGH. Horsburgh.
LONDON. Smiley.
OXFORD. Hardy.

INDIA. (3)

ALLAHABAD CITY. Mitra.
CALCUTTA. Prasad.
MADURA. Lockwood.

ITALY. (7)

BOLOGNA. Bortolotti, Enriques, Pincherle
MESSINA. Crudeli.
PISA. Bianchi.
ROME. Laboccetta.
TURIN. Fubini.

JAPAN. (3)

SENDAI. Hayashi.
TOKYO. Mikami.
PYENGYANG, KOREA. Parker.

NEW ZEALAND. (1)

DUNEDIN. Martyn.

POLAND. (1)

WARSAW. Dickstein.

PORTUGAL. (1)

LISBON. da Cunha.

SOUTH AFRICA. (4)

BLOEMFONTEIN. Arndt.
JOHANNESBURG. Dalton, Frecheville.
RONDEBOSCH. Muir.

SOUTH AUSTRALIA. (1)

ADELAIDE. Wilton.

SPAIN. (1)

MADRID. de Toledo.

SWITZERLAND. (3)

FRIBOURG. Bays.
GENEVA. Fehr.
NEUCHATEL. DuPasquier.

SYRIA. (1)

BEIRUT. Jurdak.

TURKEY. (1)

CONSTANTINOPLE. Mourad.

UKRAINE. (1)

KIEFF. Kryloff.

RECAPITULATION OF MEMBERSHIP

Individual members November 20, 1929	1,964	
Institutional members November 20, 1929	133	
Total membership November 20, 1929		2,097
Total membership November 15, 1927		1,976

CHARTER MEMBERSHIP

Individual charter members	1,045	
Institutional charter members	52	
Total charter membership		1,097
Net gain in individual members	919	
Net gain in institutional members	81	
Total net gain over charter membership		1,000
Total net gain since November 15, 1927		121

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED).

Article I—Name, Purpose and Corporate Seal.

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED).

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

Article II—Membership.

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association. Such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

3. Election to membership shall be by vote of the Board upon written application from the individual or institution seeking admission, endorsed in the case of individuals by two members of the Association.

4. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

Article III—Board of Trustees and Officers.

1. The Officers of the Association shall be a President, two (2) Vice-Presidents, a Secretary-Treasurer, a Librarian and three (3) members of a Committee on Official Journal.

2. The control and management of the affairs and funds of the Association shall be vested in a Board of twenty (20) Trustees (hereinafter called the "Board"), who shall be members of the Association. This Board shall consist of the officers of the Association and twelve (12) additional members.

3. The President shall be elected by the Association's members biennially for a term of two years and shall be ineligible for reëlection. The Vice-Presidents shall be elected by the Association's members annually for a term of one year, and four members of the Board shall be elected by the Association's members annually for a term of three years. They shall be eligible for reëlection, but not for more than two (2) consecutive terms. The Secretary-Treasurer, the Librarian, and the Committee on Official Journal, consisting of the Editor-in-Chief, the Manager and one other member, shall be appointed by the Board. All Officers and other Trustees shall hold over until their respective successors are elected or appointed and qualify.

4. The Board shall transact the official business of the Association and shall report its actions at the annual business meeting of the Association and in the official journal. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board and in the Committee on Official Journal, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Trustees a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. Approximately two months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Board shall announce two candidates for each office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

8. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Trustees and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Trustees.

9. The Vice-Presidents shall, in the absence of the President, have and exercise the powers of the President, their order being determined alphabetically. The Board of Trustees may assign to the Vice-President such duties as may from time to time be determined.

10. The Secretary-Treasurer shall have the usual duties pertaining to the Office of Secretary and of Treasurer, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Trustees and of the annual meeting and special meetings of members, and giving of due notice of all regular and special meetings of the Association and of the Board of Trustees and the supervision and safe-keeping of the funds of the Association. The Secretary-Treasurer shall also have the duty of seeing that whenever Trustees are elected, including the election of Trustees to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificate shall be signed by the Secretary-Treasurer and verified by oath of the President.

11. The Committee on Official Journal shall have supervision of the official journal subject to the control of the Board of Trustees.

12. The Librarian shall have general charge of the library of the Association and shall direct its affairs, including the exchange of the publications of the Association, subject to the control of the Board.

Article IV—Meetings.

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The outgoing Board shall hold a meeting immediately preceding the annual meeting of the Association next succeeding their election, and the members of the new Board shall hold a meeting and organize, by completing the Board, immediately succeeding the annual meeting of the Association at which the new members thereof were elected. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary-Treasurer at least thirty (30) days prior to the date set for each meeting. Notice of all meetings of the Board other than the regular meetings, provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

Article V—Sections.

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections.

Article VI—Official Publications.

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. The official journal shall be under the general management of the Committee on Official Journal. There shall also be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal and under the direction of the Committee on Official Journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

Article VII—Dues.

1. Individual members of the Association shall pay an initiation fee of Two Dollars ($2) at the time of election.

2. The annual dues of each individual member shall be Four Dollars ($4), including a subscription to the official journal.

3. The annual dues of each institutional member shall be Seven Dollars ($7), including two (2) subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a resonable time, his name shall be dropped from the list after due notice.

5. New members entering the Association after April 1 of any year shall have their dues pro-rated for the balance of the year, except when they desire to receive the full current volume of the official journal.

6. The life membership fee shall be the present value, according to McClintock's Male Annuitant Table based upon four (4) per cent interest, of an annuity due of Four Dollars $(4) a year at the attained age of the member; an annual valuation of the life membership fund shall be made under the McClintock Male Four (4) Per Cent Table; and the reserve thus computed shall be held as a liability.

Article VIII—Amendments to the Articles of Association and By-Laws.

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session thereof, or at any special meeting of the Association called for such purpose, by a two-thirds ($\frac{2}{3}$) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

PERIODS OF SERVICE OF THE OFFICERS OF THE ASSOCIATION

Presidents.

E. R. Hedrick....................1916
Florian Cajori....................1917
E. V. Huntington....................1918
H. E. Slaught....................1919
D. E. Smith....................1920
G. A. Miller....................1921
R. C. Archibald....................1922
R. D. Carmichael....................1923
H. L. Rietz....................1924
J. L. Coolidge....................1925
Dunham Jackson....................1926
W. B. Ford....................1927–1928
J. W. Young....................1929–1930

Vice-Presidents.

E. V. Huntington....................1916
G. A. Miller....................1916
D. N. Lehmer....................1917, 1918
Oswald Veblen....................1917
J. W. Young....................1918, 1926
R. G. D. Richardson....................1919
H. L. Rietz....................1919
Helen A. Merrill....................1920
E. J. Wilczynski....................1920
R. C. Archibald....................1921
R. D. Carmichael....................1921, 1922
B. F. Finkel....................1922
A. B. Chace....................1923
L. P. Eisenhart....................1923
J. L. Coolidge....................1924
Dunham Jackson....................1924, 1925
A. A. Bennett....................1925
W. B. Ford....................1926
A. J. Kempner....................1927, 1928
Clara E. Smith....................1927
F. D. Murnaghan....................1928.
E. T. Bell....................1929
W. C. Graustein....................1929

Secretary-Treasurer.

(Appointed by the Trustees after 1918)

W. D. Cairns....................1916–

Committee on Official Journal.

(Appointed by the Trustees.)

H. E. Slaught....................1916–
R. D. Carmichael....................1916–1918
W. H. Bussey....................1916–1918
R. C. Archibald....................1919–1921
W. A. Hurwitz....................1919–1921
A. A. Bennett....................1922
H. P. Manning....................1921–1922
W. B. Ford....................1923–1925
J. L. Coolidge....................1923
A. J. Kempner....................1924–
W. H. Bussey....................1926–

Elected Members of the Board.

D. N. Lehmer....................1916–1918, 1922–1924
R. E. Moritz....................1916–1918
K. D. Swartzel....................1916
Oswald Veblen....................1916, 1920–1922, 1926–
R. C. Archibald....................1916–1917, 1923–
Florian Cajori....................1916, 1918–1923, 1926–
M. B. Porter....................1916–1917
J. W. Young....................1916–1917, 1920–1922
B. F. Finkel....................1916–1921
E. H. Moore....................1916–1921, 1923–1928
Alexander Ziwet....................1916–1918
E. R. Hedrick....................1917–1922, 1924–
J. N. Van der Vries....................1916–1918
Helen A. Merrill....................1917–1919
D. E. Smith....................1917–1919, 1921–1926
Elizabeth B. Cowley....................1918–1920
G. A. Miller....................1918–1920, 1922–1924
E. J. Wilczynski....................1918–1919, 1922–1926
L. P. Eisenhart....................1919–1922, 1925–
E. V. Huntington....................1917, 1919–1927
E. L. Dodd....................1920
R. D. Carmichael....................1920, 1924–
A. A. Bennett....................1921
H. L. Rietz....................1921–1923, 1925–
C. F. Gummer....................1921–1925
Dunham Jackson....................1923–
Clara E. Smith....................1923–1925
A. B. Chace....................1924–1925
J. L. Coolidge....................1926–
E. T. Bell....................1927–1928
E. P. Lane....................1928–
W. B. Ford....................1929–
E. R. Smith....................1929–

The Carus Mathematical Monographs

The Carus Monographs are already fulfilling their mission as intended by the generous donor, Mrs. Mary Hegeler Carus, and her son, Dr. Edward H. Carus.

Somewhat more than one-half the members of the Association have taken advantage of the distribution at cost of the first three Monographs already published. Those who neglected to do so at the start may still have the privilege by applying to the Secretary. Each member is entitled to one copy of each Monograph at this special price.

It would be a great tribute to the donor and an honor to the Association if a large majority of the members would subscribe for the complete series.

It is believed that the Association is rendering a great service to mathematics by this enterprise, and a liberal support from the membership constitutes an appropriate vote of confidence in the undertaking.

Monographs Thus Far Published

No. 1. *Calculus of Variations,* by Professor G. A. Bliss. (First Impression, 1925; Second Impression, 1927.)

No. 2. *Analytic Functions of a Complex Variable,* by Professor D. R. Curtiss. (First Impression, 1926.)

No. 3. *Mathematical Statistics,* by Professor H. L. Rietz. (First Impression, March, 1927; Second Impression, October, 1929.)

No. 4. *Projective Geometry,* by Professor J. W. Young.

No. 5. In preparation.